Exponential, Trigonometric and Log Functions

Definion 1.1: $exp(z)$ is defined by its power series

$$exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1}$$

Remark 1.1: Series in (1) converges for any z.

Lemma 1.1:

$$exp(z_1 + z_2) = exp(z_1)\ exp(z_2) \tag{2}$$

Proof: Rearrange the double summation in the product of power series representation in (1) and use binomial theorem (Details left as an exercise)

Corollary 1.1: $exp(-z) = 1/exp(z)$.

Proof: $exp(z)\ \ exp(-z)\ = \ exp(0) = 1$ from (1). **QED**

Lemma 1.2 For real k, $(exp(z))^k = exp(kz)$.
Proof: For integer k, proof follows from Lemma 1.1 and Corollary 1.1, through induction. Proof can be extended simply for rational k and from continuity of exponential function to real k.

Remark: Since, clearly $exp(1) = e$, the properties above confirm that $exp(x) \ = \ e^x$, the well known exponential function for real x. There is no point distinguishing e^z and $exp(z)$, since defn. 1.1 can be thought of as defining e^z for complex z.

Definition 1.2:

$$cos\ z \ = \ \frac{1}{2}[exp(iz)\ +\ exp(-iz)] \tag{3}$$

$$sin\ z \ = \ \frac{1}{2i}[exp(iz)\ -\ exp(-iz)] \tag{4}$$

Remark : Both series convergent for all z.
Remark: All familiar trigonometric identities derivable from (3)-(4).
Remark: Note from the power series of e^z that

$$cos\ z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \tag{5}$$

$$sin\ z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{(2n+1)}}{(2n+1)!}, \tag{6}$$

each of which has an infinite radius of convergence.

Lemma 1.3:

$$e^{x+iy} = e^x \ (cos\ y \ + \ i \ sin\ y) \tag{7}$$

Proof: $exp(x+iy) \ = \ exp(x) \ exp(iy)$. But using the series representation, it is clear $exp(iy) = cos\ y \ + \ i \ sin\ y$. **QED**.

Corollary 1.2: If $z = x \ + \ iy$, with x and y real, then $|e^z| = e^x \ = \ e^{Re\ z}$.
Proof: Note from Lemma 1.3 that $|e^{x \ + \ iy}| = e^x \sqrt{cos^2\ y \ + \ sin^2\ y} \ = \ e^x$.

Lemma 1.4: If $exp(z) = 1$, then $z = 2\ \pi\ i\ n$, for integer n.
Proof: Let $z = x + iy$. From previous lemma, $e^x\ cos\ y \ = \ 1$, $e^x\ sin\ y \ = \ 0$. The second relation implies $y = m\ \pi$ for integer m. Hence $cos\ (m\ \pi)\ e^x \ = \ 1$. There are no real x roots for m odd. Hence n must be even, i.e. $m = 2\ n$ and in that case $x = 0$. **QED**.

Definition 1.3: The logarithmic function, ln (actually, a set of functions on the plane) is defined so that

$$exp\ (\ln\ z) \ = \ z \tag{8}$$

, i.e. inverse of exp.

Remark: Note that if z is written in its polar representation: $z = r\ e^{i\theta}$, where $r = \ |z|$ and $\theta \ = \ arg\ z$, then

$$\ln\ z \ \equiv \ ln\ r \ + \ i\ \theta \ + \ 2in\pi \tag{9}$$

for integer n is consistent with definition (1.3). This is easily shown through substitution. However, (9) is not unambiguous on the plane. To make (9) a single-valued function (for a particular n) on the complex plane, one needs to restrict θ to a specified interval of $2\ \pi$ interval:

$$(\theta_0,\ 2\ \pi + \theta_0] \tag{10.1}$$

or

$$[\theta_0\ ,\ 2\ \pi \ + \ \theta_0) \ \text{ where } \theta_0 \ \text{ is some constant} \tag{10.2}$$

Remark: One might wonder if (9) provides the most general representation of the inverse of exp function. The following lemma answers this in the positive.

