

Week 4 Lectures, Math 716, Tanveer

1 Diffusion in \mathbb{R}^n

Recall that for scalar x ,

$$S(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{x^2}{4\kappa t}\right] \quad (1)$$

is a special solution to 1-D heat equation with properties

$$\int_{\mathbb{R}} S(x, t) dx = 1 \text{ for } t > 0, \text{ and yet } \lim_{t \rightarrow 0^+} S(x, t) = 0 \text{ for fixed } x \neq 0 \quad (2)$$

This was called a source solution of heat equation with source at the origin.

We now claim that the product $\mathcal{S}(\mathbf{x}, t) \equiv S(x_1, t)S(x_2, t)S(x_3, t)\dots S(x_n, t)$ is a solution to the heat equation in \mathbb{R}^n :

$$u_t = \kappa\Delta \text{ for } \mathbf{x} \in \mathbb{R}^n \quad (3)$$

and satisfies property

$$\int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x}, t) d\mathbf{x} = 1 \text{ for } t > 0, \text{ and yet } \lim_{t \rightarrow 0^+} \mathbf{S}(\mathbf{x}, t) = 0 \text{ for fixed } \mathbf{x} \neq \mathbf{0} \quad (4)$$

We note that by using product rule

$$\frac{\partial \mathcal{S}}{\partial t} = \sum_{j=1}^n \frac{\partial}{\partial t} S(x_j, t) \prod_{i \neq j} S(x_i, t) = \kappa \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} S(x_j, t) \prod_{i \neq j} S(x_i, t) = \kappa \Delta \mathcal{S} \quad (5)$$

Further,

$$\int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x}, t) d\mathbf{x} = \int_{\mathbb{R}} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} S(x_1, t) S(x_2, t) \dots S(x_n, t) dx_1 dx_2 \dots dx_n = 1 \quad (6)$$

Further if $\mathbf{x} \neq \mathbf{0}$, then direct examination of

$$\mathbf{S}(\mathbf{x}, t) = \frac{1}{(4\kappa\pi t)^{n/2}} \exp\left[-\frac{x_1^2 + x_2^2 + \dots + x_n^2}{4\kappa t}\right] = \frac{1}{(4\kappa\pi t)^{n/2}} \exp\left[-\frac{\mathbf{x}^2}{4\kappa t}\right] \quad (7)$$

shows that $\lim_{t \rightarrow 0^+} \mathcal{S}(\mathbf{x}, t) = 0$ for fixed $\mathbf{x} \neq \mathbf{0}$. The solution \mathcal{S} is the source solution in \mathbb{R}^n . Analogous to 1-D, we have the following theorem:

Theorem 1 *The solution to the heat equation in \mathbb{R}^n that satisfies initial condition*

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}) \text{ for } \phi \in \mathbf{C}^0(\mathbb{R}^n) \quad (8)$$

is given by

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t) \phi(\mathbf{y}) d\mathbf{y} \quad (9)$$

2 Diffusion in the half-line

2.1 Dirichlet Boundary condition

We consider solution to heat equation in 1-D, with $x \in \mathbf{R}^+$ and take the *Dirichlet* boundary condition at $x = 0$. So the problem is

$$v_t - \kappa v_{xx} = 0 \text{ for } x > 0, \ t > 0 \quad (10)$$

$$v(x, 0) = \phi(x) \quad (11)$$

$$v(0, t) = 0 \quad (12)$$

We seek to find a solution to this problem explicitly. If it exists, the classical solution for which $v(x, t) \rightarrow 0$ as $x \rightarrow \infty$ is unique by applying maximum principle or energy method.

Now the initial data $\phi(x)$ is only specified for $x > 0$.

