Solution to Test 2, Math 415, Tanveer

1a. Find solution $y(t)$ satisfying to the initial value problem:

$$y'' + 4y' + 4y = 0 \ , \ y(0) = 0 \ , \ y'(0) = 1$$

Solution: Since constant coefficient, homogeneous solution, we look at the characterestic equation $r^2+4r+4 = 0$, or $(r+2)^2 = 0$ or $r = -2$ twice. Since roots are coincident, two independent solutions are e^{-2t} and te^{-2t}. So,

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Since $y(0) = 0$, implies $c_1 = 0$. Since $y'(t) = c_2 e^{-2t} - 2c_2 t e^{-2t}$, $y'(0) = 1 = c_2$. Therefore, solution to IVP

$$y(t) = te^{-2t}$$

1b. Find **form** of the particular solution in each of the two cases below (**Note:** not asking to find the coefficients):

i. $y'' + 4y' + 4y = e^{-2t}(1 + t^2)$

Solution: Since $r = -2$ is root of the characteristic equation twice, it follows that we must have

$$y_p = t^2 e^{-2t}(A_0 t^2 + A_1 t + A_2)$$

ii. $y'' + y = e^t \sin(t) + t$

Solution: Since $1 + i$ is not a root of characteristic equation $r^2 + 1 = 0$ (which has roots $\pm i$), it follows $y_{p,1} = A_0 e^t \sin t + B_0 e^t \cos t$ is a particular solution to

$$y'' + y = e^t \sin(t)$$

Again since $r = 0$ is not a root of the characteristic equation $r^2+1 = 0$, it follows that particular solution to

$$y'' + y = t$$

is of the form $y_{p,2} = A_1 t + B_1$. So particular solution to the problem (**ii**) is

$$y_p = y_{p_1} + y_{p_2} = A_0 e^t \sin t + B_0 e^t \cos t + A_1 t + B_1$$

2. An LCR electrical circuit has an external voltage source $2\cos t$ volts, in addition to inductance of 2 Henry, resistance 2 Ohm and capacitance 0.5 Farad (see Fig. 1). Determine the steady state charge Q on the capacitor plate.

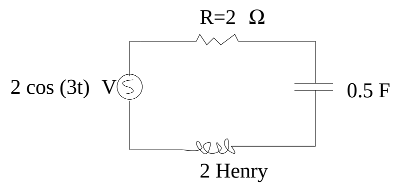

FIGURE 1. Circuit for Problem 5.

Equation $LQ'' + RQ' + \frac{Q}{C} = E_0 \cos(\omega t)$ in this case becomes

$$2Q'' + 2Q' + 2Q = 2\cos t$$

or

$$Q'' + Q' + Q = \cos t$$

Since i is not a root of associated characteristic polynomial $r^2 + r + 1 = 0$, steady state (which is the particular solution we talked about in class) has the form

$$Q(t) = A\cos t + B\sin t$$

So, $Q' = -A\sin t + B\cos t$, $Q'' = -A\cos t - B\sin t$. So,

$$Q'' + Q' + Q = B\cos t - A\sin t = \cos t$$

So, $A = 0$ and $B = 1$. So, steady state solution is

$$Q(t) = \sin t$$

Note of interest: Characteristic polynomial roots $-1/2 \pm \sqrt{3}/2$. So, independent solution to associated homogeneous equation is $y_1 = e^{-t/2}\cos\left(\sqrt{3}t/2\right)$ and $y_2 = e^{-t/2}\sin\left(\sqrt{3}t/2\right)$. Each tend to zero as $t \to \infty$ and hence the general solution $\sin t + c_1 y_1 + c_2 y_2$ approaches steady solution for large time.

3a. Suppose $f(x) = x$ for $-\pi \le x < \pi$, and $f(x + 2\pi) = f(x)$. Determine Fourier Series of f. What does the Fourier Series converge to at $x = \pi$?
Solution: $L = \pi$ since period is 2π. x is an odd function in $(-\pi, \pi)$; so $a_m = 0$ and

$$b_m = \frac{2}{\pi}\int_0^{\pi} x\sin(mx)dx$$

Calculate indefinite integral:

$$\int x\sin(mx)dx = -\frac{1}{m}\int xd\cos(mx) = -\frac{x}{m}\cos(mx) + \frac{1}{m}\int\cos(mx)$$
$$= -\frac{x}{m}\cos(mx) + \frac{1}{m^2}\sin(mx)$$

No contribution from sine term above at the end point $x = 0$ or $x = \pi$ as it is zero. So,

$$b_m = -\left[\frac{x}{m}\cos(mx)\right]_0^2 = -\frac{2}{m}\cos(m\pi) = -\frac{2(-1)^m}{m}$$

Fourier series of f is

$$-2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sin(mx)$$

Periodic extension of f is discontinuous at $x = \pi$, with $f(\pi^-) = \pi$ and $f(\pi^+) = f(-\pi^+) = -\pi$, follows from theory that Fourier series converges at $x = \pi$ to 0, which is the average.