Lemma: 1.5 If $\ln_1$ and $\ln_2$ are two functions satisfying definition (1.3), then then for a particular z,

$$ln_1\ z - ln_2\ z \ = \ 2\ \pi\ i\ n \tag{11}$$

where n is an integer.
Proof: Let $w_1 = \ln_1\ z$ and $w_2 \ = \ \ln_2\ z$. Then, $exp(w_1 - w_2) = exp(w1)/exp(w2) = 1$. From previous lemma, $w_1 - w_2 = 2\ \pi\ i\ n$. **QED**

Remark: It is possible that $\ln_1\ z\ =\ \ln_2\ z$ for some set of z in the plane and not for others. For instance: if we define

$$\ln_1\ z\ =\ \ln\ r\ +\ i\ \theta, \quad \text{with}\ \ \theta\ \ \text{in}\ \ (-\pi,\ \ \pi]\ , \tag{12}$$

$$\ln_2\ z\ =\ \ln\ r\ +\ i\ \theta, \quad \text{with}\ \ \theta\ \ \text{in}\ \ [0,\ 2\ \pi)\ , \tag{13}$$

then clearly $\ln_1\ z\ =\ \ln_2\ z$ for z in the first and second quadrant but not equal in the third and fourth quadrant. In general, the integer n appearing in (11) varies with choice of z unless θ_0 is the same for $\ln_1$ and $\ln_2$ functions.

Definition 1.4: The particular choice $n = 0$ in (9), with $\theta_0 = 0$ in (10.1) defines a particular logarithmic function, called the *principal branch* of the logarithm, with the notation $\ln_p\ z$ or $Ln\ z$.

Definition 1.5 The set of all z for which ln is undefined are called it's branch points: (0 and ∞ in this case).

Definition 1.6 The set of points across which ln function undergoes a discontinuity is called a branch cut, i.e. $\{z \mid arg\ z = \theta_0\}$ is the branch cut.

Definition 1.7 For specificied branch cut, i.e. θ_0, the value of n is referred to as the branch of the logarithm.

Remark: Branch cuts (and therefore discontinuities) of ln function cannot be avoided if $ln\ z$ is to be uniquely defined in the complex plane. However, a branch cut is quite arbitrary. It can be completely avoided if we equate the domain through the polar pair $(r,\ \theta)$, with no restriction on θ. In that case, one can define uniquely

$$\ln\ z = \ln\ r\ +\ i\theta \tag{14}$$

with

$$-\infty\ <\ \theta\ <\ \infty \tag{15}$$

In this representation, specification of z is not enough; we need to specify θ. This domain, based on choice of θ, is referred to a *Riemann surface*. Note that while $ln\ z$ is not continuous (across cuts), when restricted on the plane; it is indeed continuous on the Riemann surface.

Remark: Note that in general, $ln_p\ (z_1 z_2)\ \neq\ ln_p\ z_1\ +\ ln_p\ z_2$, though this is true on the Riemann surface, since on a Riemann surface, with appropriate choice of the argument,

$$ln\ (z_1\ z_2)\ =\ ln\ \left[r_1 r_2\ e^{i(\theta_1+\theta_2)}i\right]\ =\ ln\ r_1\ +\ ln\ r_2\ +\ i\ \theta_1\ +\ i\theta_2\ =\ ln\ z_1\ +\ ln\ z_2 \tag{16}$$

Continued discussion of ln and z^k functions

Previously, branch points of logarithmic function were defined as points where the ln is undefined. A more general definition, as follows, is needed for other functions.

Definition 2.1 A finite point z_0 is a branch point of a function $f(z)$ if for all sufficiently small $\epsilon > 0$,

$$f(z_0 + \epsilon\, e^{i(\phi + 2\pi)}) \neq f(z_0 + \epsilon\, e^{i\phi})$$

i.e. if we circle around $z = z_0$, we do not return to the same value of z.

Remark: In the above definition, we are assuming $f(z_0 + \epsilon\, e^{i\phi})$ is a continuous function of ϕ.

Eg. $ln\ (z+1)$ at $z = -1$. If $z + 1 = \epsilon\, e^{i\phi}$, then $ln\ (z+1) = ln\ \epsilon + i\ \phi + 2\ i\ n\ \pi$. When θ is replaced by $\theta + 2\ \pi$, we do not return to the same value.