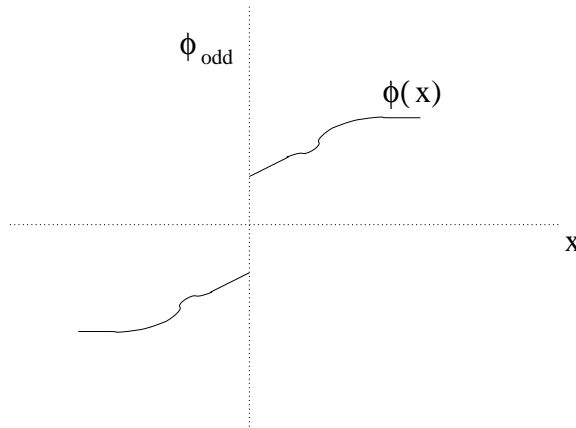


Figure 1: Odd Extension of $\phi(x)$ to $x \in \mathbb{R}$

We do an *odd extension*, *i.e.* define an extended function $\phi_{odd}(x)$ in \mathbb{R} (see Fig. 1) so that

$$\phi_{odd}(x) = \phi(x) \text{ for } x > 0 \ ; \ \phi_{odd}(x) = -\phi(-x) \text{ for } x < 0 \quad (13)$$

Let $u(x, t)$ be a solution to heat equation so as to satisfy

$$u(x, 0) = \phi_{odd}(x) \text{ for } x \in \mathbb{R} \quad (14)$$

Then, applying equation (15) of week 3 notes,

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi_{odd}(y) dy \quad (15)$$

Breaking up the integral into two parts $\int_{-\infty}^0$ and \int_0^{∞} and changing variables $y \rightarrow -y$ in the first and using (14), we note that (15) implies that

$$u(x, t) = \int_0^{\infty} [S(x - y, t) - S(x + y, t)] \phi(y) dy \quad (16)$$

We note that this solution automatically satisfies *Dirichlet* boundary condition $u(0, t) = 0$, and therefore from uniqueness, is the desired solution $v(x, t)$. Therefore, using expressions for $S(x, t)$ from last week notes, we obtain

$$v(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_0^{\infty} \left\{ \exp\left[-\frac{(x-y)^2}{4\kappa t}\right] - \exp\left[-\frac{(x+y)^2}{4\kappa t}\right] \right\} \phi(y) dy \quad (17)$$

We have just illustrated the method of obtaining solution through *odd-extension* or reflection about the origin; this is applicable to many other half-line problems involving *Dirichlet* boundary conditions.

2.2 Neumann Boundary Condition and even extension

We now consider the diffusion problem on a half-line but with *Neumann* condition. The problem becomes

$$w_t - \kappa w_{xx} = 0 \text{ for } x > 0, \quad t > 0 \quad (18)$$

$$w(x, 0) = \phi(x) \quad (19)$$

$$w_x(0, t) = 0 \quad (20)$$

In this case, it is more convenient to find solution through an *even extension*. We define $\phi_{\text{even}}(x)$

$$\phi_{\text{even}}(x) = \phi(x) \text{ for } x > 0 ; \quad \phi_{\text{even}}(x) = \phi(-x) \text{ for } x < 0, \quad (21)$$

and solve the initial value problem

$$u_t = \kappa u_{xx} ; \quad u(x, 0) = \phi_{\text{even}}(x) \quad (22)$$

The solution to this is

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi_{\text{even}}(y) dy \quad (23)$$

Once again breaking up the above integral into $\int_{-\infty}^0 + \int_0^{\infty}$ and using change of variable $y \rightarrow -y$ in the first integral, and using relation (21), one finds

$$u(x, t) = \int_0^{\infty} [S(x - y, t) + S(x + y, t)] \phi(y) dy \quad (24)$$

On differentiating (24) with respect to x and noting that $S_x(-y, t) = -S_x(y, t)$, it follows that $u_x(0, t) = 0$ for all $t > 0$. Thus the solution (24) indeed solves the Neumann problem for w . Using energy method again, we can prove that the classical solution to the initial value problem (18)-(20) is unique. Hence, using expressions for $S(x, t)$, we obtain from (24) solution to (18)-(20) in the form:

$$w(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_0^{\infty} \left\{ \exp\left[-\frac{(x-y)^2}{4\kappa t}\right] + \exp\left[-\frac{(x+y)^2}{4\kappa t}\right] \right\} \phi(y) dy \quad (25)$$

