

Sharp bounds for the heat kernel on certain symmetric spaces of non-compact type

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ABSTRACT. We discuss a sharper than Gaussian bound for the heat kernel (acting on functions) of a split rank or rank one symmetric space of non-compact type. The proof uses a modified Minakshisundaram-Pleijel parametrix and a very explicit expression for the Laplacian of the Jacobian of the exponential map in terms of the restricted roots. The motivation is to generalize the first author's "phase space bounds" to the setting of symmetric spaces.

1. Introduction

In [Gr] Leonard Gross introduced an analog for a compact group K of the classical Hermite expansion for \mathbf{R}^n . In Gross's result the role of the Gaussian measure on \mathbf{R}^n is played by the heat kernel measure on K . This result of Gross has generated a new field of study that may be called "harmonic analysis with respect to heat kernel measure." See the article [Ha3] for a survey of results obtained so far in this field.

One development motivated by the results of Gross is a generalized Segal–Bargmann transform. This was done first for the compact group case in the paper [Ha1], which is based on the first author's Ph.D. thesis, written under the direction of Gross. As shown in Section 11 of [Ha1] the Segal–Bargmann transform for compact groups "descends" in a straightforward way to the case of compact symmetric spaces (or more generally compact normal homogeneous spaces). A better description of the Segal–Bargmann transform for compact symmetric spaces was given in [St]. In particular the description in [St] demonstrates the important role played by the dual non-compact symmetric space and the heat kernel thereon.

The paper [Ha2] gives certain sharp "phase bounds" in the setting of the Segal–Bargmann transform for compact Lie groups. The results of [St] make it natural to try to generalize these results to the case of general compact symmetric spaces. To do this one needs a uniform, small-time estimate on the analytic continuation of the heat kernel of a symmetric space of compact type into its complexification (the "compact side") and a uniform, small-time estimate on the heat kernel of the

2000 *Mathematics Subject Classification.* Primary 58J35; Secondary 22E30, 43A85.

Key words and phrases. Heat kernel, symmetric spaces, split-rank, rank one.

non-compact dual symmetric space (the “non-compact side”). The contribution of this paper is to give the estimate on the non-compact side for the special case when the symmetric space has split rank or is rank one.

Let X be a simply connected symmetric space of Helgason’s non-compact type which is either “split rank” (see Section 4) or of rank one. Let Δ be the non-negative Laplacian (acting on functions) and let $E(x, y, t)$ be the fundamental solution to the heat equation $(\partial_t + \Delta)u(x, t) = 0$. The goal of this paper is to prove the following sharp estimate on $E(x, y, t)$.

THEOREM 1. *For all $T > 0$ there is a constant $C > 0$ such that for all $(x, y, t) \in X \times X \times (0, T]$,*

$$E(x, y, t) \leq (4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \theta^{-1/2}(x, y) (1 + Ct).$$

Here θ is the “Jacobian of the exponential map” (see Section 2), $d(x, y)$ is the geodesic distance from x to y , and $n = \dim X$.

There is an extensive body of literature concerning estimates of the heat kernel on a symmetric space of non-compact type. For the heat kernel of any complete, simply connected manifold of dimension n and sectional curvature less than or equal zero one has the “Euclidean” estimate [DGM]: for all $(x, y, t) \in X \times X \times (0, \infty)$,

$$E(x, y, t) \leq (4\pi t)^{-n/2} e^{-d^2(x,y)/4t}.$$

Molchanov [M] has shown that for any Riemannian manifold

$$\lim_{t \rightarrow 0^+} E(x, y, t) (4\pi t)^{n/2} e^{d^2(x,y)/4t} \theta^{1/2}(x, y) = 1$$

for x, y fixed and $d(x, y)$ sufficiently small. The estimate in Theorem 1 is sharper than both of these because it is a uniform global estimate and $\theta^{-1/2}$ has exponential decay at infinity in X . For example, the heat kernel for three dimensional hyperbolic space (with sectional curvature -1) is

$$E(x, y, t) = (4\pi t)^{-3/2} e^{-d^2(x,y)/4t} \theta^{-1/2}(x, y) e^{-t}$$

where $\theta^{-1/2}(x, y) = d(x, y) / \sinh d(x, y)$ (see [D, p. 179]). A similar formula holds for symmetric spaces of the form K_C/K where K is a compact Lie group and K_C its complexification [G]. This shows our estimate in Theorem 1 is essentially sharp.

Our result is related to the “Anker conjecture” which gives very precise upper and lower bounds for the heat kernel on symmetric spaces of the non-compact type. (See Section 3 of [AJ] for a precise statement.) The conjecture has been verified for certain cases, including the ones we consider here, namely, the rank one case [A] and the split rank case [S]. For general symmetric spaces the Anker conjecture has been proved when $t > d(x, y)$ [AJ]. (This last result does not help us in the problem we are studying here, namely, the behavior of the heat kernel at a fixed time as the distance tends to infinity.)

When applied at a fixed time as the distance tends to infinity, the Anker conjecture tells us that the heat kernel is bounded by an expression similar to the one in Theorem 1, except with the factor of $(1 + Ct)$ replaced by C . Thus our result is slightly stronger than what one gets from the Anker conjecture, since the factor of $(1 + Ct)$ tends to 1 as t tends to zero. This slight improvement over the Anker conjecture is necessary to get the non-compact part of the phase space bounds we are aiming for.

The idea of the proof in the split rank case is to exploit the fact that a modified Minakshisundaram-Pleijel parametrix for the heat equation gives the exact heat kernel [C]. We then estimate the growth of the (finite number) of individual terms in the parametrix. To do this we use a recursive formula for the terms and a very explicit expression for the Laplacian of $\theta^{-1/2}$. For the rank one cases which are not split rank this approach fails because the modified parametrix does not give the exact heat kernel. Our approach then is to estimate the convolutions which measure the difference between the modified parametrix and the actual heat kernel. An essential ingredient is the fact that the terms in the parametrix decay at infinity. This decay property does not hold in the higher rank cases.

The authors would like to thank the referee for many helpful comments.

2. The Laplacian of $\theta^{-1/2}$

We will follow the notation and conventions of Helgason [He1, Chapter VI]. Let X be an irreducible, simply connected Riemannian globally symmetric space of the non-compact type. Let G denote the identity component of the isometry group of X and K the isotropy subgroup at a chosen origin, o . G acts transitively on X and so X can be identified with G/K equipped with a G -invariant metric. Let \mathfrak{g}_o be the Lie algebra of G and \mathfrak{k}_o the Lie algebra of K . The geodesic symmetry about o in X induces an involution of \mathfrak{g}_o and a decomposition, $\mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{p}_o$, into $+1$ and -1 eigenspaces, respectively, for the involution. The tangent map at the identity of the projection $\pi : G \rightarrow G/K \cong X$ identifies \mathfrak{p}_o with T_oX . The Riemannian exponential map at the origin $o \in X$ is $\text{Exp}_o V = \exp(V) \cdot K$ where \exp is the exponential map of G and T_oX is identified with \mathfrak{p}_o as above. Let $\tau(g)$ (or just g) denote the action of $g \in G$ on X . The tangent map to the exponential map at the origin is

$$(d \text{Exp}_o)_V = d\tau(\exp V)_o \circ \sum_0^\infty \frac{(T_V)^n}{(2n+1)!}, \quad V \in \mathfrak{p}_o,$$

where $T_V = \text{ad}(V)^2$ restricted to \mathfrak{p}_o . We have identified both T_oX and $T_V(T_oX)$ with \mathfrak{p}_o .

We define the function $\theta : X \times X \rightarrow (0, \infty)$ as follows. The Riemannian exponential map at any point $x \in X$ is a diffeomorphism from T_xX to X . Thus for any x, y in X we can consider the invertible linear map

$$d(\text{Exp}_x)_{\text{Exp}_x^{-1}y} : T_{\text{Exp}_x^{-1}y}(T_xX) \cong T_xX \rightarrow T_yX.$$

Using orthonormal bases in T_xX and T_yX we define $\theta(x, y)$ by

$$\theta(x, y) = \left| \det \left(d(\text{Exp}_x)_{\text{Exp}_x^{-1}y} \right) \right|.$$

This is well-defined because different choices of orthonormal bases will only change the sign of the determinant. The function θ is invariant under the diagonal action of G and symmetric in (x, y) . Our goal in this section is to compute $\Delta_2 \theta^{-1/2}(x, y)$ where Δ_2 is the Laplacian acting on the second variable.

