Homework 4, Math 804 Tanveer
Due date: Nov. 9th, '09

1. Use Schwartz Reflection principle to show that an analytic function f in a domain D can be analytically continued across part of its boundary ∂D where $\Im\ f$ (or $Re\ f$) is a constant provided the the boundary segment is analytic. (Recall an analytic curve is the graph of some analytic function $\Gamma(t)$ for t real, $a \leq t \leq b$). **Hint:** You may want to think of an appropriate composition of analytic functions.

2. Show that the function

$$w(z) = \frac{1}{2} \left(z + \frac{1}{z}\right)$$

maps the exterior of a unit circle centered around the origin in the z plane to the exterior of a straight line cut from -1 to 1 in the w plane. What is the inverse of this mapping. Be specific on the choice of branch if the inverse function involves branches.

3. Think of solving $\Delta\phi = 0$ in the strip domain $\{x + iy, 0 < y < 1\}$, with boundary conditions $\phi = 1$ on $y = 1$ and $\phi = 0$ on $y = 0$. What *a priori* assumption do you need on the growth of $\phi(x, y)$ as $x \to \pm\infty$ to assure that the only solution to this problem is $\phi(x, y) = y$? (**Hint:** You might want to think of transformation and Phragmen-Lindeloff principle.

4. Solve the potential problem for $\phi(x, y)$

$$\Delta\phi = 0 \ \text{ in } \ y > 0$$

with

$$\phi(x, 0) = 1 \ \text{ for } x > 1, \phi(x, 0) = -1 \ \text{ for } x < -1 \ \text{ and } \ \frac{\partial\phi}{\partial y}(x, 0) = 0 \ \text{ for} |x| < 1$$

State and prove conditions that will make the solution you find unique. **Hint:** Think of mapping to a semi-infinite strip and Schwarz Reflection Principle.

5. Complete the proof of Lemma 11.1 in Week 5 notes, page 1.

6. a. Find a linear fractional transformation that maps the unit disk centered at 0 to the upper-half plane. **b.** Find a 1-1 analytic mapping of the interior of $|z - 1 - 2i| = 2$ to the interior of the unit disk centered at $i/2$.