3 Half-line problem for linear wave equation

Now, we try the same type of reflection approach for second order wave equation. Consider first *Dirichlet* boundary condition. Thus the problem (IVP) is

$$v_{tt} - c^2 v_{xx} = 0 \quad \text{for } x > 0, -\infty < t < \infty \quad (26)$$

$$v(x, 0) = \phi(x), \quad v_t(x, 0) = \psi(x) \quad \text{for } x > 0 \quad (27)$$

$$v(0, t) = 0 \quad (28)$$

where $\phi \in \mathbf{C}^2$, $\psi \in \mathbf{C}^1$.

As for the diffusion equation on a line, we carry out an odd extension over \mathbb{R} , in this case both for $\phi(x)$ and $\psi(x)$. We define the oddly extended functions to be $\phi_{odd}(x)$ and $\psi_{odd}(x)$. We seek solution $u(x, t)$ to wave equation for $x \in \mathbb{R}$ so that it satisfies

$$u(x, 0) = \phi_{odd}(x), \quad u_t(x, 0) = \psi_{odd}(x) \quad \text{for } x \in \mathbb{R} \quad (29)$$

From d'Alembert formula, the solution is

$$u(x, t) = \frac{1}{2} [\phi_{odd}(x+ct) + \phi_{odd}(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{odd}(y) dy \quad (30)$$

We verify that $u(0, t) = 0$ is indeed satisfied by this expression. Also, using energy arguments, we can prove uniqueness of solution satisfying (26)-(28). Hence, desired $v(x, t)$ is given by (30).

Now the formula (30) can be re-expressed in terms of $\phi(x)$ and $\psi(x)$, but the expression is different in different regimes of (x, t) .

First, for $x > c|t|$, we notice that each of the arguments $x-ct$ and $x+ct$ are positive. Hence, (30) reduces to

$$v(x, t) = \frac{1}{2} [\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy \quad (31)$$

The second regime is $0 < x < ct$. We have for $\phi_{odd}(x-ct) = -\phi(ct-x)$, and $\psi_{odd}(y) = -\psi(-y)$ for $y < 0$. Hence

$$\begin{aligned} v(x, t) &= \frac{1}{2} [\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c} \left[\int_0^{x+ct} \psi(y) dy + \int_{x-ct}^0 [-\psi(-y)] dy \right] \\ &= \frac{1}{2} [\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c} \left[\int_{ct-x}^{x+ct} \psi(y) dy \right] \end{aligned} \quad (32)$$

Graphically, the result (32) can be interpreted as follows. Draw the backward characteristics from the point (x, t) (see Fig.2). In the regime $0 < x < ct$, such a backward characteristic intersects the t -axis, before crossing the x -axis at $(x-ct, 0)$. The formula (32) shows that the reflection induces a change of sign. The value of $v(x, t)$ now depends on the values of ϕ at the pair of points $ct \pm x$ and on ψ over the interval $(ct-x, ct+x)$, which is shorter than $(x-ct, x+ct)$. This is because the integral of $\psi_{odd}(y)$ over the symmetric interval $(x-ct, ct-x)$ is zero.

If $0 < x < -ct$, using similar arguments, we can check that (30) reduces to

$$v(x, t) = \frac{1}{2} [-\phi(-ct-x) + \phi(-ct+x)] - \frac{1}{2c} \left[\int_{-ct-x}^{-ct+x} \psi(y) dy \right] \quad (33)$$

The case of *Neumann problem* on a half-line for the wave equation is very similar, with an even extension of each of ϕ and ψ .