We first recall a useful expression for θ . Choose a maximal abelian subspace $\mathfrak{h}_{\mathfrak{p}_o}$ of \mathfrak{p}_o and let Σ be the roots of \mathfrak{g}_o with respect to $\mathfrak{h}_{\mathfrak{p}_o}$ (i.e., the ‘‘restricted roots’’). Choose an ordering for the dual of $\mathfrak{h}_{\mathfrak{p}_o}$ and let Σ^+ be the positive restricted roots with respect to this ordering. Let $\mathfrak{h}_{\mathfrak{p}_o}^+ = \{H \in \mathfrak{h}_{\mathfrak{p}_o} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+\}$. Every $p \in X$ can be written as $p = \text{Exp}_o(\text{Ad}(k)H) = k \exp(H) \cdot o$ with $H \in \mathfrak{h}_{\mathfrak{p}_o}^+$,

$k \in K$. The H (but not the k) is uniquely determined by p [**He1, Theorem 1.1, Chap. IX**]. Finally let m_α be the multiplicity of $\alpha \in \Sigma$, i.e., the dimension of the joint eigenspace for α . The following expression for θ is well known (see [**He1, p. 294**] for the proof in the compact case; the non-compact case is essentially the same).

LEMMA 1. *Let $x, y \in X$, $x = g_1 \cdot o$, $y = g_2 \cdot o$ and write $g_1^{-1}g_2 \cdot o = k \exp(H) \cdot o$ with $H \in \mathfrak{h}_{\mathfrak{p}_o}^+$, $k \in K$. Then*

$$\theta(x, y) = \theta(o, \exp(H) \cdot o) = \prod_{\alpha \in \Sigma^+} \left(\frac{\sinh \alpha(H)}{\alpha(H)} \right)^{m_\alpha}.$$

The remainder of this section is devoted to proving Proposition 1 (see below). We will need a preliminary lemma to facilitate the computations. The proof can be found in the Appendix.

LEMMA 2. *On the open dense subset of $\mathfrak{h}_{\mathfrak{p}_o}$ where they are defined,*

$$\begin{aligned} \text{i)} \quad & \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+}} m_\alpha m_\beta \langle \alpha, \beta \rangle \frac{1}{\alpha(H)\beta(H)} = 0 \\ \text{ii)} \quad & \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+}} m_\alpha m_\beta \langle \alpha, \beta \rangle (\coth \alpha(H) \coth \beta(H) - 1) = 0. \end{aligned}$$

We recall that if two roots are proportional, then the constant of proportionality must be ± 1 , ± 2 , or $\pm 1/2$ [**He1, Chapter X, Section 3**]. A root α is said to be multipliable if 2α is a root. Let F be the function which is invariant under the diagonal action of G and defined, using the conventions of Lemma 1, on an open dense subset of $X \times X$ by

$$\begin{aligned} F(x, y) = F(o, \exp(H) \cdot o) &= \sum_{\alpha \in \Sigma^+} \frac{m_\alpha(m_\alpha - 2)}{4} |\alpha|^2 \left(\operatorname{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) \\ &+ 2 \sum_{\text{multipliable } \alpha \in \Sigma^+} \frac{m_\alpha m_{2\alpha}}{4} |\alpha|^2 \left(\operatorname{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right). \end{aligned}$$

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. The Riemannian metric on X gives an inner product on $\mathfrak{h}_{\mathfrak{p}_o}$ and $\mathfrak{h}_{\mathfrak{p}_o}^*$. We use this inner product to define $|\rho|^2$. The main result of this section is the following computation of $\Delta_2 \theta^{-1/2}$.

PROPOSITION 1. $\Delta_2 \theta^{-1/2}(x, y) = \theta^{-1/2}(x, y)(|\rho|^2 + F(x, y))$ where Δ_2 is the non-negative Laplacian acting on the second variable.

PROOF. By continuity it suffices to prove the proposition for $H \in \mathfrak{h}_{\mathfrak{p}_o}^+$. From the G -invariance of Δ_2 and the invariance of θ under the diagonal action of G we have $\Delta_2 \theta^{-1/2}(x, y) = \Delta_2 \theta^{-1/2}(o, \exp(H) \cdot o)$. The function $\theta_o(y) := \theta(o, y)$ is invariant under the action of K , so

$$\overline{\Delta_2 \theta^{-1/2}(o, \cdot)} = \operatorname{Rad}(\Delta)_2 \overline{\theta^{-1/2}(o, \cdot)}$$

where the overline indicates restriction to $\exp(\mathfrak{h}_{\mathfrak{p}_o}^+) \cdot o$ and $\operatorname{Rad}(\Delta)$ is the radial part of Δ for the action of K on $X = G/K$ with $\exp(\mathfrak{h}_{\mathfrak{p}_o}^+) \cdot o$ as a transversal

manifold. Let H_1, \dots, H_l be an orthonormal basis of $\mathfrak{h}_{\mathfrak{p}_o}$. Using the proof of [He2, Proposition 3.9, Chap. II] we have

$$\text{Rad}(\Delta) = - \sum_{i=1}^l \left(D_{H_i}^2 + \sum_{\alpha \in \Sigma^+} m_\alpha (\coth \alpha(\cdot)) \alpha(H_i) D_{H_i} \right)$$

where D_{H_i} is the directional derivative in the direction of H_i (again, we use the non-negative Laplacian) so that

$$\begin{aligned} \Delta_2 \theta^{-1/2}(o, \exp(H) \cdot o) \\ = - \sum_{i=1}^l \left(D_{H_i}^2 + \sum_{\alpha \in \Sigma^+} m_\alpha (\coth \alpha(H)) \alpha(H_i) D_{H_i} \right) \overline{\theta^{-1/2}}(o, \exp(H) \cdot o). \end{aligned}$$

To compute the action of the D_{H_i} on $\overline{\theta^{-1/2}}$ we use Lemma 1: for $H \in \mathfrak{h}_{\mathfrak{p}_o}$,

$$\begin{aligned} D_{H_i} \overline{\theta^{-1/2}}(o, \exp(H) \cdot o) &= \frac{d}{dt} \Big|_{t=0} \theta^{-1/2}(o, \exp(H + tH_i) \cdot o) \\ &= \frac{d}{dt} \Big|_{t=0} \prod_{\alpha \in \Sigma^+} \left(\frac{\sinh \alpha(H + tH_i)}{\alpha(H + tH_i)} \right)^{-m_\alpha/2}. \end{aligned}$$

Thus

$$\begin{aligned} D_{H_i} \overline{\theta^{-1/2}}(o, \exp(H) \cdot o) \\ = - \overline{\theta^{-1/2}}(o, \exp(H) \cdot o) \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{2} \left(\coth \alpha(H) - \frac{1}{\alpha(H)} \right) \alpha(H_i), \\ \sum_i D_{H_i}^2 \overline{\theta^{-1/2}}(o, \exp(H) \cdot o) = \overline{\theta^{-1/2}}(o, \exp(H) \cdot o) \\ \times \left[\sum_{\alpha, \beta \in \Sigma^+} \frac{m_\alpha m_\beta}{4} \left(\coth \alpha(H) - \frac{1}{\alpha(H)} \right) \left(\coth \beta(H) - \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle \right. \\ \left. + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{2} \left(\text{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \right], \\ \sum_i \sum_{\alpha \in \Sigma^+} m_\alpha (\coth \alpha(H)) \alpha(H_i) D_{H_i} \overline{\theta^{-1/2}}(o, \exp(H) \cdot o) \\ = - \overline{\theta^{-1/2}}(o, \exp(H) \cdot o) \sum_{\alpha, \beta \in \Sigma^+} \frac{m_\alpha m_\beta}{2} \coth \alpha(H) \left(\coth \beta(H) - \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle. \end{aligned}$$

Multiplying out and canceling terms gives

$$\begin{aligned} \Delta_2 \theta^{-1/2}(o, \exp(H) \cdot o) &= - \overline{\theta^{-1/2}}(o, \exp(H) \cdot o) \\ &\times \left[\sum_{\alpha, \beta \in \Sigma^+} \frac{m_\alpha m_\beta}{4} \left(- \coth \alpha(H) \coth \beta(H) + \frac{1}{\alpha(H) \beta(H)} \right) \langle \alpha, \beta \rangle \right. \\ &\left. + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{2} \left(\text{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \right]. \end{aligned}$$

Decomposing the first sum into “diagonal” and “off-diagonal” terms shows that the terms in the brackets above can be written as

$$-\left[\sum_{\alpha \in \Sigma^+} \left(\frac{m_\alpha |\alpha|}{2} \right)^2 + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha (m_\alpha - 2)}{4} \left(\operatorname{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \right. \\ \left. + \sum_{\substack{\alpha \neq \beta \\ \alpha, \beta \in \Sigma^+}} \frac{m_\alpha m_\beta}{4} \left(\coth \alpha(H) \coth \beta(H) - \frac{1}{\alpha(H)\beta(H)} \right) \langle \alpha, \beta \rangle \right].$$

