Homework Set 2, Math 804, Due October 19th

1. Let f have simple poles at $z = z_n$, $n = 1, 2, ..N$, with corresponding residues a_n, and analytic everywhere else. Show that

   a.
$$\frac{1}{2\pi i}\oint_{C_N} \frac{f(z')}{z'-z} = f(z) + \sum_{n=1}^{N} \frac{a_n}{z_n - z},$$
   where C_N is a large circle of radius R_N enclosing all the poles.

   b. Use result in **a.**, and assuming $z_n \neq 0$, show that
$$\frac{1}{2\pi i}\oint_{C_N} \frac{z f(z')}{z'(z'-z)} = f(z) - f(0) + \sum_{n=1}^{N} a_n \left(\frac{1}{z_n - z} - \frac{1}{z_n}\right) \tag{1}$$

   c. Assume f has simple poles at $\{z_n\}_{n=1}^{\infty}$ with corresponding residues $\{a_n\}_{n=1}^{\infty}$, with $z_n \neq 0$. Assume f is uniformly bounded on a sequence of contours $\{C_N\}_{n=N_0}^{\infty}$, with $R_N \to \infty$, and that the sum on the right of (1) converges for fixed z as $N \to \infty$. Then show that
$$f(z) = f(0) + \sum_{n=1}^{N} a_n \left(\frac{1}{z - z_n} + \frac{1}{z_n}\right)$$

   d. Use (c) for $f(z) = \pi \cot \pi z - \frac{1}{z}$ to show
$$\pi \cot \pi z = \frac{1}{z} + {\sum}'^{\infty}_{n=-\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right),$$
   where $\sum'$ means that $n = 0$ term is missing in the summation.

2. Consider the analytic function $\Gamma(z)$ defined in $Re\ z > 0$ through
$$\Gamma(z) = \int_0^{\infty} t^{z-1}\ e^{-t}\ dt$$
   (This is analytic because $P(t, z) = t^{z-1}e^{t}$ is analytic in z and $\mathcal{L}_1$ in t; hence $\int P(t, z)dt$ is analytic from a theorem in complex analysis). Show that
$$\Gamma(z) = \frac{1}{2i \sin\ (\pi z)} \int_C t^{z-1} e^{t} dt,$$
   with C given in Fig. 1, provides for the analytic continuation to $Re\ z \leq 0$, and that $\Gamma(z)$ has only simple pole singularities when z is a non-positive integer.

3. Specify cuts and branches for ln so that
$$\int_1^z \frac{1}{z'} dz' = \ln\ z$$
   for each of the three integration contours (a)-(c) in Figures 2-4:

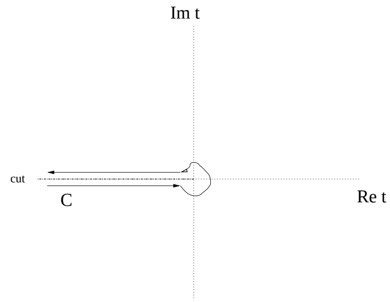

Figure 1: Integration path C in t-plane in defining $\Gamma(z)$.

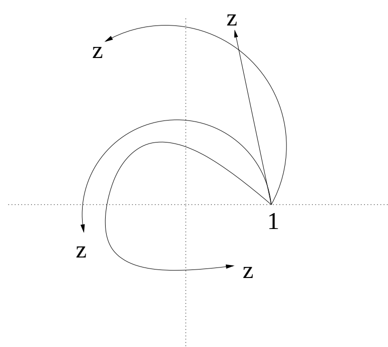

Figure 2: Integration paths joining 1 to z in different quadrants for case (a)

4. Recall $\sin\ z = \frac{e^{iz}-e^{-iz}}{2i}$. Using this, determine $\sin^{-1}\ z$ in terms of log and specify different choice of branch cuts and branches. On the Riemann surface, describe how branches are connected to each other as we cross branch cuts. Determine cuts and branch so that $\sin^{-1}\ (1) = \frac{\pi}{2}$.

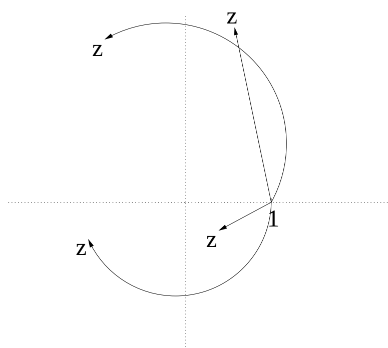

Figure 3: Integration paths joining 1 to z in different quadrants for case (b)

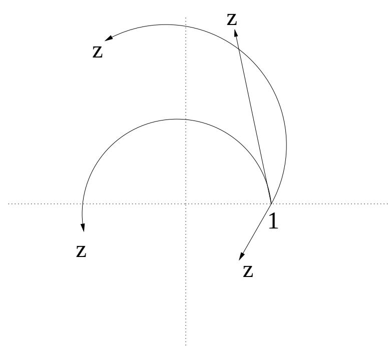

Figure 4: Integration paths joining 1 to z in different quadrants for case (c)