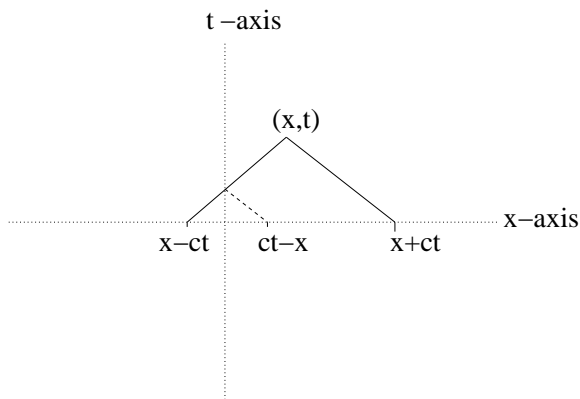


Figure 2: Backward characteristics from (x, t) for $0 < x < ct$

4 Wave Equation in a finite line

Consider the 1-D wave equation on a finite interval in x with homogeneous *Dirichlet* boundary conditions, that would correspond to say a guitar string with fixed ends. The initial value problem is given by:

$$v_{tt} = c^2 v_{xx}, \quad v(x, 0) = \phi(x), \quad v_t(x, 0) = \psi(x) \quad \text{for } 0 < x < l \quad (34)$$

and

$$v(0, t) = v(l, t) = 0 \quad (35)$$

We can extend the data for each of $\phi(x)$ and $\psi(x)$ by first doing an *odd* extension to get a function over $(-l, l)$ and then define periodical extensions $\phi_{ext}(x)$ and $\psi_{ext}(x)$ over \mathbb{R} . Thus

$$\phi_{ext}(x) = \phi(x) \quad \text{for } x \in (0, l) \quad , \quad \phi_{ext}(x) = -\phi(-x) \quad \text{for } x \in (-l, 0) \quad \text{and} \quad \phi_{ext}(x + 2l) = \phi_{ext}(x) \quad (36)$$

and a similar formula is valid for $\psi_{ext}(x)$. It is then possible to write solution as

$$v(x, t) = \frac{1}{2} [\phi_{ext}(x + ct) + \phi_{ext}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{ext}(s) ds \quad (37)$$

It can be checked directly that this satisfies $v(0, t) = 0$. To check $v(l, t) = 0$, we note that $\phi_{ext}(l - ct) = \phi_{ext}(-l - ct) = -\phi_{ext}(l + ct)$. Again, using periodicity and oddness of ψ_{ext} , we can prove that $\int_{l-ct}^{l+ct} \psi(y) dy = 0$.

While the expression (37) for the solution is relatively simple in terms of ϕ_{ext} and ψ_{ext} , it is much more complicated if we want to write it in terms of ϕ and ψ . There are different regimes of expression, depending on how many times reflection was possible at the end points $x = 0$ and $x = l$. I will refer you to Fig. 4 of the text on page 62 for a graphical illustration.

We will find later in this course alternate expressions of solution in terms of a Fourier Series.

5 Diffusion with a source

In this section, we solve the *inhomogeneous* diffusion equation in \mathbb{R}^n , for any dimension $n \geq 1$. So, the initial value problem of interest is

$$u_t - \kappa \Delta u = f(\mathbf{x}, t) \quad (38)$$

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}) \quad (39)$$

where $\phi, f \in \mathbf{C}^0$. We will prove that the solution of the problem (38)-(39) is given by

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t) \phi(\mathbf{y}) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} d\tau \quad (40)$$

This is an example of application of so-called *Duhammel's* principle where solution to inhomogeneous linear autonomous differential equation is expressed in terms of appropriate solution to the *homogeneous* solution, which in this case is \mathcal{S} . See text page 65 for analogy to ODEs.

It is convenient to define

$$v(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} d\tau \quad (41)$$

Then noting

$$\lim_{\tau \rightarrow t^-} \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} = f(\mathbf{x}, t)$$

we obtain

$$v_t - \kappa \Delta v = \lim_{\tau \rightarrow t^-} \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} [\mathcal{S}_t(\mathbf{x} - \mathbf{y}, t - \tau) - \kappa \Delta_{\mathbf{x}} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau)] = f(\mathbf{x}, t) \quad (42)$$