We have, since $\rho = \frac{1}{2} \sum_{\Sigma^+} m_\alpha \alpha$,

$$\langle \rho, \rho \rangle = \sum_{\alpha, \beta \in \Sigma^+} \frac{m_\alpha m_\beta}{4} \langle \alpha, \beta \rangle = \sum_{\alpha \in \Sigma^+} \left(\frac{m_\alpha |\alpha|}{2} \right)^2 + \sum_{\substack{\alpha \neq \beta \\ \alpha, \beta \in \Sigma^+}} \frac{m_\alpha m_\beta}{4} \langle \alpha, \beta \rangle$$

so that $\Delta_2 \theta^{-1/2}(o, \exp(H) \cdot o)$ is equal to

$$\theta^{-1/2}(o, \exp(H) \cdot o) \left[\langle \rho, \rho \rangle + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha (m_\alpha - 2)}{4} \left(\operatorname{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \right. \\ \left. + \sum_{\substack{\alpha \neq \beta \\ \alpha, \beta \in \Sigma^+}} \frac{m_\alpha m_\beta}{4} \left(\coth \alpha(H) \coth \beta(H) - \frac{1}{\alpha(H)\beta(H)} - 1 \right) \langle \alpha, \beta \rangle \right].$$

By Lemma 2 and the remark following it, the last summation above is equal to the sum over the positive roots α, β with $\alpha = k\beta$, $k = 1$ or $\frac{1}{2}$. We split the last summation above into the sum over $\alpha = 2\beta$ plus the sum over $\alpha = \frac{1}{2}\beta$. Since the summation is symmetric in α and β , the last summation in the displayed equation above can be written as

$$2 \sum_{\{\alpha \in \Sigma^+ : 2\alpha \in \Sigma^+\}} \frac{m_\alpha m_{2\alpha}}{4} \left(\coth \alpha(H) \coth 2\alpha(H) - \frac{1}{2\alpha(H)^2} - 1 \right) 2 \langle \alpha, \alpha \rangle.$$

The identity $2(\coth \alpha \coth 2\alpha - 1/(2\alpha^2) - 1) = \operatorname{csch}^2 \alpha - 1/\alpha^2$ gives the Proposition. \square

3. The modified Minakshisundaram-Pleijel parametrix

From Proposition 1 we have $\Delta_2 \theta^{-1/2} = \theta^{-1/2}(|\rho|^2 + F)$ where F is given explicitly. We look for a parametrix $S_k(x, y, t) \in C^\infty(X \times X \times (0, \infty))$ for the heat equation of the form

$$S_k(x, y, t) = (4\pi t)^{-n/2} e^{-|\rho|^2 t - d(x, y)^2/4t} \theta^{-1/2}(x, y) \left(\sum_{i=0}^k t^i v_i(x, y) \right)$$

such that for $k = 0, 1, \dots$,

$$(1) \quad (\partial_t + \Delta_2) S_k(x, y, t) = \\ (4\pi t)^{-n/2} e^{-|\rho|^2 t - d(x, y)^2/4t} \theta^{-1/2}(x, y) t^k (F + (\operatorname{grad}_2 \log \theta) \cdot \operatorname{grad}_2 + \Delta_2) v_k(x, y),$$

where grad_2 is the gradient in the second variable (and \cdot is the inner product induced by the metric), and

$$(2) \quad \lim_{t \rightarrow 0^+} S_k(x, y, t) = \delta_x(y).$$

Our parametrix differs from the usual one in that we have factored out $e^{-|\rho|^2 t}$ and $\theta^{-1/2}(x, y)$. Let $r = d(x, y)$ and let y_τ denote the unit speed geodesic from x to y with parameter τ .

PROPOSITION 2. *If $v_0(x, y) = 1$ and*

$$v_i(x, y) = -r^{-i} \int_0^r (F + (\text{grad}_2 \log \theta) \cdot \text{grad}_2 + \Delta_2) v_{i-1}(x, y_\tau) \tau^{i-1} d\tau$$

for $1 \leq i \leq k$, then (1), (2) hold.

PROOF. We follow [BGM, Chapter III, E.III]. We compute

$$\begin{aligned} \partial_t S_k(x, y, t) &= (4\pi t)^{-n/2} e^{-|\rho|^2 t - r^2/4t} \theta^{-1/2}(x, y) \\ &\quad \times \left(\left(-\frac{n}{2t} - |\rho|^2 + \frac{r^2}{4t^2} \right) \sum_{i=0}^k t^i v_i + \sum_{i=0}^k i t^{i-1} v_i \right). \end{aligned}$$

Using

- a. $\Delta_2(fh) = h\Delta_2 f - 2g(d_2 f, d_2 h) + f\Delta_2 h$ ([BGM, Chapter III, G.III.6]; recall we use the non-negative Laplacian. Here d_2 means differentiation with respect to the second variable.)
- b. $g(d_2 e^{-r^2/4t}, d_2 f) = \partial_r e^{-r^2/4t} \partial_r f$ (this follows from the Gauss Lemma, c.f. [BGM, page 207], last displayed line. Here $\partial_r f(x, y)$ means differentiation along the unique geodesic joining x to y with x held fixed.)
- c. $\Delta_2 e^{-r^2/4t} = e^{-r^2/4t} \left(-\frac{r^2}{4t^2} + \frac{n}{2t} + \frac{r}{2t} \cdot \frac{\partial_r \theta}{\theta} \right)$ (see the expression for $\Delta_2 G$ in [BGM, page 207])

we compute

$$\begin{aligned} \Delta_2(e^{-r^2/4t} \theta^{-1/2} v_i) &= e^{-r^2/4t} \theta^{-1/2} \left(\left(-\frac{r^2}{4t^2} + \frac{n}{2t} + \theta^{1/2} \Delta_2 \theta^{-1/2} \right) v_i \right. \\ &\quad \left. + \frac{1}{t} r \partial_r v_i - 2(\text{grad}_2 \log \theta^{-1/2}) \cdot \text{grad}_2 v_i + \Delta_2 v_i \right). \end{aligned}$$

Using that $\Delta_2 \theta^{-1/2} = \theta^{-1/2} (|\rho|^2 + F)$ (Proposition 1) we get

$$\begin{aligned} (\partial_t + \Delta_2) S_k &= (4\pi t)^{-n/2} e^{-|\rho|^2 t - r^2/4t} \theta^{-1/2} \\ &\quad \times \left(\sum_0^k t^i \left(F + \frac{1}{t} (i + r \partial_r) \right) + (\text{grad}_2 \log \theta) \cdot \text{grad}_2 + \Delta_2 \right) v_i \right). \end{aligned}$$

The term of order t^{-1} in the sum vanishes if and only if $r \partial_r v_0 = 0$. If we take $v_0 = 1$, then S_k satisfies (2) as in the usual Minakshisundaram-Pleijel expansion. The term of order t^{-1+i} in the sum vanishes if and only if

$$(i + r \partial_r) v_i + (F + (\text{grad}_2 \log \theta) \cdot \text{grad}_2 + \Delta_2) v_{i-1} = 0.$$

Having chosen v_0, v_1, \dots, v_{i-1} , the term of order t^{-1+i} will vanish if

$$v_i = -r^{-i} \int_0^r (F + (\text{grad}_2 \log \theta) \cdot \text{grad}_2 + \Delta_2) v_{i-1}(x, y_\tau) \tau^{i-1} d\tau.$$

Choosing the v_i , $i = 1, \dots, k$, in this way leaves only the term of order t^k in the sum. \square

4. Estimate of the heat kernel in the split-rank case

A symmetric space $X = G/K$ is said to have split-rank if $\text{rank } G = \text{rank } G/K + \text{rank } K$; equivalently the multiplicities of the restricted roots are all even. This is equivalent to the condition that all Cartan subalgebras of the (real) Lie algebra of G are conjugate under the adjoint group. By the classification of symmetric spaces, the irreducible, simply connected symmetric spaces of non-compact type with split rank are the odd dimensional hyperbolic spaces, K_C/K where K is a compact Lie group, $SU^*(2n)/Sp(n)$, and $E_{6(-26)}/F_4$.

In the split-rank case the modified Minakshisundaram-Pleijel (M-P) coefficients, $v_i(x, y)$, vanish for i sufficiently large [C, §9]. Thus the modified M-P expansion gives the *exact* heat kernel. To prove Theorem 1 in the split-rank case it suffices to show that each $v_i(x, y)$ is bounded by a constant C_i . The rest of this section is devoted to proving this estimate.

