Homework Set 6: Math 716, Due Friday, March 9th

1. Determine the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ for Dirichlet condition for Laplace's equation in 3-D dimensions in the hemispherical domain:

$$\Omega = \{\mathbf{x} : |\mathbf{x}| < 1, \quad x_3 > 0\}$$

Use this to determine an integral expression for $u(r, \theta, \phi)$ in spherical polar-coordinates satisfying

$$\Delta u = 0 \ \text{ in}\Omega \quad \text{, with } \ u = 0 \ \text{ for } \theta = 0, \ \text{ and } \ u(1, \theta, \phi) = \psi(\theta, \phi) \ \text{ for } \theta \in [0, \frac{\pi}{2}], \phi \in [0, 2\pi]$$

Hint: Better not to use spherical polar coordinates until after you get an integral representation.

2. If $\mathcal{S}(\mathbf{x}, t)$ is the source solution to the heat equation given by

$$\mathcal{S}(\mathbf{x}, t) = \left(\frac{1}{4\pi\kappa t}\right)^{n/2} \ \exp\left[-\frac{|\mathbf{x}|^2}{4\kappa t}\right],$$

then show that

$$R(\mathbf{x}, t; \mathbf{x}_0, t_0) = \mathcal{S}(\mathbf{x} - \mathbf{x}_0, t - t_0) \ \text{ for } t > t_0 \ \text{ and } \ R = 0 \ \text{ for } t < t_0 \quad \mathbf{x} \in \mathbb{R}^n$$

satisfies

$$R_t - \kappa \Delta R = \delta(\mathbf{x} - \mathbf{x}_0)\delta(t - t_0)$$

3. Consider the wave equation

$$u_{tt} - \Delta u = 0 \ \text{ for } \mathbf{x} \in \mathbb{R}^n, \quad t > 0, \quad \text{with } u(\mathbf{x}, 0) = \phi(\mathbf{x}), u_t(\mathbf{x}, 0) = \psi(\mathbf{x}),$$

with compactly supported ϕ and ψ.

a. Show that for $n = 3$, then there exists a constant C such that

$$|u(\mathbf{x}, t)| < \frac{C}{1 + t} \ \text{ for } \ \mathbf{x} \in \mathbb{R}^3, \quad t > 0$$

b. Show that for $n = 3$, at a fixed point $\mathbf{x}$ outside the support of ϕ and ψ, there exists an earliest impact time $t_i > 0$ that depends on $\mathbf{x}$ so that $u(\mathbf{x}, t) = 0$, for $0 \leq t < t_i$, and that there also exists a final impact time $t_f > t_i$ so that for $t > t_f$, $u(\mathbf{x}, t) = 0$.

c. Show that for $n = 2$, for $\mathbf{x}$ outside the support of ψ and ϕ, there exists an initial impact time $t_i > 0$ depending on $\mathbf{x}$ but no final final impact time t_f.

Hint: Examine carefully the consequences of Poisson and Kirchoff formula.