Complex variable method for particular solutions

1. PARTICULAR SOLUTIONS TO INHOMOGENEOUS CONSTANT COEFFICIENT 2ND ORDER ODE

We already know how to use method of undetermined coefficients to find solution to the forced oscillator problem

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t) \tag{1.1}$$

We can use the same method for the L-C-R circuit problem:

$$LQ'' + RQ' + \frac{Q}{C} = E_0 \cos(\omega t) \tag{1.2}$$

In the following we discuss an alternate approach using complex variables that makes things algebraically simpler, though it requires you to know a bit about how to handle complex functions and differentiate them. The starting point of analysis in this case is to consider instead the complex forcing problem

$$mv'' + \gamma v' + kv = F_0 e^{i\omega t}, \tag{1.3}$$

realizing that the right hand side of (1.1) is simply the real part of the right hand side of (1.3), since

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

Therefore, the particular solution v_p satisfying the complex ODE (1.3) has real part equalling u_p a particular solution to (1.1). This assertion follows by realizing that the algebraic operation of taking the real part of a complex function commutes with taking derivatives, *i.e.* you can first take the real part and then differentiate or first differentiate and then take the real part, you will get the same thing in each case.

To solve (1.3), we seek a particular soluition in the form of

$$v_p = Ae^{i\omega t} \tag{1.4}$$

where A is some complex constant. We plug this into (1.3) and and note that $\frac{d}{dt}Ae^{i\omega t} = A\frac{d}{dt}e^{i\omega t} = i\omega Ae^{i\omega t}$, since

$$\frac{d}{dt}\left[\cos(\omega t) + i\sin(\omega t)\right] = -\omega\sin(\omega t) + i\omega\cos(\omega t) = i\omega\left[\cos(\omega t) + i\sin(\omega t)\right]$$

Further, in a similar manner, it follows that

$$\frac{d^2}{dt^2}\left[Ae^{i\omega t}\right] = -\omega^2 Ae^{i\omega t}$$

So, substituting $v = v_p$ from (1.4) into (1.3), we obtain

$$\left[-m\omega^2 + i\gamma\omega + k\right] Ae^{i\omega t} = F_0 e^{i\omega t} \tag{1.5}$$

Therefore, the complex amplitude A is

$$A = \frac{F_0}{(k - m\omega^2) + i\gamma\omega} \tag{1.6}$$

We note from polar representation of complex number

$$k - m\omega^2 + i\gamma\omega = \sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}\; e^{i\delta} \text{ where } \delta = \arctan \frac{\gamma\omega}{k - m\omega^2} \tag{1.7}$$

So, using (1.7) in (1.6), we obtain from (1.4) the particular solution

$$v_p(t) = \frac{F_0 e^{i\omega t - i\delta}}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}} \tag{1.8}$$

So, taking the real part, we have

$$u_p(t) = \frac{F_0 \cos{(\omega t - \delta)}}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}} \text{ where } \delta = \arctan \frac{\gamma\omega}{k - m\omega^2} \tag{1.9}$$

This is the same result as quoted in the text in equation (11) in section on forced oscillator (§3.9), provided you substitute $k/m = \omega_0^2$. However, unlike method of undetermined coefficients for real functions, the complex derivation is much more terse, though it requires you to understand differentiation of complex exponentials. Your professors in Engineering and Physics might use this derivation rather than the one I talked about in class. I wanted you to be aware of this.