Let F be the function on $X \times X$ defined before Proposition 1. By Proposition 1, $\Delta_2 \theta^{-1/2}(x, y) = \theta^{-1/2}(x, y)(|\rho|^2 + F(x, y))$. From Proposition 2 we have $v_0 = 1$ and, after a change of variables,

$$v_i(x, y) = - \int_0^1 (F + \Delta_2 + (\text{grad}_2 \log \theta) \cdot \text{grad}_2) v_{i-1}(x, y_{rs}) s^{i-1} ds$$

for $i \geq 1$. Here y_τ is the unit speed geodesic from x to y parameterized by τ and $r = d(x, y)$. Since the v_i are invariant under the diagonal action of G we can define functions on $\mathfrak{h}_{\mathfrak{p}_o}$, still denoted v_i , which are invariant under the action of the Weyl group, W , by

$$v_i(H) = v_i(o, \exp(H) \cdot o).$$

Then for $H \in \mathfrak{h}_{\mathfrak{p}_o}^+$,

$$v_i(H) = - \int_0^1 (F + \Delta + (\text{grad} \log \theta) \cdot \text{grad}_2)_{\text{rad}} v_{i-1}(sH) s^{i-1} ds$$

where the subscript “rad” indicates the radial part of the operator for the action of K on $X = G/K$ with $A^+ \cdot o := \exp(\mathfrak{h}_{\mathfrak{p}_o}^+) \cdot o$ as a transversal manifold. Here F is thought of as a W -invariant function on $\mathfrak{h}_{\mathfrak{p}_o}$. We will identify $A \cdot o := \exp(\mathfrak{h}_{\mathfrak{p}_o}) \cdot o$ with $\mathfrak{h}_{\mathfrak{p}_o}$ since \exp is an isometry between the two. It suffices to show that the v_i , thought of as W -invariant functions on $\mathfrak{h}_{\mathfrak{p}_o}$, are bounded by constants C_i .

LEMMA 3. $(F + \Delta + (\text{grad} \log \theta) \cdot \text{grad}_2)_{\text{rad}} = F + \Delta_{\mathfrak{h}_{\mathfrak{p}_o}} - \sum_{\beta \in \Sigma^+} \frac{m_\beta}{\beta} A_\beta$ where

A_β is the differential operator $\text{grad} \beta \cdot \text{grad}$ and $\Delta_{\mathfrak{h}_{\mathfrak{p}_o}}$ is the Laplacian, both with respect to the induced flat metric on $\mathfrak{h}_{\mathfrak{p}_o}$.

PROOF. We will first show that $((\text{grad}_2 \log \theta) \cdot \text{grad}_2)_{\text{rad}} = \sum_{\beta \in \Sigma^+} m_\beta (\coth \beta - 1/\beta) A_\beta$, i.e., if $f(x, \cdot)$ is a locally K -invariant function on an open subset of X and $H \in \mathfrak{h}_{\mathfrak{p}_o}^+$, then

$$(3) \quad (\text{grad}_2 \log \theta) \cdot \text{grad}_2 f(x, \exp(H) \cdot o) = \sum_{\beta \in \Sigma^+} m_\beta (\coth \beta - \frac{1}{\beta}) A_\beta (f(x, \cdot)|_{A^+ \cdot o})(\exp(H) \cdot o).$$

Choose coordinates on X near a point $a_o \in A^+$ as in [He2, p. 262, Eq. (35)] (except here $r = \text{rank } X = \dim A$):

$$\begin{aligned} & (x_1(b \cdot a), \dots, x_{n-r}(b \cdot a), x_{n-r+1}(b \cdot a), \dots, x_n(b \cdot a)) \\ &= (y_1(b), \dots, y_{n-r}(b), z_{n-r+1}(a), \dots, z_n(a)) \end{aligned}$$

where a in a neighborhood A_o of a_o in A^+ and B is a relatively compact submanifold of K forming a local cross section through e over a neighborhood of eK^{a_o} in K/K^{a_o} ; the y_i are local coordinates on B and the z_α are local coordinates on A . By [loc. cit., Eq. (37)] we have

$$(\text{grad}_2 \log \theta) \cdot \text{grad}_2 = \sum_{i,j=1}^{n-r} g^{ij} (\partial_i \log \theta) \partial_j + \sum_{\alpha, \beta=n-r+1}^n g^{\alpha\beta} (\partial_\alpha \log \theta) \partial_\beta.$$

For f locally invariant under K we have $\partial_j f = 0$, $j = 1, \dots, n-r$. Thus

$$((\text{grad}_2 \log \theta) \cdot \text{grad}_2 f)|_{A^+ \cdot o} = (\text{grad}_{A \cdot o} \log \theta) \cdot \text{grad}_{A \cdot o} (f|_{A^+ \cdot o})$$

where $\text{grad}_{A \cdot o}$ is the gradient with respect to the induced flat metric on $A \cdot o$. Identifying $A \cdot o$ (isometrically) with $\mathfrak{h}_{\mathfrak{p}_o}$ and using Lemma 1 gives (3). To prove the lemma we recall

$$(\Delta)_{\text{rad}} = \Delta_{A \cdot o} - \sum_{\beta \in \Sigma^+} m_\beta (\coth \beta) A_\beta$$

[He2, Proposition 3.9, Chap. II]. Note the Laplacian in [He2] is non-positive; we use the non-negative Laplacian. \square

Let $C_{bW}^\omega(\mathfrak{h}_{\mathfrak{p}_o})$ denote the set of real-analytic, bounded, W -invariant functions f on $\mathfrak{h}_{\mathfrak{p}_o}$ such that for any constant coefficient differential operator D on $\mathfrak{h}_{\mathfrak{p}_o}$ there exists a $C_{f,D}$ such that $|Df| \leq C_{f,D}$. We will say that a real analytic function f on $\mathfrak{h}_{\mathfrak{p}_o}$ extends to a uniform tube about $\mathfrak{h}_{\mathfrak{p}_o}$ if there is an $\epsilon > 0$ such that f can be analytically continued to a tubular neighborhood of radius ϵ about $\mathfrak{h}_{\mathfrak{p}_o}$ in the complexification $\mathfrak{h}_{\mathfrak{p}_o \mathcal{C}}$. By Lemma 3 we have

$$v_i(H) = - \int_0^1 (F + \Delta_{\mathfrak{h}_{\mathfrak{p}_o}} - \sum_{\beta \in \Sigma^+} \frac{m_\beta}{\beta} A_\beta) v_{i-1}(sH) s^{i-1} ds.$$

To prove Theorem 1 in the split-rank case it suffices to prove

PROPOSITION 3. For all i , $v_i \in C_{bW}^\omega(\mathfrak{h}_{\mathfrak{p}_o})$. Furthermore v_i extends to a uniform tube about $\mathfrak{h}_{\mathfrak{p}_o}$.

PROOF. It is clear that $v_0 \in C_{bW}^\omega(\mathfrak{h}_{\mathfrak{p}_o})$. Using the expression above for v_i we compute $v_1 = -F$. Since $F \in C_{bW}^\omega(\mathfrak{h}_{\mathfrak{p}_o})$ it is also clear that $v_1 \in C_{bW}^\omega(\mathfrak{h}_{\mathfrak{p}_o})$. From the explicit form of F given prior to Proposition 1 we see that F and hence v_1 extends to a uniform tube about $\mathfrak{h}_{\mathfrak{p}_o}$. Suppose inductively that $v_{i-1} \in C_{bW}^\omega(\mathfrak{h}_{\mathfrak{p}_o})$ and extends to a uniform tube about $\mathfrak{h}_{\mathfrak{p}_o}$. Then:

- v_i is real analytic. It clearly suffices to show that $\sum_{\beta \in \Sigma^+} (m_\beta A_\beta / \beta) v_{i-1}$ is real analytic. For the reflection s_α we have $s_\alpha(A_\beta v_{i-1}) = A_{s_\alpha \beta} s_\alpha v_{i-1}$. Since v_{i-1} is W -invariant, we have $s_\alpha(A_\beta v_{i-1}) = -A_\beta v_{i-1}$. Thus $A_\beta v_{i-1}$ vanishes on the hyperplane $\beta = 0$. By Lemma 10 (see Appendix) there is a real analytic function h on $\mathfrak{h}_{\mathfrak{p}_o}$ such that $A_\beta v_{i-1} = \beta h$. It follows that v_i is real analytic.

- v_i extends to a uniform tube about $\mathfrak{h}_{\mathfrak{p}_o}$. It suffices to show this for $(A_\beta / \beta) v_{i-1}$. By a rotation we may choose coordinates, (x_1, \dots, x_r) for $\mathfrak{h}_{\mathfrak{p}_o}$ such that $\beta = x_1$ and $A_{x_1} v_{i-1}$ vanishes on $x_1 = 0$. By Lemma 10 there is a real analytic function, h , on $\mathfrak{h}_{\mathfrak{p}_o}$ such that $A_{x_1} v_{i-1} = x_1 h$. Then h extends to some (not necessarily uniform) tube about $\mathfrak{h}_{\mathfrak{p}_o}$. We can then extend h to the uniform tube by setting $h = (A_{z_1} v_{i-1}) / z_1$ away from $\mathfrak{h}_{\mathfrak{p}_o}$.