Definition 2.2 $z = \infty$ is a branch point of $f(z)$ if $f(1/w)$ has a branch point at $w = 0$.
Eg. $ln\ z$ has a branch point at ∞ since $ln\ 1/w = -\ ln\ w + 2in\pi$ has a branch point at $w = 0$. However, $ln\ \left(\frac{z+1}{z-1}\right)$ has no branch points at $z = \infty$.

Recall also from last lecture that in general, for a specific branch, we may not have

$$ln(z_1\ z_2) = ln(z_1) + ln(z_2) \tag{1}$$

However, if we do not assign any explicit restriction on $arg\ (z_1\ z_2)$, but instead define it as:

$$arg\ (z_1\ z_2) = arg\ z_1 + arg\ z_2 \tag{2}$$

then (1) will hold for the $n = 0$ branch. For instance, if we take $arg\ (-1) = \pi$, and apply definition (2), $arg\ (-1)^2 = 2\ arg(-1) = 2\ i\ \pi$ and hence $ln\ (-1)^2 = 2\ i\ \pi$ and not 0.

In defining a composite function involving logarithm, *say.* $\ln\ (z^2 - 1)$, it is convenient to place a $2\ \pi$ interval restriction on $arg\ (z-1)$ and $arg\ (z+1)$, but not on $arg\ (z^2 - 1)$; instead we require

$$arg\ (z^2 - 1) = arg\ (z-1) + arg(z+1)\ , \tag{3}$$

in accordance to (2). Thus, if $z - 1 = r_1 e^{i\theta_1}$ and $z_2 = r_2\ e^{i\theta_2}$, then, using (3),

$$\ln\ (z^2 - 1) = \ln\ (r_1\ r_2) + iarg(z^2 - 1) + 2in\pi = \ln\ r_1 + \ln\ r_2 + i(\theta_1 + \theta_2) + 2in\pi \tag{4}$$

If θ_1, θ_2 chosen to be within $(-\pi,\ \pi]$, the branch cuts for $ln\ (z^2 - 1)$ will be those shown in Fig. 1. $z = \pm 1$ and $z = \infty$ are branch points as may be readily checked. There is

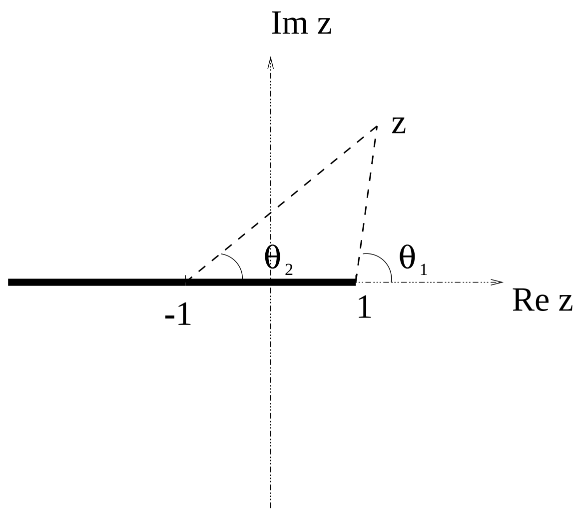

Figure 1: Branch cuts for ln $(z^2 - 1)$

of course a denumerably infinite set of branches of the function, corresponding to differing integral values of the integer n.

Alternately, if we restricted θ_1 to $[0,\ 2\,\pi)$ and θ_2 to $(-\pi,\ \pi]$, then the corresponding branch cuts are as given below in Fig. 2. Once again we have infinitely many branches characterized by integer n in (4).

