Week 1 notes:

1. Differential equations and mathematical models

1.1. Basic Definitions and Examples. An equation involving a function and its derivatives is called a *differential equation.*

If we use the notation x and y for independent and dependent variables, respectively, then any of the following equation is an example of an *ordinary*[1] differential equations (ODE) for $y(x)$:

(1) $y' + y - 2 = 0$

(2) $2y'' + e^x y' + 2y = \sin x$

(3) $\frac{d^2y}{dx^2} + 2\sin y = 0$

Note: $y(x)$ is typically *not* given to us and we want to find it, once the ODE is given. Further, there is no particular significance to using notation x and y for independent and dependent variables. For particle motion, it is customary x for displacement as a function of t

Definition A *first order* ordinary differential equation (ODE) involves only the *unknown function*, say $y(x)$, and its first derivative. In explicit representation,

$$y' = f(x, y) \tag{1.1}$$

where f is a given function. Example (1) is clearly a first order ODE, with $f(x, y) = -y + 2$.
Definition More generally, an *n-th order* ODE has the following explicit representation:

$$y^{(n)} = f\left(x, y', y'', ..y^{(n-1)}\right) \tag{1.2}$$

where f is given. Examples (2) and (3) above are both examples of second order ODE. For example (2),

$$f(x, y, y') = -\frac{1}{2}\sin x - y - \frac{e^x}{2}y'$$

and in example (3),

$$f(x, y, y') = -2\sin y$$

Definition A ODE *solution* is any expression $y(x)$ that satisfies the given ordinary differential equation (ODE).

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[1] as versus partial (PDE) when more than one independent variable is involved

Example: For instance $y(x) = e^{-x} + 2$ is a solution to $y' = -y + 2$. On substituting $y = e^{-x} + 2$, into the equation, the left side is $-e^{-x}$ as is the right side. Hence $e^{-x} + 2$ is a solution.

Example: Verify that $y(x) = c \sin x$ satisfies the following ODE for *any* constant c:

$$y'' = -y$$

Clearly, on substituting $y = c \sin x$, the left side of the differential equation equals $c\frac{d^2}{dx^2} \sin x = -c \sin x$. The right hand side $-y = -c \sin x$. Therefore, the left and right side of the equation are equal, and the verification is complete. Similarly, you can verify that $y = d \cos x$ for any d is also a solution. **Note** however that if we substituted some random function, say $y(x) = x^2$, into the above equation, the left and right side are not the same for arbitrary x. Therefore $y(x) = x^2$ is *not* a solution to *this* ODE.

1.2. **Modeling through differential equations.** Differential Equations arise frequently in the physical sciences and engineering. It also arises in quantitative predictions in ecology, biology as well as social sciences.

We breakdown the process of converting statements about physical/biological processes into differential equations into a sequence of steps:

(1) Identify dependent and independent variables as well as constant parameters.
(2) Determine units of measurement for variables and constants.
(3) Seek to relate the rate of change of dependent variable to given statement about the physical/biological process and express it as an equation.
(4) Typically, the equation in (3) will involve unknowns other than simply the dependent variable. Look for additional relations between the unknowns and the dependent variable.
(5) Using relation (4) in (3), we should come up with a differential equation, which must be dimensionally consistent.

1.3. **Motion of falling object with air friction.** We seek to predict the downwards velocity v of a falling object of mass m (see Fig. 1) as a function of time t, assuming constant gravity constant g and a friction force opposing motion that is proportional to the velocity.

Step 1: We recognize velocity v and time t to be dependent and independent variables respectively. m and g are only constants. An additional constant describing air-friction will be needed later.