- v_i is W -invariant. Again it suffices to show that $\sum_{\beta \in \Sigma^+} (m_\beta A_\beta / \beta) v_{i-1}$ is W -invariant. For $\gamma \in \Sigma$ we have

$$s_\gamma \left(\sum_{\beta \in \Sigma^+} \frac{m_\beta}{\beta} A_\beta v_{i-1} \right) = s_\gamma \left(\frac{1}{2} \sum_{\beta \in \Sigma} \frac{m_\beta}{\beta} A_\beta v_{i-1} \right) = \frac{1}{2} \sum_{\beta \in \Sigma} \frac{m_{s_\gamma \beta}}{s_\gamma \beta} A_{s_\gamma \beta} v_{i-1}.$$

Since s_γ permutes the set of all roots this amounts to a rearrangement of the sum. It follows that v_i is W -invariant.

- v_i and all its derivatives are bounded. It is clear from the inductive hypothesis that all derivatives of $\int_0^1 (F + \Delta_{\mathfrak{h}_{\mathfrak{p}_o}}) v_{i-1} (sH) s^{i-1} ds$ are bounded. It remains to show that for each $\beta \in \Sigma^+$, $(A_\beta / \beta) v_{i-1}$ and all its derivatives are bounded. We may choose coordinates (x_1, \dots, x_r) as above such that $\beta = x_1$. After the coordinate change, $(A_{z_1} / z_1) v_{i-1}$ extends to a uniform tube about $\mathfrak{h}_{\mathfrak{p}_o}$. We can then use Cauchy's integral formula to estimate $(A_{z_1} / z_1) v_{i-1}$ and all its derivatives on a slightly smaller, but still uniform, tube. \square

This concludes the proof of Theorem 1 in the split-rank case.

5. Estimate of the heat kernel in the rank one case

We now assume the rank of X is one. In general the Minakshisundaram-Pleijel series is neither terminating nor convergent, so we need to estimate the convolutions which give the fundamental solution (see below). Let $F(x, y) = \theta^{1/2} \Delta_2 \theta^{-1/2}(x, y) - |\rho|^2$ as above. In the rank one case, F depends only on $r = d(x, y)$. We first derive an estimate on the decay of the modified M-P coefficients at infinity.

LEMMA 4. *For every non-negative integer l there are positive constants C_1, C_2 such that for all $r \geq 0$, $|\partial_r^l F(r)| \leq C_1 (C_2 + r)^{-2-l}$.*

PROOF. From Proposition 1 we have

$$F(x, y) = F(r) = C_3 \left(C_4 \left(\operatorname{csch}^2 ar - \frac{1}{a^2 r^2} \right) + C_5 \left(\operatorname{csch}^2 2ar - \frac{1}{4a^2 r^2} \right) + C_6 \left(\coth ar \coth 2ar - \frac{1}{2a^2 r^2} \right) \right)$$

(for some constant a) which satisfies the estimate given above. \square

We stress that Lemma 4 is false in the higher rank case: there is no decay at infinity along a wall. Next we estimate the decay of the modified M-P coefficients as $r \rightarrow \infty$. In the rank one case the v_i only depend on r :

$$v_0(r) = 1, \quad v_i(r) = -r^{-i} \int_0^r \left(F - \partial_\tau^2 - \frac{n-1}{\tau} \partial_\tau \right) v_{i-1}(\tau) \tau^{i-1} d\tau, \quad i \geq 1.$$

LEMMA 5. *For every positive integer i and non-negative integer l there are constants C_1, C_2 such that for all $r \geq 0$, $|\partial_r^l v_i(r)| \leq C_1(C_2 + r)^{-i-l}$.*

PROOF. The estimate holds for $i = 1$ because $F(\tau)$ is integrable over $[0, \infty)$ by Lemma 4. Assume the estimate holds for $v_{i-1}(r)$. Then the integrand $(F - \partial_\tau^2 - \frac{n-1}{\tau} \partial_\tau) v_{i-1}(\tau) \tau^{i-1}$ is integrable over $[0, \infty)$ by the inductive hypothesis and Lemma 4. Differentiating v_i improves the estimate as indicated for the same reason. \square

COROLLARY 1. *For all positive integers k there are constants C_1, C_2 such that for all $(x, y, t) \in X \times X \times (0, \infty)$,*

$$|(\partial_t + \Delta_2) S_k(x, y, t)| \leq e^{-|\rho|^2 t - d(x, y)^2 / 4t} \theta^{-1/2}(x, y) t^{k-n/2} C_1 (C_2 + d(x, y))^{-k-2}.$$

PROOF. Follows immediately from (1), Lemma 5, and the explicit expression for $\Delta + (\operatorname{grad}_2 \log \theta) \cdot \operatorname{grad}_2$ in the rank-one case. (There is no problem at $r = 0$ because v_i is an even function of r , as can easily be seen by induction and the fact that $F(r)$ is even). \square

The fundamental solution of the heat equation is constructed from the modified parametrix in the same way as for the usual M-P expansion. We do not need the cut-off functions that enter into the usual construction because here the exponential map is a global diffeomorphism. The extra decay of the M-P coefficients given in Lemma 5 allows us to estimate directly the error in the modified M-P parametrix approximation to the fundamental solution. Define, for functions A, B for which the integrals converge,

$$A * B(x, y, t) = \int_0^t ds \int_X A(x, m, s) B(m, y, t - s) dm$$

where dm is the Riemannian volume measure. Let

$$G_k(x, y, t) = (\partial_t + \Delta_2) S_k(x, y, t).$$

For $k \gg n/2$ and λ a positive integer the convolutions $(G_k)^{* \lambda}$, $(G_k)^{* \lambda} * S_k$ make sense. The heat kernel is given by, for $k \gg n/2$,

$$(4) \quad E(x, y, t) = S_k(x, y, t) + \sum_{\lambda=1}^{\infty} (-1)^\lambda (G_k)^{* \lambda} * S_k(x, y, t).$$

See [CdV, Section 1]. The goal of the remainder of this section is to prove the rank one case of Theorem 1 by estimating (4). We will think of the roots α and $\rho = \frac{1}{2} \sum_{\Sigma^+} m_\alpha \alpha$ as functions on $X \times X$ by defining $\alpha(g_1 \cdot o, g_2 \cdot o)$ to be $\alpha(H)$, where H is the unique vector in $\overline{\mathfrak{h}_{\mathfrak{p}_o}^+}$ such that $g_1^{-1} g_2 = k_1 \exp(H) k_2$ with $k_1, k_2 \in K$. Note $|H| = d(x, y)$ and α, ρ are symmetric in x and y in the rank one case.

LEMMA 6. *There is a constant C such that for all $(x, y) \in X \times X$,*

$$C e^{-\rho} \prod_{\alpha \in \Sigma^+} (1 + 2\alpha)^{m_\alpha/2} \leq \theta^{-1/2} \leq e^{-\rho} \prod_{\alpha \in \Sigma^+} (1 + 2\alpha)^{m_\alpha/2}.$$

PROOF. There is a constant C such that for all $\alpha \in [0, \infty)$,

$$C(1 + 2\alpha)e^{-\alpha} \leq \frac{\alpha}{\sinh \alpha} \leq (1 + 2\alpha)e^{-\alpha}.$$

The lemma follows immediately. \square

PROPOSITION 4. *There are constants $k \gg n$, C_4 such that for all $(x, y, t) \in X \times X \times (0, \infty)$,*

$$|G_k(x, y, t)| \leq C_4 t^{k-n/2} e^{-|\rho|^2 t - d^2(x, y)/4t - \rho(x, y)}.$$

PROOF. If we choose k large enough that

$$\prod_{\alpha \in \Sigma^+} (1 + 2\alpha)^{m_\alpha/2} (C_2 + d(x, y))^{-k-2} \leq C_3,$$

then the proposition follows from Corollary 1 and Lemma 6. \square

To prove Theorem 1 in the rank one case we will obtain a similar estimate for $(G_k)^{* \lambda} * S_k$, show that $\sum (-1)^\lambda (G_k)^{* \lambda} * S_k$ converges and estimate the sum. We will follow very closely [CdV, Section 1]. In the rank one case, $\rho(x, y)$ is just a positive multiple of $d(x, y)$. We will use the following fact, which seems to be true only for rank one: for all $x, y, m \in X$,

$$(5) \quad \rho(x, y) \leq \rho(x, m) + \rho(m, y).$$

We first estimate the integral over X in the convolution. Let σ be the sectional curvature of X and choose A so that $-A^2 \leq \sigma \leq 0$.

