Solution to Quiz 3, Math 415

1. Find solution to the wave equation $u_{tt} = u_{xx}$ for $0 < x < \pi$ satisfying initial and boundary conditions:

$$u(x,0) = \sin x + \frac{1}{7}\sin(2x) \quad , \quad u_t(x,0) = 0 \quad , \quad u(0,t) = 0 = u(\pi,t)$$

Solution: Separation of variable gives $u_n(x,t) = X_n(x)T_n(t)$ with $X_n(x) = \sin(nx)$, $T_n(t) = c_n\cos(nt) + k_n\sin(nt)$. Since $u_t(x,0) = 0$, $k_n = 0$.

$$u_n(x,t) = c_n\sin(nx)\cos(nt)$$

Noting only two modes are present corresponding to $c_1 = 1$, $c_2 = \frac{1}{7}$, we obtain

$$u(x,t) = \sin x\cos t + \frac{1}{7}\sin(2x)\cos(2t)$$

2. Transform the following equation into a system of first order ODEs:

$$y'' + 4y' + 4y = e^{-2t}(t+1)$$

Solution: Define $x_1 = y$, $x_2 = y'$. Then the system is:

$$x_1' = x_2 \quad , \quad x_2' = -4x_2 - 4x_1 + e^{-2t}(t+1)$$

3. Verify that $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$ is a solution to

$$\frac{d}{dt}\mathbf{x} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$$

Solution: We note that

$$LHS = \frac{d}{dt}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

$$RHS = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 2e^t - e^t \\ 3e^t - 2e^t \end{pmatrix} = LHS$$

So, verification complete.