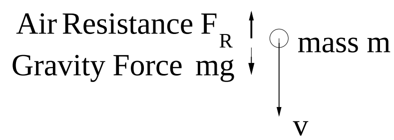

FIGURE 1. Falling object of mass m with gravity and air-resistance

Step 2: We introduce units: mass m in *kilograms* $(Kg.)$, time t in *seconds* (s), velocity v in $meters/sec$ (m/s), acceleration due to gravity g in m/s^2.
Step 3: We seek an equation for rate of change of v. Suppose we consider the small time interval $[t, t+h]$. The rate of change of downwards velocity over that time interval $\frac{v(t+h)-v(t)}{h}$. As $h \to 0$, this gives $\frac{dv}{dt}$ the instantaneous rate of change of velocity (or downwards acceleration) at time t. We know from Newton's second law:

$$m\frac{dv}{dt} = \textbf{net downwards force } F_T \tag{1.3}$$

Step 4: We have introduced additional unknown F_T. We need to relate it back to dependent variable v. We recognize that **net** downwards force equals

$$F_T = mg + F_R \tag{1.4}$$

where F_R is the air-resistance. Since we are given that air-resistance opposes motion and is proportional to velocity,

$$F_R = -\gamma v, \tag{1.5}$$

where γ is some positive constant of proportionality. Note that since F_R is has units of $Newtons$ (N), which is the same as $Kg - m/s^2$, γ has units Kg/s.
Step 5: We substitute in (1.3) expressions for F_T in (1.4), where F_R is replaced by (1.5). This gives the ODE for v:

$$m\frac{dv}{dt} = mg - \gamma v, \text{ or } \frac{dv}{dt} = g - \frac{\gamma}{m}v \tag{1.6}$$

We check that the left hand side of the latter equation has units m/s^2 as does the right side. We are done with modelling the physics into a differential equation (1.6). We note that g, γ and m are constants in (1.6). For instance, if we have $10kg$ mass, and air-resistance constant $\gamma = 2\ Kg/s$, then (1.6) becomes (for $g = 9.8$):

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{1.7}$$

1.3.1. *Direction Field for (1.7).* Noting that (1.7) gives the slope $\frac{dv}{dt}$ for given v and t, it is useful to draw the **Direction field** or **Slope Field** corresponding to the differential equation (1.7)– at each point in $v-t$ plot, we draw a little vector whose slope is equal to the right side of (1.7). This is shown in the Figure 2 below.

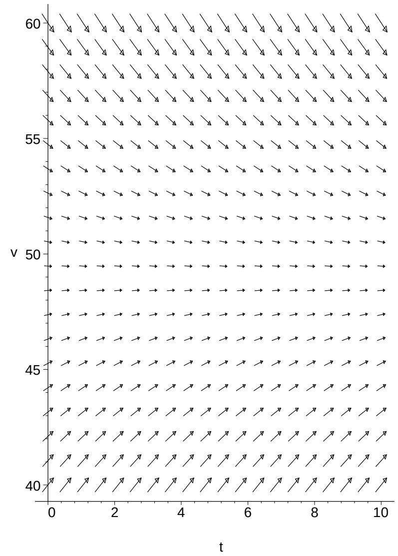

FIGURE 2. Direction Field for Differential Equation (1.7)

Therefore, depending on what $v(0)$ is we may join the direction fields in a smooth way and obtain an approximate **Solution curve** graphically. It is to be noted from (1.7) that the slope is exactly zero when $v = 49$, which is an equilibrium solution. That slope is positive for $v < 49$ and negative for $v > 49$. Therefore, if $v(0) < 49$, then solution approaches $v = 49$ from below. On the otherhand if $v(0) > 49$, it approaches $v = 49$ from above.

Exercise 1: Derive a differential equation for a falling object under gravity when air-resistance opposes motion but is proportional to the square of the velocity. For some typical value of constants, roughly sketch the direction field, solution curves and determine the behavior of the velocity as time $t \to \infty$.

1.4. **Problem of field mice population prediction.** Yet another interesting problem is one that comes from ecology. Suppose we want to predict the population of mice inhabiting some rural area. Of course the mice population is discrete. However, if the numbers are sufficiently large, we may model the mice population as a continuous variable.

Assume that the mice in some area reproduces at a rate proportional to the existing mice population, but are killed off by predators at a constant rate of 15 mice/day. We seek to determine a differential equation to predict the mice population.

First step is to identify dependent and independent variables, which clearly is the mice population p and time t.