LEMMA 7. [CdV, Lemma 2]. *(Assume rank of X is 1). There is a constant $M > 0$ depending only on n and A such that for all $a, b > 0$, $x, y \in X$,*

$$\begin{aligned} & \int_X e^{-d^2(x, m)/a - d^2(m, y)/b - \rho(x, m) - \rho(m, y)} dm \\ & \leq M e^{-d^2(x, y)/(a+b) - \rho(x, y) + ((n-1)A)^2 ab/(a+b)} \times \left[\left(\frac{1}{a} + \frac{1}{b} \right)^{-\frac{n}{2}} + \left(\frac{1}{a} + \frac{1}{b} \right)^{-n + \frac{1}{2}} \right]. \end{aligned}$$

PROOF. The only difference between our situation and [CdV, Lemma 2] is that we use the estimate, $-\rho(x, m) - \rho(m, y) \leq -\rho(x, y)$, instead of $d(x, m) + d(m, y) \leq d(x, o) + 2d(o, m) + d(y, o)$ in the notation of [CdV, Lemma 2]. \square

LEMMA 8 [CdV, Lemma 3]. *Let \mathcal{A} and $\mathcal{B}: X \times X \times (0, \infty) \rightarrow \mathbf{R}$ satisfy: there are $\alpha, \beta \geq -n/2$ and $K, L > 0$ such for all $(x, y, t) \in X \times X \times (0, \infty)$,*

$$\begin{aligned} |\mathcal{A}(x, y, t)| &\leq Kt^\alpha e^{-d(x, y)^2/4t - \rho(x, y)} \\ |\mathcal{B}(x, y, t)| &\leq Lt^\beta e^{-d(x, y)^2/4t - \rho(x, y)}. \end{aligned}$$

Fix $T > 0$. Then there is a $C_T > 0$ depending only on n, A , and T such that for all $t \in (0, T]$,

$$|\mathcal{A} * \mathcal{B}(x, y, t)| \leq KLC_T e^{-d^2(x, y)/4t - \rho(x, y)} t^{\alpha+\beta+1} \cdot \frac{1}{1 + \frac{n}{2} + \sup(\alpha, \beta)}.$$

PROOF. See [CdV, Lemma 3]; we only use the result for real z in the notation of [CdV]. \square

By induction we obtain an estimate on $(G_k)^{* \lambda}$.

COROLLARY 2. *For all $(x, y, t) \in X \times X \times (0, T]$,*

$$|(G_k)^{* \lambda}(x, y, t)| \leq (C_4 t^{k - \frac{n}{2}})^\lambda (C_T t)^\lambda e^{-d^2(x, y)/4t - \rho(x, y)} \frac{1}{\prod_{j=1}^{\lambda-1} (\frac{n}{2} + j(k - \frac{n}{2} + 1))}.$$

Now we estimate $(G_k)^{* \lambda} * S_k$.

PROPOSITION 5. *Fix $T > 0$ and $k \gg n \geq 2$. There is a constant $C_{T, k, X} > 0$ depending only on T, k , and the geometry and root space structure of X such that for all $(x, y, t) \in X \times X \times (0, T]$,*

$$\left| \sum_{\lambda=0}^{\infty} (G_k)^{* \lambda} * S_k(x, y, t) \right| \leq C_{T, k, X} e^{-d^2(x, y)/4t} \theta^{-1/2}(x, y).$$

PROOF. By Lemma 5, $\sum_{i=0}^k t^i v_i$ is bounded by a k and T -dependent constant on $X \times X \times [0, T]$. Using Lemma 6 we obtain, on $X \times X \times (0, T]$,

$$|S_k(x, y, t)| \leq C_{T, k} t^{-n/2} e^{-d^2(x, y)/4t - \rho(x, y)} \prod_{\alpha \in \Sigma^+} (1 + 2\alpha)^{m_\alpha/2}.$$

In the rank one case there are positive constants a_α such that $\prod_{\alpha \in \Sigma^+} (1 + 2\alpha)^{m_\alpha/2} = \prod (1 + 2a_\alpha d(x, y))^{m_\alpha/2}$. Using Lemma 8 and the above there is a constant $C > 0$ (depending only on T, n, k and A) such that for all $(x, y, t) \in X \times X \times (0, T]$ and all $\lambda \geq 1$,

$$\begin{aligned} |(G_k)^{* \lambda} * S_k(x, y, t)| &\leq \frac{C(C_4 C_T)^\lambda}{\prod_{j=1}^{\lambda-1} (\frac{n}{2} + j(k - \frac{n}{2} + 1))} \int_0^t s^{\lambda(k - \frac{n}{2} + 1) - 1} (t - s)^{-\frac{n}{2}} \\ &\times \int_X e^{-d^2(x, m)/4s - d^2(m, y)/4(t-s) - \rho(x, m) - \rho(m, y)} \prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha d(m, y))^{m_\alpha/2} dm ds. \end{aligned}$$

We use the following estimates. For fixed t and s , let $a = 4s$, $b = 4(t - s)$ and let o be the point on the (unique) geodesic joining x to y such that $d(x, o)/a = d(y, o)/b$ as in [CdV, Proof of Lemma 2]. Then:

1. $d(x, y) = d(x, o) + d(o, y)$ and $d(m, y) \leq d(m, o) + d(x, y)$. Thus

$$\prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha d(m, y))^{m_\alpha/2} \leq \prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha d(x, y))^{m_\alpha/2} \prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha d(o, m))^{m_\alpha/2}.$$

2. $-d^2(x, m)/a - d^2(m, y)/b \leq -d^2(o, m)(1/a + 1/b) - d^2(x, y)/(a + b)$ [CdV, Proof of Lemma 2].

3. $-\rho(x, m) - \rho(m, y) \leq -\rho(x, y)$ (see equation (5)).

We obtain, using $\prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha)^{m_\alpha/2} = \prod (1 + 2a_\alpha d(x, y))^{m_\alpha/2}$ and the lower estimate in Lemma 6, that

$$\begin{aligned} |(G_k)^{\star \lambda} * S_k(x, y, t)| &\leq e^{-d^2(x, y)/4t} \theta^{-1/2}(x, y) C(C_4 C_T)^\lambda \prod_{j=1}^{\lambda-1} \left(\frac{n}{2} + j(k - \frac{n}{2} + 1) \right)^{-1} \\ &\times \int_0^t s^{\lambda(k - \frac{n}{2} + 1) - 1} (t - s)^{-\frac{n}{2}} \int_X e^{-d^2(o, m)(1/a + 1/b)} \prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha d(o, m))^{m_\alpha/2} dm ds \end{aligned}$$

for a possibly different constant $C > 0$ depending only on T , n , k and A . We estimate the integral over X using the volume comparison $dm \leq re^{Ar} dr$ (as in [CdV, Proof of Lemma 2]) to obtain

$$\begin{aligned} &\int_X e^{-d^2(o, m)(1/a + 1/b)} \prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha d(o, m))^{m_\alpha/2} dm \\ &\leq C_7 \int_0^\infty e^{-r^2(1/a + 1/b)} e^{A(n-1)r} r^{n-1} \prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha r)^{m_\alpha/2} dr \\ &\leq C_7 \int_0^\infty e^{-r^2(1/a + 1/b)} e^{A(n-1)r} r^{n-1} (1 + 2a_{\max} r)^{n-1} dr \\ &\leq C_7 e^{\beta^2/4\gamma} \sum_{l=0}^{n-1} (2a_{\max})^l \int_0^\infty e^{-\gamma(r - \beta/2\gamma)^2} r^{n-1+l} dr. \end{aligned}$$

where $\beta = (n - 1)A$, $\gamma = 1/a + 1/b$, C_7 depends only on n and a_{\max} depends only on the root space structure of X . As in [CdV, loc. cit.] we have

$$\begin{aligned} \int_0^\infty e^{-\gamma(r - \beta/2\gamma)^2} r^{n-1+l} dr &\leq C_{8,l} (\gamma^{-(n+l)+1/2} + \gamma^{-(n+l)/2}) \\ &\leq C_9 \left((s(1 - s/t))^{n-1/2} + (s(1 - s/t))^{n/2} \right) \end{aligned}$$

where $C_{8,l} > 0$ depends only on A , n , and l and C_9 depends only on T , A and n . Since $\beta^2/4\gamma$ is bounded by a constant depending only on A , n , and T we obtain

$$\begin{aligned} &\int_X e^{-d^2(o, m)(1/a + 1/b)} \prod_{\alpha \in \Sigma^+} (1 + 2a_\alpha d(o, m))^{m_\alpha/2} dm \\ &\leq C_{10} \left((s(1 - s/t))^{n-1/2} + (s(1 - s/t))^{n/2} \right) \end{aligned}$$

where C_{10} depends only on T , n , A , and the root space structure of X . Changing variables to $u = s/t$ we obtain