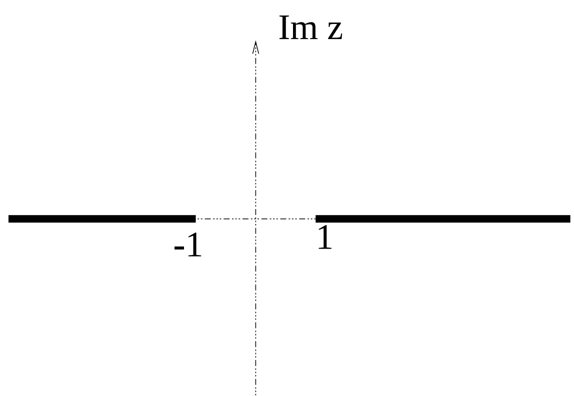

Figure 2: Another choice of cuts for ln $(z^2 - 1)$

Consider another example of of a composite function involving logarithm:

$$ln\ \left(\frac{z-1}{z+1}\right) \tag{5}$$

In this case, let $z-1\ =\ r_1\ e^{i\ \theta_1}$ and $z+1\ =\ r_2\ e^{i\theta_2}$. Then, if we put $2\,\pi$ interval restriction on $arg(z\ \pm 1)$, but avoid putting direct restrictions on $arg\ \left(\frac{z-1}{z+1}\right)$ itself, then if is possible to define

$$arg\ \left(\frac{z-1}{z+1}\right)\ =\ arg\ (z-1)\ -\ arg\ (z+1) \tag{6}$$

Then

$$ln\ \left(\frac{z-1}{z+1}\right)\ =\ ln\ r_1\ -\ ln\ r_2\ +\ i(\theta_1\ -\ \theta_2)\ +\ 2\ i\ n\ \pi \tag{7}$$

If both θ_1 and θ_2 are restricted to $(-\pi,\ \pi]$, then the corresponding branch cut is shown in Fig. 3.

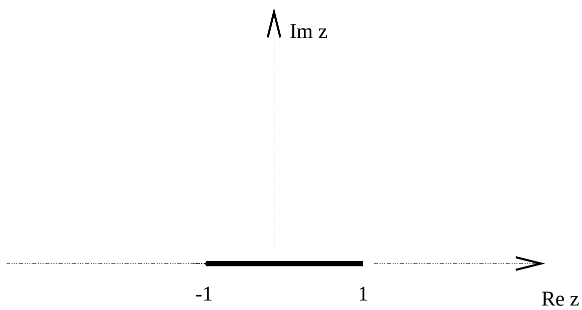

Figure 3: Branch cuts for ln $\left(\frac{z-1}{z+1}\right)$

Note that there is no cut on the real axis, left of $z\ =\ -1$. The reason in this case is that the two cuts, for ln $(z-1)$ and ln $(z+1)$ have cancelled each other out. To see this, note that if we approach from the top a point on the real axis to the left of -1, $\theta_1,\ \theta_2\ =\ \pi$; approaching from below, one gets $\theta_1,\ \theta_2\ =\ -\pi$. In either case, $(\theta_1-\theta_2)$ appearing in (7) is equal to 0. So, for any fixed branch n, the function as defined in (7) has no discontinuity on the real axis to the left of -1 and hence no cut over there. Note if we restrict θ_1 and θ_2 to be in $[0,\ 2\ \pi)$, then the branch cut situation is still the same as in Fig. 3. In this case, the branch cut cancellation is on the positive real axis to the right of the branch point $z=1$.

On the other hand if we restrict θ_1 to $[0,\ 2\pi)$ and θ_2 to $(-\ \pi,\ \pi]$, then the corresponding branch cut situation is described by Fig. 4.

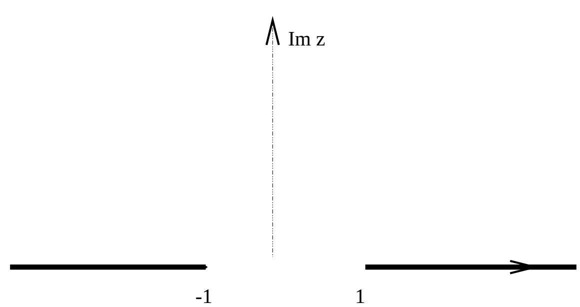

Figure 4: Another possible cut for ln $\left(\frac{z-1}{z+1}\right)$

It is to be noted that the branch cut situation described in Fig. 3 and Fig. 4 are not the only ones possible. One can have other $2\ \pi$ interval restrictions on θ_1 and θ_2 that will give rise to other kinds of branch cuts for the function in (7). In practice, specific choices are made according to the requirement that the function be continuous in some part of the complex plane.

