

Homework Set 4: Math 716, Due Wednesday, February 22nd

1. Using the two approaches listed below as (a) and (b), find two different representation of solution to the following Initial-Boundary value problem for the heat equation:

$$u_t - \kappa u_{xx} = 0 \quad \text{for } 0 < x < l \quad ; \quad u(x, 0) = \phi(x) \quad , \quad u(0, t) = 0 = u(l, t)$$

Prove that for $t > 0$, u is infinitely differentiable both in x and t . Which representation is suitable for evaluation of solution for small t ? Which one is suitable for large t ? Explain.

- (a) Use method of successive reflection of source solutions.
 (b) Separation of variables.

For the separation of variable method, determine a suitable bound for the size of the maximum error uniformly for $t > 0$, if the series representation is truncated to N terms, when $\phi \in \mathbf{C}^1(0, 1)$, with $\phi(0) = 0 = \phi(1)$.

2. Find solution to the problem of a circular vibrating membrane for 2-D :

$$u_{tt} - c^2 \Delta u = 0 \quad \text{for } |\mathbf{x}| < 1; \quad u(\mathbf{x}, 0) = \phi(\mathbf{x}) \quad , \quad u_t(\mathbf{x}, 0) = 0 \quad , \quad u(\mathbf{x}, t) = 0 \quad \text{for } |\mathbf{x}| = 1$$

What is the restriction on ϕ needed? **Hint:** You may want to use the information that the solution to $v'' + \frac{1}{r}v' + \left(\lambda^2 - \frac{m^2}{r^2}\right)v(r) = 0$ is given by a linear combination of Bessel functions $J_m(\lambda r)$ and $Y_m(\lambda r)$, where Y_m blows up at the origin. J_m is regular at the origin and has countably infinite zeros on the real line similar to the sin function.

3. Find solution in series form to Laplace's equation in 2-D in an annular domain $a < |\mathbf{x}| < 1$:

$$\Delta u = 0 \quad ; \quad u(\mathbf{x}) = \phi(\mathbf{x}) \quad \text{for } |\mathbf{x}| = a \quad \text{and} \quad u(\mathbf{x}) = \psi(\mathbf{x}) \quad \text{for } |\mathbf{x}| = 1$$

Show that $u(\mathbf{x})$ is infinitely differentiable for $a < |\mathbf{x}| < 1$, even when boundary data ϕ and ψ are simply continuous.

4. Determine a representaton for solution to Laplace's equation in a semi-circular domain

$$\mathcal{H} := \{ \mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < 1, x_2 > 0 \}$$

$$\Delta u = 0 \quad \text{with} \quad u = 0 \quad \text{for } x_2 = 0, \quad \frac{\partial u}{\partial n} + u = f(\mathbf{x}) \quad \text{on } |\mathbf{x}| = 1$$

Make appropriate assumptions on f and apply whatever theory you need to justify your solution representation and to show that the solution is \mathbf{C}^∞ inside \mathcal{H} .