3b. Express the following $f(x)$ as a Fourier cosine series with period 4. Sketch the function to which Fourier series converges to in (-4, 4).

$$f(x) = 1\ ,\ \text{for } 0 < x < 1\ ,\ f(x) = 0\ ,\ \text{for } 1 < x < 2$$

Solution: We do even extension followed by 4-periodic extension as shown in Fig. 2. Then no Fourier sine terms appear.

$$a_0 = \frac{2}{2}\int_0^1 dx = 1$$

and for $n \geq 1$,

$$a_n = \frac{2}{2}\int_0^1 \cos\frac{n\pi x}{2}dx = \frac{2}{n\pi}\sin\frac{n\pi x}{2}|_0^1 = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

Only nonzero for $n = 2j + 1$, for integer $j \geq 0$:

$$a_n = a_{2j+1} = \frac{2}{(2j+1)\pi}(-1)^j$$

So, Fourier cosine series is given by

$$f(x) = \frac{1}{2} + \sum_{j=0}^{\infty}\frac{2}{(2j+1)\pi}(-1)^j\cos\frac{(2j+1)\pi x}{2}$$

The series converges point wise to the function sketched in Fig. 1 (except at the points of discontinuity, where it converges to the mean)

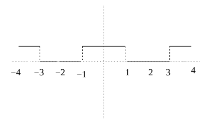

FIGURE 2. Even extension and function to which Fourier cosine series converges

4a. Consider the heat equation $u_t = u_{xx}$ for $0 < x < \pi$ for $t > 0$, with boundary conditions $u(0,t) = 0 = u_x(\pi,t)$. Use separation of variable to show that there are simple solutions $u(x,t) = X(x)T(t)$, with $X(x) = \sin\left[(n-\frac{1}{2})x\right]$ for integer $n \geq 1$. Determine corresponding $T(t)$?

Solution: Separation of variable leads to

$$X'' + \lambda X = 0\ ,\ X(0) = 0\ = X'(\pi)$$

$$T' + \lambda T = 0$$

The first is an eigen value problem. In the range $\lambda = \beta^2 > 0$, general solution X to given ODE is

$$X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$$

Since $X(0) = 0$, implies $c_1 = 0$, so $X(x) = c_2\sin(\beta x)$. $X'(\pi) = 0$, implies $c_2\beta\cos(\beta\pi) = 0$. Nonzero c_2 possible, when $\beta\pi = (2n-1)\frac{\pi}{2}$ for integer $n \geq 1$. or for $\beta = \left(n - \frac{1}{2}\right)$. So, eigenvalue $\lambda = \left(n-\frac{1}{2}\right)^2 \equiv \lambda_n$ and corresponding eigenfunction $X(x) = \sin\left\{\left(n-\frac{1}{2}\right)x\right\}$ as given. So, solving ODE for T, $T(t) = c_n e^{-\lambda_n t}$.

4b. Use (**4a.**) to determine solution to heat equation $u_t = u_{xx}$ for $0 < x < \pi$, $t > 0$ satisfying boundary and initial conditions:

$$u(0,t) = 0 = u_x(\pi,t)\ ,\ u(x,0) = \sin\frac{x}{2} - \frac{1}{4}\sin\frac{3x}{2}$$

Solution: Using a linear combinatin of solution as in **4a** we have a more general solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}\sin\left[\left(n-\frac{1}{2}\right)x\right]$$

$$u(x,0) = c_1\sin\frac{x}{2} + c_2\sin\frac{3x}{2}\ +\ .. = \sin\frac{x}{2} - \frac{1}{4}\sin\frac{3x}{2}$$

provided $c_1 = 1$, $c_2 = -\frac{1}{4}$ and all other $c_n = 0$. So, solution is

$$u(x,t) = e^{-t/4}\sin\frac{x}{2} - \frac{1}{4}e^{-9t/4}\sin\frac{3x}{2}$$

where we used $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{9}{4}$.