Homework 5, Math 804 Tanveer
Due date: Wednesday, November 23rd, '11

1. Consider the Fourier Transform

$$f(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{(x+1)^2+1}} dx$$

Determine the asymptotic series for $f(k)$ as $k \to \pm\infty$ and explicitly calculate the first two nonzero terms. **Hint**: Use Watson's Lemma after suitable contour deformation in the complex x plane.

2. Determine solution to the following singular integral equation for $x \in (-1, 1)$:

$$f(x) + \fint_{-1}^{1} \frac{f(s)}{s-x} ds = x$$

3. Obtain solution f to the singular integral equation

$$a(z)f(z) = \lambda \fint_C \frac{f(t)dt}{t-z} + g(z),$$

where C is a smooth simple closed curve. State conditions on λ, $a(z)$ and $g(z)$ for solutions to exist and be unique. (**Note:** Closed curve needs special consideration in the associated R-H problem)

4. Suppose f is Holder continuous in (0, 1). Assume near 0 and 1, $t^\beta f$ and $(1-t)^\beta f$ are, respectively, Holder continuous for some $\beta < 1$. Prove that $\Phi(z) = \frac{1}{2\pi i}\int_0^1 \frac{\phi(s)ds}{s-z}$ can at most have weak singularities at $z = 0$ and $z = 1$. **Hint:** It suffices to consider z near 0 since same arguments hold as z near 1. For z near 0, consider two separate cases: (a) $|\theta| = |\arg z| \geq \frac{\pi}{4}$ and (b) $|\arg z| < \frac{\pi}{4}$. In case (a), scale $s = |z|\tau$ and note that $|\tau - e^{i\theta}|$ is bounded away from zero. For case (b), with $x = \Re z$, convenient to decompose given integral as

$$\int_0^1 \frac{s^\beta\phi(s) - x^\beta\phi(x)}{s^\beta(s-z)} ds + x^\beta\phi(x)\int_0^1 \frac{s^{-\beta} - x^{-\beta}}{s-z} ds + \phi(x)\int_0^1 \frac{ds}{s-z}$$

Use Holder continuity in the first, scaling $s = x\tau$ in the second and explicit calculation in the third.

5. Use result in the last exercise to characterize the most general solution for the Riemann Hilbert problem

$$\Phi_+(t) - \Phi_-(t) = \phi(t) \text{ , for } t \in (0, 1)$$

where ϕ satisfies conditions in the last exercise with growth conditions

$$\lim_{z\to\infty} z^{-n}\Phi(z) = 0$$