Second, note that p has no units and we may measure t in some reasonable units, say month.

Step 3: We ask what is the rate of change of population, *i.e.* $\frac{dp}{dt}$. We recognize it changes because of new mice addition and because some mice are killed:

$$\frac{dp}{dt} = \text{new births/month} - \text{mice killed/month } K \tag{1.8}$$

Step 4: We recognize from the given statement that new births per month A is given by

$$A = rp \tag{1.9}$$

where r is a constant, which is called the growth rate per month. Clearly it has units of *per month per mice* From the statement about 15 mice getting killed per-day, we have over a month

$$K = 450/month \tag{1.10}$$

Step 5: Using (1.9) and (1.10) in (1.8) helps us complete the differential equation

$$\frac{dp}{dt} = rp - 450 \tag{1.11}$$

We note the dimensional consistency of both sides Here r is a constant and has to be provided. To be more specific, say, $r = 1/2$. Then, we have (1.11) reduce to

$$\frac{dp}{dt} = \frac{p}{2} - 450 \tag{1.12}$$

The direction field in the $t - p$ plane is shown in Figure 3. As before, we can join the direction field in a smooth way and determine graphically in an approximate manner the solution for given starting value $p(0)$.

Note that slope is zero at $p = 900$, which is an equilibrium solution. The slope is negative for $p < 900$ and positive for $p > 900$. In this if $p(0) < 900$, then the mice population $p(t)$ for large t eventually crosses $p = 0$ axis (extinction of mice), beyond which the solution does not make sense. On the other hand, if $p(0) > 900$, then the mice population keeps growing, despite losing 450 a month.

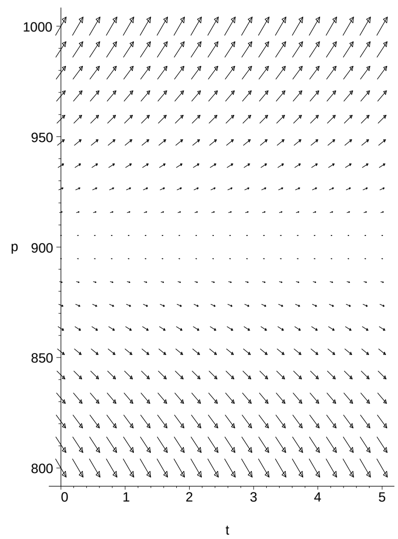

FIGURE 3. Direction Field for Differential Equation (1.12)

1.5. $R-C$ **Electrical Circuits.** We seek to determine current flowing through an electrical circuit (see Fig 4), with a capacitance C (Farads F), resistance R ($Ohms$) and constant voltage source V ($Volts$).

It is known from Physics that the total voltage drop across a circuit is zero and that voltage drop V_R across the resistor is IR, I being the current ($Amps$), while the voltage drop (volts) across the capacitor is Q/C, $\pm Q$ ($Coulombs$) being the charge on the two capacitor plates (see Figure 4).

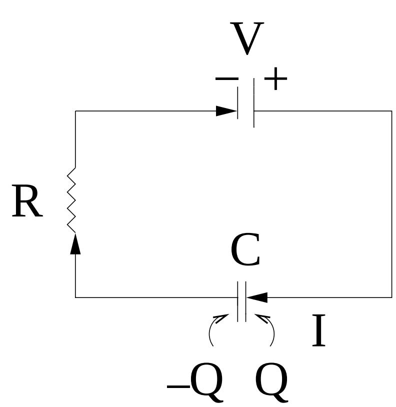

FIGURE 4. $R-C$ circuit

First step is to recognize that we are seeking to determine current I as a function of time t in second (s). C, R and V are merely constants, that will be given for a particular problem.

Since the units are already identified, we can skip the second step.