z^k for nonintegral k

Definition 2.3: For nonintegral k

$$z^k = exp[k\ ln\ z]\ =\ exp[k\ (ln\ r\ +\ i\theta\ +\ 2\ i\ n\ \pi)] \tag{8}$$

where for the purposes of uniqueness on a plance (for particular n), θ is restricted in the interval

$$[\theta_0,\ 2\ \pi + \theta_0) \quad \text{or} \quad (\theta_0,\ 2\ \pi + \theta_0\ ] \tag{9}$$

Remark: Since k is not an integer, $exp[2\ i\ n\ k\ \pi]\ \neq\ 1$; hence z^k can have more than one values for the same z, depending on n. A fixed choice of n in (8) and θ_0 in (9) specifies as unique function z^k on the plane. The value of n characterizes the branch, and the choice θ_0 defines the branch cut. Note however, that unlike the case of ln , z^k need not have infinite set of distinct branches since differing values of n can give the same value of $exp[2\ i\ n\ k\ \pi]$. In particular, if $k = p/q$, p and q being integers, then clearly $exp[2\ i\ n\ k\ \pi]$ can have q distinct values corresponding to the choice $n = 0, 1, 2, .., (q-1)$. Note that $n = q$ gives rise to the same value as $n = 0$, $n = q + 1$ the same as $n = 1$ and so on. For instance $z^{1/2}$ has only two distinct branches corresponding to even and odd n in (8).

$$z^{1/2} = r^{1/2}\ e^{i\theta/2} \tag{10}$$

$$z^{1/2}\ =\ -\ r^{1/2}\ e^{i\theta/2} \tag{11}$$

Remark: We note that for a particular choice of n (i.e. branch) and a specific 2π restriction on the arg (...), it is possible that

$$(z_1\ z_2)^k\ \neq\ z_1^k\ z_2^k \tag{12}$$

On the other hand, if we only impose $2\ \pi$ interval restriction on $arg\ z_1$ and $arg\ z_2$, and not on $arg\ (z_1\ z_2)$, then as before in the context of ln function, it is possible to define $arg\ (z_1\ z_2)$ so that

$$(z_1\ z_2)^k\ \ =\ z_1^k\ z_2^k \tag{13}$$

Exercise for reader: Determine appropriate branch cuts and branches for (i) $(z^2 - 1)^{1/2}$ and (ii) $\left(\frac{z+1}{z-1}\right)^{1/2}$. Suppose we have a branch for which $f(2) > 0$. Determine what is the value of $f(-2)$ consistent with your choice of cuts.

Remark: Sometimes in determining branch cuts and branches of a composite function, involving logarithm and nonintegral powers in a complicated manner, it is useful to introduce suitable intermediate transformation $w = g(z)$ and determine the branch cut in the w variable. The pre-image of that cut in the z-variable gives the cut location in the z-plane. The example below illustrates this point.

Example: Describe the branch cuts, branch points and branches for the function

$$f(z) = \ln\ (1 + z^{1/2})$$

Solution: Let $w = g(z) = z^{1/2}$. Then $f(z) = \ln\ [1 + g(z)]$. If we restrict $arg\ z = \theta$ to $(-\pi,\ \pi]$ (corresponding cut shown in Fig. 5), then there are two *distinct* possibilities for $g(z)$:

$$g_1(z) = r^{1/2}\ e^{i\theta/2},\quad g_2(z) = -\ r^{1/2}\ e^{i\theta/2} \tag{14}$$

where $r = |z|$. There is a branch point of each of $g_1(z)$ and $g_2(z)$ at $z = 0$ and $z = \infty$. These are clearly branch points of $f(z) = \ln(1 + g(z))$ as well.