$$\begin{aligned} & \int_0^t s^{\lambda(k-\frac{n}{2}+1)-1} (t-s)^{-\frac{n}{2}} \int_X e^{-d^2(o,m)(1/a+1/b)} \prod_{\alpha \in \Sigma^+} (1+2a_\alpha d(o,m))^{m_\alpha/2} dm ds \\ & \leq C_{10} t^{\lambda(k-\frac{n}{2}+1)} \int_0^1 u^{\lambda(k-\frac{n}{2}+1)-1} (1-u)^{-\frac{n}{2}} \left(t^{\frac{n-1}{2}} (u(1-u))^{n-\frac{1}{2}} + (u(1-u))^{\frac{n}{2}} \right) du \\ & \leq C_{11} t^{\lambda(k-\frac{n}{2}+1)} \int_0^1 u^{\lambda(k-\frac{n}{2}+1)-1} (1-u)^{-\frac{n}{2}} \left((u(1-u))^{n-\frac{1}{2}} + (u(1-u))^{\frac{n}{2}} \right) du \end{aligned}$$

where C_{11} depends only on T , n , A , and the root space structure. We use the estimate

$$\int_0^1 u^p (1-u)^q du \leq \frac{1}{1 + \sup(p, q)}$$

to estimate the integrals above (assuming $n \geq 2$):

$$\begin{aligned} & \int_0^1 \left(u^{\lambda(k-n/2+1)-3/2+n} (1-u)^{(n-1)/2} + u^{\lambda(k-n/2+1)-1+n/2} \right) du \\ & \leq \frac{1}{n-1/2 + \lambda(k-n/2+1)} + \frac{1}{n/2 + \lambda(k-n/2+1)} \leq \frac{2}{n/2 + \lambda(k-n/2+1)}. \end{aligned}$$

Finally,

$$\begin{aligned} & \left| \sum_{\lambda=1}^{\infty} (-1)^\lambda (G_k)^{* \lambda} * S_k(x, y, t) \right| \\ & \leq e^{-d^2(x,y)/4t} \theta^{-1/2}(x, y) 2C C_{11} \sum_{\lambda=0}^{\infty} \frac{(C_4 C_T t^{k-n/2+1})^\lambda}{\prod_{j=1}^{\lambda} (n/2 + j(k-n/2+1))}. \end{aligned}$$

Since the sum is bounded by $\exp(C_4 C_T t^{k-n/2+1})$, the proof of the Proposition is complete. \square

PROOF OF THEOREM 1 IN THE RANK ONE CASE. Fix $T > 0$. By (4) and Proposition 5 there is a $C > 0$ (depending on T , k , and X) such that for all $(x, y, t) \in X \times X \times (0, T]$, $E(x, y, t)$ is bounded above by

$$(4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \theta^{-1/2}(x, y) e^{-|\rho|^2 t} \left(1 + \sum_{i=1}^k t^i |v_i| + C(4\pi t)^{n/2} e^{|\rho|^2 t} \right).$$

The theorem now follows from Lemma 5. \square

6. Appendix

PROOF OF LEMMA 2. The root system Σ decomposes into a disjoint union of orthogonal irreducible root systems Σ_k on orthogonal subspaces $V_k \subset \mathfrak{h}_{\mathfrak{p}_\sigma}^*$. Because of the inner products in the sum, it suffices to prove the identity for each irreducible component of the root system. So we assume that Σ is an irreducible root system. (We use this in the proof of Lemma 9, cases 2 and 3). The root system Σ is not necessarily reduced, i.e., there can be roots α for which 2α is also a root. However we recall that if $\alpha, \beta \in \Sigma^+$ and $\alpha = k\beta$, then $k = 1/2, 1$, or 2 .

Proof of i). Let l be the number of elements in Σ^+ . If $l = 1$ then the sum is empty. If $l = 2$, then the only root system that gives a non-empty sum is $\mathfrak{a}_1 \times \mathfrak{a}_1$ which is not irreducible (note the two roots are orthogonal so it is clear the sum is zero). Assume $l > 2$. Let

$$(6) \quad \begin{aligned} P(\alpha_1, \dots, \alpha_l) &= \left(\prod_{\gamma \in \Sigma^+} \gamma \right) \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+}} m_\alpha m_\beta \langle \alpha, \beta \rangle \frac{1}{\alpha\beta} \\ &= \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+}} m_\alpha m_\beta \langle \alpha, \beta \rangle \prod_{\gamma \in \Sigma^+ \setminus \{\alpha, \beta\}} \gamma. \end{aligned}$$

P is a homogeneous polynomial of degree $l - 2$. We will show that P is divisible by $\prod_{\gamma \in \Sigma^+} \gamma$, a homogeneous polynomial of degree l , so the quotient must be zero.

To accomplish this we must show that if $\eta \in \Sigma^+$ is unmultipliable and indivisible, then P is divisible by η ; and if η is divisible or multipliable, then P is divisible by η^2 . This will show that P is divisible by $\prod_{\gamma \in \Sigma^+} \gamma$ since the non-proportional roots are relatively prime as polynomials.

The Weyl group, $W(\Sigma)$, is the group of orthogonal linear transformations of $\mathfrak{h}_{\mathfrak{p}_o}^*$ generated by the reflections s_η , $\eta \in \Sigma$. The Weyl group acts on $\mathfrak{h}_{\mathfrak{p}_o}$ via the Killing form identification with $\mathfrak{h}_{\mathfrak{p}_o}^*$. The action on $\mathfrak{h}_{\mathfrak{p}_o}$ is compatible with the action on $\mathfrak{h}_{\mathfrak{p}_o}^*$ in the sense that if $\eta \in \Sigma$, $s \in W(\Sigma)$ and $H \in \mathfrak{h}_{\mathfrak{p}_o}$, then $s\eta(H) = \eta(sH)$. The Weyl group acts on all functions on $\mathfrak{h}_{\mathfrak{p}_o}$ by pullback: $sf := s^*f$. Note the kernel of the root η is the fixed point set of s_η acting on $\mathfrak{h}_{\mathfrak{p}_o}$. We will need the following result about the action of the Weyl group on certain sums.

LEMMA 9. *Let $f, g: \mathbf{R} \rightarrow \mathbf{R}$ be an odd, resp. even smooth real valued function. Let*

$$R(\alpha_1, \dots, \alpha_l) = \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+}} m_\alpha m_\beta \langle \alpha, \beta \rangle g(\alpha)g(\beta) \prod_{\gamma \in \Sigma^+ \setminus \{\alpha, \beta\}} f(\gamma).$$

Let $\eta \in \Sigma^+$ and let s_η be the reflection across η . Then $s_\eta R = R$ if η is multipliable or divisible and $s_\eta R = -R$ otherwise.

REMARK. We will use this lemma in the following two cases: $f(t) = t$, $g(t) = 1$ and $f(t) = e^t - e^{-t}$, $g(t) = e^t + e^{-t}$.

PROOF. Let $M(\alpha_1, \dots, \alpha_l) = \prod_{\gamma \in \Sigma^+} f(\gamma)$ and let

$$N := R \div M = \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+}} m_\alpha m_\beta \langle \alpha, \beta \rangle \frac{g(\alpha)g(\beta)}{f(\alpha)f(\beta)}.$$

We first claim that N is $W(\Sigma)$ -invariant. The reflections across simple roots generate $W(\Sigma)$. If η is a simple root, then s_η permutes the set $\Sigma^+ \setminus \{\eta, 2\eta\}$ and sends $\eta, 2\eta$ to $-\eta, -2\eta$ respectively (omit 2η if 2η is not a root). The Killing form on $\mathfrak{h}_{\mathfrak{p}_o}^*$ is invariant under the action of $W(\Sigma)$. The multiplicities m_α are also invariant, i.e., if $s \in W$, $\alpha \in \Sigma$, then $m_\alpha = m_{s\alpha}$. Let η be a simple root. To show that N is $W(\Sigma)$ -invariant, decompose the sum into terms that involve $\eta, 2\eta$ (if 2η is a root)

and those that do not, then apply s_η . One obtains

$$\begin{aligned} s_\eta N &= 2 \sum_{\beta \in \Sigma^+ \setminus \{\eta, 2\eta\}} m_\eta m_{s_\eta \beta} < -\eta, s_\eta \beta > \frac{g(-\eta)g(s_\eta \beta)}{f(-\eta)f(s_\eta \beta)} \\ &+ 2 \sum_{\beta \in \Sigma^+ \setminus \{\eta, 2\eta\}} m_{2\eta} m_{s_\eta \beta} < -2\eta, s_\eta \beta > \frac{g(-2\eta)g(s_\eta \beta)}{f(-2\eta)f(s_\eta \beta)} \\ &+ \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+ \setminus \{\eta, 2\eta\}}} m_{s_\eta \alpha} m_{s_\eta \beta} < s_\eta \alpha, s_\eta \beta > \frac{g(s_\eta \alpha)g(s_\eta \beta)}{f(s_\eta \alpha)f(s_\eta \beta)}. \end{aligned}$$

This is the same as N by the parity properties of f , g and because s_η is a permutation of $\Sigma^+ \setminus \{\eta, 2\eta\}$ (and s_η preserves the non-proportionality of roots). Since the reflections across simple roots generate $W(\Sigma)$, N is $W(\Sigma)$ -invariant.