In the third step, we use the statement about voltage drop across the circuit:

$$IR + \frac{Q}{C} - V = 0 \tag{1.13}$$

As a fourth step, we recognize that in (1.13) that while we want I as a function of t, we have an unknown variable Q, which we need to some how relate to I. Here, we recognize that the current shown in Figure 4 is what leads to build up of charge Q on the capacitor plate. Indeed, the number of Coulombs per second flowing is the rate of change the charge Q on the capacitor plate (see Figure); hence

$$\frac{dQ}{dt} = I, \tag{1.14}$$

the current flowing through the circuit. Using (1.14), in (1.13), we obtain

$$R\frac{dQ}{dt} + \frac{Q}{C} = V \text{ or } \frac{dQ}{dt} + \frac{Q}{RC} = \frac{V}{R}, \tag{1.15}$$

which is a differential equation for charge $Q(t)$ as a function of t.

If indeed, the constants $\frac{1}{RC} = \frac{1}{5}$ and $\frac{V}{R} = 9.8$, we obtain

$$\frac{dQ}{dt} + \frac{Q}{5} = 9.8, \text{ or} \frac{dQ}{dt} = 9.8 - \frac{Q}{5}, \tag{1.16}$$

we end up with exactly the same plot of direction field given for the falling body, with Q replacing v. We only note at the end that in this problem, our quantity of interest is not $Q(t)$, but the current $I(t) = \frac{dQ}{dt}$, which is the slope of Q versus t curve. Note from the plots, that as $t \to \infty$, slope of Q against t (which is $I(t)) \to 0$; *i.e.* there is no steady current in a $D - C$ circuit with a capacitor.

2. Explicit Solutions to some first order ODEs

In the context of falling bodies and field mice population:

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v \tag{2.17}$$

and

$$\frac{dp}{dt} = rp - k \text{ Note : we took } \; k = 450, \; r = 1/2 \text{ as special case} \tag{2.18}$$

In the context of $R - C$ electrical circuit, with capacitance and resistance, we had

$$R\frac{dQ}{dt} + \frac{Q}{C} = V \tag{2.19}$$

for constants R, C and V. Each of these equations are mathematically similar and is of the form

$$\frac{dy}{dt} = ay - b \tag{2.20}$$

with given constants a and b. Notice, $y = v$, $a = -\frac{\gamma}{m}$, $b = -g$ for falling body problem; $y = p$, $a = r$, $b = k$ for field mice problem and $y = Q$, $a = -\frac{1}{RC}$ and $b = -\frac{V}{R}$ for the R-C circuit. Thus knowing how to solve one of these problems gives us the methodology for all the other problems of this type.

Consider the field mice problem for specific choice of constant $r = \frac{1}{2}$, $k = 450$: We rewrite it as

$$\frac{1}{p - 900}\frac{dp}{dt} = \frac{1}{2} \tag{2.21}$$

We recall from chain rule that $\frac{d}{dt}g(p) = g'(p)\frac{dp}{dt}$. So, the left side of (2.21) can be associated with $\frac{d}{dt}g(p)$, if we choose $g'(p) = \frac{1}{p-900}$, *i.e.* $g(p) = \ln|p - 900|$ So,

$$\frac{d}{dt}\ln|p-900| = \frac{1}{2}\,, \text{implying } \ln|p-900| = \frac{t}{2}+c\,, \text{implying } |p-900| = e^{t/2+c} = e^c e^{t/2}$$

for some arbitrary constant c. So

$$p - 900 = \pm e^c e^{t/2} = Ce^{t/2} \tag{2.22}$$

where C is some arbitrary constant. So,

$$p = 900 + Ce^{t/2} \equiv p(t)$$

So we have a family of curves that depend on the choice of the constant C. If we specifiy initial condition; say the value of p at $t = 0$, then C can be uniquely determined. If $p(0) = 850$, then $850 = 900 + C$; implying $C = -50$ and the specific solution curve in that case is

$$p = 900 - 50e^{t/2}$$

If $p(0) = 950$, then $C = 50$ and we get the solution curve

$$p = 900 + 50e^{t/2}$$

If we draw a bunch of solution curves for different value of C, we obtain the figure 5 shown below:

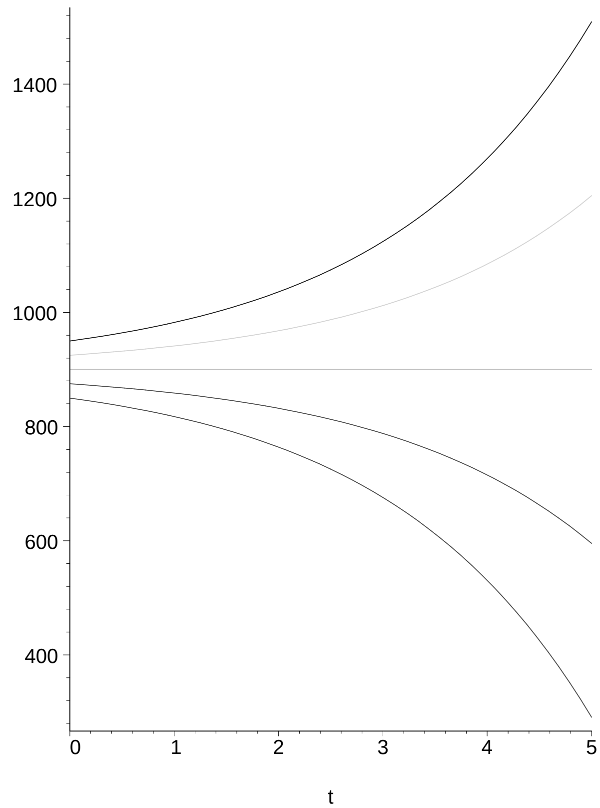

FIGURE 5. Plot of $p(t) = 900 + Ce^{t/2}$ for $C = 50, 25, -25, -50$

It is prudent to solve the more general differential equation (2.20): $\frac{dy}{dt} = ay - b = a(y - b/a)$ as it covers all the different examples. We rewrite it as

$$\frac{1}{y - b/a}\frac{dy}{dt} = a,$$

implying on integration

$$\ln|y - b/a| = at + c,$$

or

$$|y - b/a| = e^{t+c} = e^c e^{at},$$

or

$$y - b/a = \pm e^c e^t = Ce^{at},$$

or

$$y = \frac{b}{a} + Ce^{at}$$

for some arbitrary constant C that depends on initial condition $y(0)$.

For the falling body problem, recall $y = v$, $a = -\frac{\gamma}{m}$, $b = -g$; so

$$v = \frac{mg}{\gamma} + Ce^{-\gamma t/m} \equiv v(t) \tag{2.23}$$

If $v(0) > \frac{mg}{\gamma}$, $C > 0$, while if $v(0) < \frac{mg}{\gamma}$, $C < 0$. In all cases, notice that if $t \to \infty$, $v \to \frac{mg}{\gamma}$, which is the terminal velocity. For the special case of $m = 10kg$, $\gamma = 2kg/sec$ and $g = 9.8m/sec^2$, we obtain

$$v = 49 + Ce^{-t/5} \equiv v(t) \tag{2.24}$$

and this is plotted in Fig. 6 for different values of initial velocity and hence C (note $v(0) = 49 + C$).

