Homework Set 3: Math 716, Due: Friday, February 10th

1. Use energy method to prove uniqueness of classical solution to the initial value problem for the damped wave equation ($\epsilon > 0$):

   $$u_{tt}+\epsilon u_t = \Delta u \text{ for } \mathbf{x} \in \Omega \subset \mathbb{R}^n,\ t > 0 \text{, with } u(\mathbf{x},0) = \phi(\mathbf{x}),\ u_t(\mathbf{x},0) = \psi(\mathbf{x})\ ,\ u(\mathbf{x},0) = 0 \text{ on } \partial\Omega$$

2. Find representation of solution to heat equation with Neumann boundary conditions on a half-line

   $$u_t = u_{xx} \quad 0 < x < \infty,\ t > 0, \quad \text{with} \quad u(x,0) = 0, \quad u_x(0,t) = \sin\ t$$

3. Find a solution to the inhomogeneous wave equation

   $$u_{tt} = c^2 u_{xx} + f(x,t), \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

   for $x \in \mathbb{R}$ with $c \neq 0$. Assume $\phi \in \mathbf{C}^2$, $\psi \in \mathbf{C}^1$, and $f \in \mathbf{C}^0$ are given bounded functions.

   Hint: Use Duhammel's principle, which for ODEs, says the following: If $v(t;\tau)$ is the solution to

   $$\frac{dv}{dt} = Av, \quad v(\tau;\tau) = f(\tau) \quad \text{then} \quad u(t) = \int_0^t v(t;\tau)d\tau$$

   is a solution to $\frac{du}{dt} = Au + f(t)$, $u(0) = 0$.

4. Prove the weak maximum principle for Laplace's equation. Thus, assume that $\mathcal{D}$ is an open and bounded subset of $\mathbb{R}^n$ with boundary ∂D. Assume also that $u(\mathbf{x})$ is a solution to Laplace's equation $\Delta u = 0$ in $\mathcal{D}$ and that u is continuous on $\bar{\mathcal{D}}$, twice differentiable in D. Show that

   $$\sup_{\bar{D}} u = \sup_{\partial D} u$$

5. a. Prove that if there exists a solution of the Neumann problem

   $$\Delta u = f \quad \text{for} \quad x \in \mathcal{D} \subset \mathbb{R}^n, \quad \frac{\partial u}{\partial n} = h(x) \quad \text{for} \quad x \in \partial D,$$

   then it is unique up to adding an arbitrary constant.

   b. Consider the Robin problem

   $$\Delta u = f \quad \text{for} \quad x \in \mathcal{D}, \quad \frac{\partial u}{\partial n} + a(x)u = h(x) \quad \text{for} \quad x \in \partial D, \quad \text{with} \quad a(x) > 0$$

   Show that its solutions are unique.