

Homework Set 2: Math 716, Due: Monday, January 30th

1. Consider the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$$

for $x \in \mathbb{R}$ with constant $c \neq 0$. Assume $\phi \in \mathbf{C}^2$ and $\psi \in \mathbf{C}^1$ have compact support (*i.e.* there is a number $R > 0$ such that $\phi(x) = \psi(x) = 0$ for $|x| > R$). Denote d'Alembert's solution of (1) by $u(x, t)$. Using d'Alembert's solution, show that the solutions to (1) depend in a stable fashion on the initial data ϕ and ψ . More precisely: if u_1 and u_2 satisfy (1) with initial data (ϕ_1, ψ_1) and (ϕ_2, ψ_2) , respectively, show that

$$\sup_{x \in \mathbb{R}} |u_1(x, t) - u_2(x, t)| \leq C(t) \left[\sup_{x \in \mathbb{R}} |\phi_1(x) - \phi_2(x)| + \sup_{x \in \mathbb{R}} |\psi_1(x) - \psi_2(x)| \right]$$

for some constant $C(t)$ that may depend on t , but not on the initial data. What is a good estimate for $C(t)$? Can you find similar estimates for $\partial_x(u_1 - u_2)$ and $\partial_{xx}(u_1 - u_2)$?

2. Find an integral representation of the solution to advection-diffusion equation

$$u_t + au_x = u + \kappa u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = \phi(x), \quad -\infty < x < \infty$$

where a and $\kappa > 0$ are constants.

Hint: Transform independent variable $(x, t) \rightarrow (x - at, t)$, and replace u by $v = e^{-t}u$ as the dependent variable.

3. Consider the equation

$$u_t = u_{xx} \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0 = u(1, t) \quad t > 0$$

$$u(x, 0) = 4x(1 - x) \quad 0 < x < 1$$

Show that

- 1) $0 < u(x, t) < 1$ for all $0 < x < 1$ and $t > 0$.
 - 2) $u(x, t) = u(1 - x, t)$ for all $0 < x < 1$ and $t > 0$.
 - 3) The energy $\int_0^1 u^2(x, t) dx$ decreases strictly with t for $t > 0$.
4. Prove the comparison principle: If $u(x, t)$ and $v(x, t)$ are two solutions of the heat equation for $0 < x < l$ and $t \geq 0$ such that $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for all $0 \leq x \leq l$ and $t > 0$.