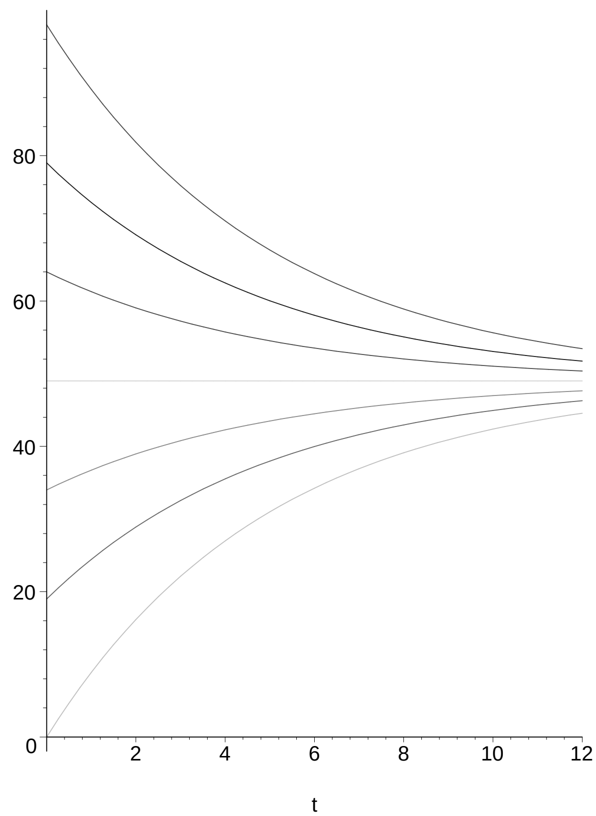

FIGURE 6. Plot of $v(t) = 49 + Ce^{-t/5}$ for $C = 49, 30, 15, 0, -15, -30, -49$

3. Classification of Differential Equations

3.1. **Ordinary and Partial Differential Equations.** So far all the examples we have given involve differential equations that involve only one independent variable (t or x). These are called ordinary differential equations (ODEs). In applications we sometimes encounter differential equations with more than one independent variable. These are called *partial differential equations* (PDEs). As an example, consider the equation governing temperature distribution $u(x,t)$ in a one-dimensional rod at location x at time t is given by the so-called heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

This is a PDE with two independent variables (x,t). The **order** of a PDE is defined to be the order of the highest derivative in any variable that occurs in the equation. This concept is similar to that for ODE.

We will mainly talk about ODEs in this course, though towards the end we will also talk a little bit about how to solve partial differential equations.

3.2. **Linear and Nonlinear ODEs.** In the last section, we already mentioned that an n-th order ODEs say for the dependent variable y as a function of t in the form

$$y^{(n)} = f(t, y, y', y'', ..y^{(n-1)}) \tag{3.25}$$

for known f, *e.g*

$$y'' = -y' - y - \sin x$$

More generally, we may have simply an implicit expression such as

$$F(t, y, y', y'', ...y^{(n-1)} = 0, \tag{3.26}$$

for instance

$$[y'']^2 + y^2 = 9 - 9t^2$$

In such cases, we are sometimes able to solve for $y^{(n)}$ and convert the equation to the form (3.25).

If further, the equation (3.27) is in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} \ + \ ...a_n(t)y = g(t) \tag{3.27}$$

then the equation is term **linear**. Otherwise it is termed **nonlinear**. For example

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{3.28}$$

is a **linear second order** ODE for dependent variable θ as a function of t, where g and l are constants. This equation arises in the motion of pendulum for small amplitudes, where θ is the angle made by the pendulum from the vertical. On the otherhand,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

is a nonlinear equation, as we are unable to write the equation in the form (3.27).

3.3. **Linear 1st order ODE and method of integrating factors.** From last class, we know that linear first order ODEs generally look like

$$a_0(t)y' + a_1(t)y = \tilde{g}(t) \tag{3.29}$$

where $a_0(t)$ is not identically zero. Dividing by $a_0(t)$, we obtain

$$y' + p(t)y = g(t) \tag{3.30}$$

where $p(t) = a_1(t)/a_0(t)$, $g(t) = \tilde{g}(t)/a_0(t)$. A special case of this is when $p(t) = a$, $g(t) = b$ are constants. In that case we obtain

$$y' = -ay + b = -a(y - b/a) \tag{3.31}$$

We may rewrite this as

$$\frac{1}{y - b/a}\frac{dy}{dt} = -a \tag{3.32}$$

and integrate both sides with respect to t (as done last class) to obtain

$$\ln|y - \frac{b}{a}| = -at \tag{3.33}$$

from which we obtain (see algebra from last time).

$$y = \frac{b}{a} + Ce^{-at} \tag{3.34}$$

Now, suppose we have more general case where $p(t)$ and $g(t)$ are not constants. In that case, the above solution is invalid.