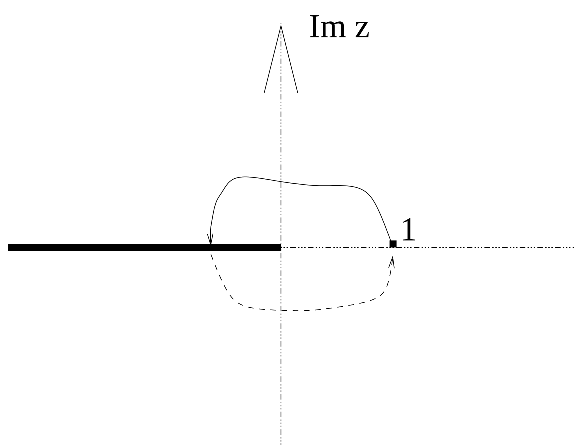

Figure 5: Branch cuts for $z^{1/2}$ and $f_{1,m}(z)$. A cut-crossing path also shown

We also note that $w = -1$ is not in the range of g_1, but it is of g_2 (since $g_2\ (1) = -1$). Now, consider $\ln\ (1 + w)$. We choose the restriction $arg\ (1 + w)$ in $(-\pi, \pi]$, with the cut in Fig. 6. we define

$$\ln(1 + w) = \ln|1 + w| + i\ arg(1 + w) + i\ 2\ m\ \pi \tag{15}$$

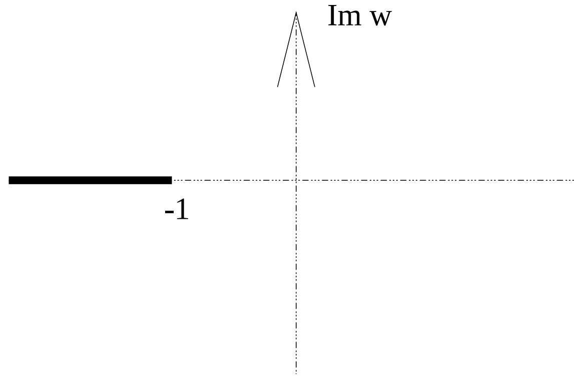

Figure 6: Branch cut in the w plane for $\ln\ (1+w)$

Since no point on the chosen cut in the w plane is in the range of g_1, it is clear that

$$f_{1,m}(z) \;=\; \ln_m\ (1\ +\ g_1(z)) \tag{16}$$

will have no branch points and cuts in the z plane corresponding to the branch point and cut in the w-plane. So, the only branch point and branch cut for the function $f_{1,m}(z)$ are those shown in Fig. 5– only the ones corresponding to $g_1(z)$.

However, the pre-image in the z-plane of the chosen branch cut in the w-plane under g_2 is the positive real z axis, to the right of the branch point $z = 1$ (corresponds to the cut in the w-plane in Fig. 6). Thus, the function

$$f_{2,m}(z) \;=\; \ln_m\ (1\ +\ g_2(z)) \tag{17}$$

will have branch cuts shown in Fig. 7.

The function $f(z)$ has two set of infinite number of branches given by (16) and (17), corresponding to integer m. The cuts are different for (16) and (17). The reader might be interested to note that if we start from $z = 1$ where $f(1) \;=\; \ln\ 2$, i.e. choose $f_{1,0}(z)$ for this point evaluation, and follow the change of $f(z)$ on the Riemann surface along the path shown in Fig. 5, then on crossing the cut in Fig. 5, $f(z)$ ceases to be $f_{1,0}(z)$. Instead, it becomes $f_{2,0}(z)$ for which Fig. 7 describes the branch cuts and points. We have used dotted lines on the path in Fig. 5 to denote our journey to the other sheet for which Fig. 7 is applicable.

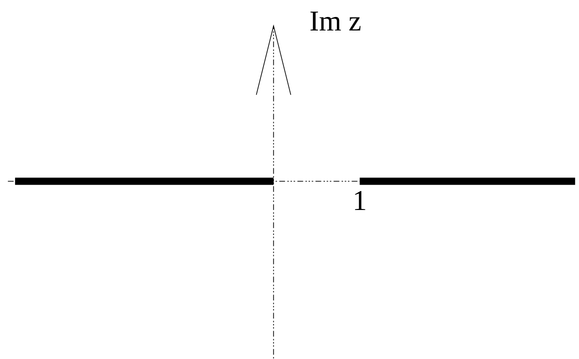

Figure 7: Branch cut for $f_{2,m}(z)$, as given by (17)