Further, since

$$\left| \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} \right| \leq \sup_{\mathbf{y} \in \mathbb{R}^n} |f(\mathbf{y}, \tau)| \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) d\mathbf{y} = \|f(\cdot, \tau)\| \quad \text{for } \tau < t$$

it follows $\lim_{t \rightarrow 0^+} v(\mathbf{x}, t) = 0$. Since, we know from before that

$$w(\mathbf{x}, t) = \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t) \phi(\mathbf{y}) d\mathbf{y}$$

solves the heat equation without any source $w_t - \kappa \Delta w = 0$ and initial condition $w(\mathbf{x}, 0) = \phi(\mathbf{x})$, it follows that $u(\mathbf{x}, t) = v(\mathbf{x}, t) + w(\mathbf{x}, t)$ given by (40) solves the *inhomogeneous* heat equation (38) and satisfies the initial condition (39).

6 Inhomogenous Heat Equation in a half space \mathbb{H}

Note that the text on page 67 talks about solving inhomogenous diffusion equation on a half-line. There is no problem extending this idea to n -dimensions. We define half space

$$\mathbb{H} \equiv \{\mathbf{x} \in \mathbb{R}^n : x_1 > 0\} \quad (43)$$

Our problem is to solve

$$v_t - \kappa \Delta v = f(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in \mathbb{H} \quad (44)$$

$$v(\mathbf{x}, 0) = \phi(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{H} \quad (45)$$

$$v(\mathbf{x}, t) = 0 \quad \text{for } \mathbf{x} \in \partial\mathbb{H} \quad (46)$$

First, we note using odd-extension of $\phi(\mathbf{x})$ in the component x_1 , as in 1-D, we can obtain a *source* type solution that satisfies the boundary condition on $x_1 = 0$ (which is the same as $\partial\mathbb{H}$). It is convenient to define

$$\mathbf{y}_- = (-y_1, y_2, y_3, \dots, y_n) \quad , \quad \text{where } \mathbf{y} = (y_1, y_2, \dots, y_n)$$

Then, it is easy to verify that

$$\mathcal{T}(\mathbf{x}, \mathbf{y}, t) \equiv \mathcal{S}(\mathbf{x} - \mathbf{y}, t) - \mathcal{S}(\mathbf{x} - \mathbf{y}_-, t) \quad (47)$$

is a solution of the heat equation in the half-space \mathbb{H} corresponding to a unit source at \mathbf{y} that satisfies the homogenous *Dirichlet* condition (36). Therefore, using the same ideas as in the last section, we can verify that the solution to the problem posed in (44)-(46) is given by

$$v(\mathbf{x}, t) = \int_{\mathbb{H}} \mathcal{T}(\mathbf{x}, \mathbf{y}, t) \phi(\mathbf{y}) d\mathbf{y} + \int_0^t \int_{\mathbb{H}} \mathcal{T}(\mathbf{x}, \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} d\tau \quad (48)$$

7 Note on Inhomogenous Dirichlet or Neumann conditions

So far, our boundary conditions, either for the heat or wave equation involved homogeneous *Dirichlet* or *Neumann* boundary conditions. However, solution for an inhomogenous condition is not a serious problem. For the sake of being definite, consider for instance

$$u_t - \kappa u_{xx} = f(x, t); \quad \text{for } x > 0 \quad , \quad u(0, t) = h(t) \quad , \quad u(x, 0) = \phi(x) \quad (49)$$

We assume $h \in \mathbf{C}^1$. Noting that the function $w(x, t) \equiv e^{-x} h(t)$ satisfies the boundary condition at $x = 0$ and well-behaved in x as $x \rightarrow +\infty$, it follows that by decomposing $u(x, t) = w(x, t) + v(x, t)$, $v(x, t)$ satisfies

$$v_t - \kappa v_{xx} = f(x, t) - w_t + \kappa w_{xx} \equiv \tilde{f}(x, t) \quad (50)$$

$$v(0, t) = 0 \quad ; \quad v(x, 0) = \phi(x) - h(0)e^{-x} \equiv \tilde{\phi}(x) \quad (51)$$

Equations (38)-(39) defines an inhomogeneous heat equation with homogeneous *Dirichlet* condition and is of the type for which we have a general representation of solution, as seen in the last section.