Quiz 1, Math 415, Tanveer, 25 minutes
Instructions: Closed book and notes. Show work. Each problem worth the same

1. (a) Determine with good reasons if the following equation is linear or nonlinear. (here x is the independent variable). (b) Determine the expression for the general solution.

$$\frac{y'}{1+y} = x$$

Solution: (a). On cross multiplication, we obtain $y' = x + xy$, or $y' - xy = x$; of the form $y' + p(x)y = g(x)$, which is linear. (b) Now, integrating factor is $e^{-\int x dx} = e^{-x^2/2}$. Multiplying both sides of equation by this factor, we obtain

$$e^{-x^2/2}\left[y' - xy\right] = xe^{-x^2/2}$$

$$\frac{d}{dx}\left[e^{-x^2/2}y\right] = xe^{-x^2/2}$$

$$e^{-x^2/2}y = \int xe^{-x^2/2}dx = -e^{-x^2/2} + c$$

So, general solution

$$y = -1 + ce^{x^2/2}$$

2. Solve the initial value problem

$$y' = x(1+y^2) \quad , \quad y(0) = 1$$

Solution: Equation separable. We write $\frac{dy}{1+y^2} = xdx$; or

$$\arctan y = \frac{x^2}{2} + c$$

$$y = \tan\left(\frac{x^2}{2} + c\right)$$

Since $y(0) = 1$ implies $\arctan 1 = c$. So, $c = \frac{\pi}{4}$. So, solution to the initial value problem in the implicity form is

$$y = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right)$$

3. People move out of a city at the rate of 10 thousand a year. However, new babies are born a rate proportional to the existing population, with birth rate being 20 babies for every thousand people each year. (a) Find a differential equation for the population of the city. (b) What is the minimum initial population needed for the city not to be completely depopulated ever.

Solution: (a) We model population of city $p(t)$ as a continuous variable and t is measured in years. Population changes because of babies born. If 20 babies are born for every 1000 citizens, $20/1000 * p(t)$ babies will be born every year, which will add to the population. Decrease will be 10,000 per year. So

$$\frac{dp}{dt} = -10^4 + 0.02p$$

(b) Solving the equation as for the mice population problem in class:

$$p(t) = 5 \times 10^5 + Ce^{0.02t}$$

If $p(0) < 5 \times 10^5$, then $C < 0$ and then the for large enough time, there is no population left in time because the exponential blows up negatively and clearly sooner or later becomes larger than 5×10^5. So, threshold city population that will sustain itself is 5×10^5 or half a million people.