To prove Lemma 9 it suffices to show that for every $\eta \in \Sigma^+$ (not necessarily simple), $s_\eta M = M$ if η is multipliable or divisible and $s_\eta M = -M$ otherwise.

Case 1. Σ is a reduced root system, i.e., if $\alpha, \beta \in \Sigma$ and $\alpha = k\beta$, then $k = \pm 1$. If η is simple then $s_\eta M = -M$ because s_η permutes $\Sigma^+ \setminus \{\eta\}$ and sends η to $-\eta$. Since the simple roots generate $W(\Sigma)$ it follows that for every $s \in W(\Sigma)$, $sM = (\det s)M$. Since the reflections have determinant -1 we are done in this case.

If Σ is not reduced, then by the classification of irreducible root systems Σ must be the root system

$$(\mathbf{bc})_l = \{\pm e_k, \pm 2e_k \ (1 \leq k \leq l), \pm e_i \pm e_l \ (1 \leq i < j \leq l)\}$$

where e_i are the standard coordinate functions on \mathbf{R}^l [He1, Chapter X, Theorem 3.25]. The $W(\Sigma)$ -invariant inner product is unique up to a constant multiple; we will use the standard Euclidean inner product. Decompose $\Sigma = \Sigma_{\text{indiv}} \cup \Sigma_{\text{div}}$ into a disjoint union of indivisible and divisible roots. Then Σ_{indiv} is an irreducible, reduced root system on \mathbf{R}^l and Σ_{div} is the reducible root system $\mathbf{a}_1 \times \cdots \times \mathbf{a}_1$ (l times).

Case 2. $\eta \in \Sigma^+$ is multipliable or divisible. Since $s_{k\eta} = s_\eta$ we can assume that $\eta \in \Sigma_{\text{indiv}}^+$ and $2\eta \in \Sigma_{\text{div}}^+$. Using case 1 applied to the irreducible reduced root system Σ_{indiv}^+ , the orthogonality of the roots in Σ_{div}^+ and the parity of f we have

$$s_\eta \left(\prod_{\gamma \in \Sigma_{\text{indiv}}^+} f(\gamma) \right) = - \prod_{\gamma \in \Sigma_{\text{indiv}}^+} f(\gamma) \quad \text{and} \quad s_\eta \left(\prod_{\gamma \in \Sigma_{\text{div}}^+} f(\gamma) \right) = - \prod_{\gamma \in \Sigma_{\text{div}}^+} f(\gamma)$$

which implies the lemma in this case.

Case 3. Σ is not reduced and $\eta \in \Sigma^+$ is indivisible and unmultipliable. Since Σ_{indiv}^+ is an irreducible, reduced root system it suffices to show that

$$s_\eta \left(\prod_{\gamma \in \Sigma_{\text{div}}^+} f(\gamma) \right) = + \prod_{\gamma \in \Sigma_{\text{div}}^+} f(\gamma).$$

The indivisible and unmultipliable positive roots are $e_i + e_j$, $e_i - e_j$, $1 \leq i < j \leq l$, and the divisible positive roots are $2e_k$, $1 \leq k \leq l$. We compute directly that

$$s_{e_i + e_j}(2e_k) = \begin{cases} 2e_k & i, j \neq k \\ -2e_i & j = k \\ -2e_j & i = k \end{cases} \quad \text{and} \quad s_{e_i - e_j}(2e_k) = \begin{cases} 2e_k & i, j \neq k \\ 2e_i & j = k \\ 2e_j & i = k \end{cases}$$

which proves Lemma 9. \square

We continue with the proof of Lemma 2, part i).

Case 1. η is indivisible and unmultipliable. By Lemma 9 (with $f(t) = t$, $g(t) = 1$), $s_\eta P = -P$. Then P vanishes on the kernel of η . Since η acquires no extra zeros over the complex numbers, Hilbert's Nullstellensatz says that some positive integer power of P is divisible by η . By the unique factorization property P is divisible by η .

Case 2. η is multipliable or divisible. Then P is *a priori* divisible by η because no term in (6) omits both η and $k\eta$. Let $P_2 = P/\eta$. By Lemma 9, $s_\eta P_2 = -P_2$. As in case 1 we conclude that P_2 is divisible by η . (If $l = 3$ then P_2 is a constant, which must be zero by Lemma 9.) This concludes the proof of Lemma 2, part i).

Proof of Lemma 2, part ii). It suffices to show that $\sum_{\alpha \neq k\beta \in \Sigma^+} m_\alpha m_\beta < \alpha, \beta > \coth \alpha \coth \beta$ is constant because $\coth \alpha \rightarrow 1$ as $\alpha \rightarrow \infty$. Write

$$(7) \quad \prod_{\gamma \in \Sigma^+} (e^\gamma - e^{-\gamma}) \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+}} m_\alpha m_\beta < \alpha, \beta > \coth \alpha \coth \beta \\ = \sum_{\substack{\alpha \neq k\beta \\ \alpha, \beta \in \Sigma^+}} m_\alpha m_\beta < \alpha, \beta > (e^\alpha + e^{-\alpha})(e^\beta + e^{-\beta}) \prod_{\gamma \in \Sigma^+ \setminus \{\alpha, \beta\}} (e^\gamma - e^{-\gamma}).$$

Let $P_{\coth}(x, y)$ be the polynomial in $2l$ variables obtained by replacing e^{α_i} by x_i and $e^{-\alpha_i}$ by y_i in the right hand side of (7). P_{\coth} is homogeneous of degree l . We will show that P_{\coth} is divisible by $Q(x, y) = \prod_{i=1}^l (x_i - y_i)$, a homogeneous polynomial of degree l . The quotient is a homogeneous polynomial of degree 0, hence constant.

Lemma 9 (with $f(t) = e^t - e^{-t}$, $g(t) = e^t + e^{-t}$) shows that the right hand side of (7) is invariant under s_η if $\eta \in \Sigma^+$ is multipliable or divisible and skew under s_η otherwise.

Case 1. $\alpha_i \in \Sigma^+$ is indivisible and unmultipliable. Then the right hand of (7) is skew under the reflection s_{α_i} and so vanishes on the kernel of α_i . It follows that P_{\coth} vanishes on the intersection of the kernel of $x_i - y_i$ with the positive "octant." Since the restriction of P_{\coth} to the hyperplane $x_i - y_i = 0$ is a real analytic function vanishing on an open set, it follows that P_{\coth} vanishes on the kernel of $x_i - y_i$. The Nullstellensatz implies that P_{\coth} is divisible by $x_i - y_i$.

Case 2. α_i is multipliable or divisible. We must show that P_{\coth} is divisible by $(x_i - y_i)^2$. We may assume α_i is multipliable and indivisible. Then P_{\coth} is *a priori* divisible by $x_i - y_i$ because no term in (7) omits both $e^{\alpha_i} - e^{-\alpha_i}$ and $e^{2\alpha_i} - e^{-2\alpha_i}$. Let $\tilde{P}_{\coth} = P_{\coth}/(x_i - y_i)$. Then $\tilde{P}_{\coth}(e^\alpha, e^{-\alpha})$ is skew under s_{α_i} and so, as in case 1, $\tilde{P}_{\coth}(x, y)$ is divisible by $x_i - y_i$. This concludes the proof of Lemma 2, part ii). \square

LEMMA 10. *Let β be a non-zero linear function on \mathbf{R}^r and suppose $g \in C^\omega(\mathbf{R}^r)$ vanishes on the hyperplane $\beta = 0$. Then there is an $h \in C^\omega(\mathbf{R}^r)$ such that $g = \beta h$.*

PROOF. By a rotation of coordinates it suffices to prove the lemma for the linear function $\beta(x_1, \dots, x_r) = cx_1$. By the uniqueness of analytic continuation it suffices to prove the existence of h locally at points $p \in \{\beta = 0\}$. Extend g to a holomorphic function \tilde{g} on a polydisk \mathcal{V} centered at p in \mathbf{C}^r . For fixed (z_2, \dots, z_n) ,

\tilde{g}/z_1 is a holomorphic function of z_1 . For r sufficiently small,

$$\tilde{g}(z_1, z_2, \dots, z_r)/z_1 = \frac{1}{2\pi i} \int_{|s|=r} \frac{\tilde{g}(s, z_2, \dots, z_r)/s}{s - z_1} ds.$$

The function h defined in a neighborhood of p by the right hand side is holomorphic in (z_1, \dots, z_r) near p . \square

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