Homework 6, Math 804 Tanveer
Due date: Monday, November 19th, '07

1. Obtain solution f to the singular integral equation

$$a(z)f(z) = \lambda \fint_C \frac{f(t)dt}{t-z} + g(z),$$

where C is a smooth simple closed curve. State conditions on λ, $a(z)$ and $g(z)$ for solution to exist and for solution to be unique.

2. Consider $C = [0,1]$. Suppose f is Holder continuous except for weak singularities at end points $t = 0$ and $t = 1$, where $t^{\beta_1}\phi(t)$ is locally Holder continuous near $t = 0$ and $(1-t)^{\beta_2}\phi(t)$ is locally Holder continuous at $t = 1$ for $\beta_1, \beta_2 \in (0,1)$. Prove that $\Phi(z) = \frac{1}{2\pi i}\int_0^1 \frac{\phi(s)ds}{s-z}$ can have at most weak singularity. at $z = 0$ and $z = 1$. (**Note:** The result is true for generally smooth simple open contour C. After you are done, think how you would produce more generally)

Hint: Note it suffices to show result near $z = 0$, since the argument at $z = 1$ is similar. For analysis near $z = 0$, you may want to consider two separate cases: (a) $\theta \equiv |\arg z| \geq \frac{\pi}{4}$, (b) $|\arg z| < \frac{\pi}{4}$. For case (a) it is convenient to rescale $s = |z|\tau$ and note that the $\frac{1}{\tau - e^{i\theta}}$ is bounded away from zero. For case (b), introduce $x = \Re z$ and break up the integral

$$\int_0^1 \frac{s^{\beta_1}\phi(s) - x^{\beta_1}\phi(x)}{s_1^{\beta}(s-z)} ds + x^{\beta_1}\phi(x)\int_0^1 \frac{s^{-\beta_1} - x^{-\beta_1}}{s-z} ds + \phi(x)\int_0^1 \frac{ds}{s-z} ds$$

use Holder continuity for the first, scaling $s = x\tau$ in the second and explicit calculation in the third. You may find it convenient to get bounds in terms of x first and then use $|z| \geq |x| \geq \frac{|z|}{\sqrt{2}}$.

3. Use result in last exercise to characterize the most general solution to the Riemann Hilbert problem

$$\Phi_+(t) - \Phi_-(t) = \phi(t) \text{ for } t \in (0,1)$$

where ϕ satisfies conditions of the last exercise with growth condition

$$\lim_{z\to\infty} z^{-n}\Phi(z) = 0$$

for integer $n > 0$, *i.e.* $n \in \mathbb{Z}^+$.