In that case, we multiply equation (3.30) by some $\mu(t)$ (called integrating factor) so that we can integrate the resulting equation:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \tag{3.35}$$

We require that the above equation reads as

$$\frac{d}{dt}\left[\mu(t)y(t)\right] = \mu(t)g(t) \tag{3.36}$$

This is possible if we choose $\mu'(t) = \mu(t)p(t)$. or

$$\frac{1}{\mu}\frac{d}{dt}\mu = p(t) \text{ implying } \frac{d}{dt}\ln|\mu| = p(t) \tag{3.37}$$

So,

$$\mu(t) = ce^{\int p(t)dt} \tag{3.38}$$

Say, we choose a specific $\mu(t)$ (say by choosing $c = 1$). Then, on integrating (3.36) becomes

$$\mu(t)y(t) = \int^t \mu(t)g(t) + C \tag{3.39}$$

for some arbitrary integration constant C Therefore, the general solution will be

$$y(t) = \frac{1}{\mu(t)}\left[\int^t \mu(t')g(t')dt' + C\right] \tag{3.40}$$

As an example, consider the differential equation

$$\frac{dy}{dt} + \frac{y}{2} = 2 + t \tag{3.41}$$

We want to find the general solution, as well as identify the solution that passes through $(0, 2)$. We note that in this case, one integrating factor is $\mu = \exp\left[\int^t \frac{1}{2}dt\right] = e^{t/2}$. So that equation (3.41) on multliplcation by $e^{t/2}$ becomes:

$$\frac{d}{dt}\left[e^{t/2}y\right] = (2 + t)e^{t/2} \tag{3.42}$$

Integrating both sides we get

$$\begin{aligned} e^{t/2}y = \int^t (2 + t)e^{t/2} &= 4e^{t/2} + \int^t te^{t/2}dt \\ &= 4e^{t/2} + 2te^{t/2} - 4e^{t/2} + c = c + 2te^{t/2} \end{aligned} \tag{3.43}$$

Therefore,

$$y = ce^{-t/2} + 2t \tag{3.44}$$

This is the general solution. To pass through $(0, 2)$, *i.e.* $y(0) = 2$, we have from substituting $t = 0$, $y = 2$ in (3.44)

$$2 = c + 0 \tag{3.45}$$

So, the particular solution that passes through $(0, 2)$ is

$$y = 2e^{-t/2} + 2t \tag{3.46}$$

Example 2: Solve the initial value problem

$$ty' + 2y = 4t^2 \ , \ y(1) = 2$$

Rewriting this in the standard form, the differential equation is

$$y' + \frac{2}{t}y = 4t \tag{3.47}$$

The integrating factor is $\mu = \exp\left[\int \frac{2}{t}dt\right] = \exp\left[2\ln t\right] = t^2$. Multiplying (3.47) by the integrating factor μ, we obtain

$$t^2y' + 2ty = 4t^3 \ ; \text{ or } \frac{d}{dt}\left[t^2y\right] = 4t^3 \tag{3.48}$$

Integrating both sides with respect to t, we get

$$t^2y = t^4 + c \tag{3.49}$$

Therefore, the general solution is

$$y = t^2 + c/t^2 \tag{3.50}$$

shown in Figure 7 for a set of values of c. The solution satisfying initial condiiton $y(1) = 2$ means from (3.50) that

$$2 = 1 + c \tag{3.51}$$

So $c = 1$. Therefore, solution to initial value problem is

$$y = t^2 + \frac{1}{t^2} \tag{3.52}$$

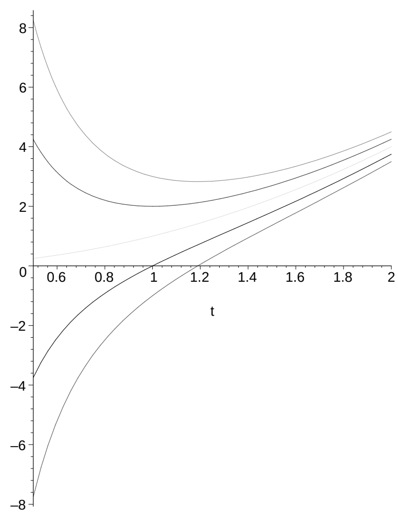

FIGURE 7. Plot of $y(t) = t^2 + c/t^2$ for $c = 1, 2, -1, -2, 0$