Solution to Homework 4, Math 804, Tanveer

1. Use Schwartz Reflection principle to show that an analytic function f in a domain D can be analytically continued across an analytic segment of its boundary ∂D, where $\Im\, f$ (or $Re\ f$) is a constant and and the intersection of D with a sufficiently small ball centered at a boundary point is a "semi-circular" type domain on one side of ∂D. (Recall an analytic curve is the graph of some analytic function $\Gamma(t)$ for t real, $a < t < b$. You may also assume that $\Gamma'(t) \neq 0$, as appropriate for given assumption.)

Solution First, there is no loss of generality taking $\Im f = 0$ on part of the boundary ∂D in question. If $\Im f = c \neq 0$, we replace f by $f - ic$ in the argument presented. If instead $\Re f = c$, we replace f by $i(f-c)$ in the argument presented below since $\Im \{i(f-c)\} = 0$.

Further, the argument will be local in a neighborhood of arbitrary point z_0 on the boundary segment where $\Im f = 0$. We may represent a local boundary segment at z_0 to be $\gamma(t) = t + i\Gamma(t)$ for $t \in (-a, a)$ for a sufficiently small, where $z_0 = \gamma(0)$. (If the boundary tangent at z_0 is vertical, repeat the same argument for $\gamma(t) = \Gamma(t) + it$). Since the boundary is analytic $\Gamma(t)$ is analytic in $(-a, a)$. Consider the image of the ball $\{t : |t| < a\}$ under the mapping γ. Since $\gamma'(0) = 1 + i\Gamma'(0) \neq 0$, it follows that there exists a ball $\mathcal{B}_\epsilon(z_0) \subset \gamma(\{t : |t| < a\})$ such that the mapping γ is an analytic homeomorphism between $\gamma^{-1}\{\mathcal{B}_\epsilon(z_0)\}$ and $\mathcal{B}_\epsilon(z_0)$ as shown in the figure below (Recall the proof in class notes that show that a map g is an 1-1 analytic map in a local neighborhood of a point where $g' \neq 0$). The subset

$$\mathcal{H} := \{\Im\ t > 0\} \cup \gamma^{-1}\{\mathcal{B}_\epsilon(z_0)\},$$

is mapped to $\gamma(\mathcal{H})$, adjacent to ∂D, each region shown in shade in Fig. 1. Therefore, $f \circ \gamma$ is a locally analytic function near $t = 0$ that maps $\mathcal{H}$ into a some region $f \circ \gamma(\mathcal{H})$ in the ζ plane adjacent to the real axis.

From Schwarz reflection principle, $f \circ \gamma$ is analytically continuable to the the unshaded region inside $\gamma^{-1}(\mathcal{B}_\epsilon(z_0))$ including parts of the real t-axis. Since γ is an analytic homeomorphism in this region, it follows that f is analytic on $\partial D \cap \mathcal{B}_\epsilon(z_0)$.

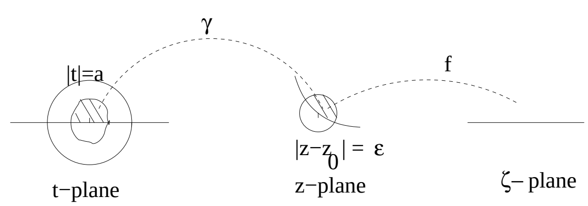

Figure 1: Analytic homeomorphism between shaded regions $\mathcal{H} = \{\Im t > 0\} \cup \gamma^{-1}(\mathcal{B}_\epsilon(z_0))$ and $\gamma(\mathcal{H})$. $f \circ \gamma$ maps shaded region adjacent to the real t to the a region next to the real axis in the ζ-plane.

2. Determine a mapping function that maps the upper-half plane into the interior of an equilateral triangle with sides of unit length. State any constraint(s) to determine parameters in the mapping.

Solution We arrange the equilateral triangle in the z-plane to have vertices at $\frac{\sqrt{3}}{2}i$, $-\frac{1}{2}$ and $\frac{1}{2}$. We look at this map so that these vertices correspond to -1, 0 and 1 in the upper-half ζ-plane. Note that the exterior angle is $\gamma_1 = \gamma_2 = \gamma_3 = \frac{2}{3}\pi$ Then from Schwartz-Christoeffel map, we have

$$z = f(\zeta) = \frac{\sqrt{3}}{2}i + C\int_{-1}^{\zeta} (\zeta'+1)^{-2/3}\zeta'^{-2/3}(\zeta'-1)^{-2/3}d\zeta'$$

$$= \frac{\sqrt{3}}{2}i + e^{i\frac{4}{3}\pi}A\int_{-1}^{\zeta} (\zeta'+1)^{-2/3}(-\zeta')^{-2/3}(1-\zeta')^{-2/3}d\zeta'$$

where $A = Ce^{-i\frac{4}{3}\pi}$. In order that $\zeta = 0$ corresponds to $z = -\frac{1}{2}$, we have the constraint

$$-\frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{i\frac{4}{3}\pi}A\int_{-1}^{0} (1+\zeta')^{-2/3}|\zeta'|^{-2/3}(1-\zeta')^{-2/3}d\zeta',$$

implying A is real and positive with

$$A = \left[\int_{-1}^{0} (1+\zeta')^{-2/3}|\zeta'|^{-2/3}(1-\zeta')^{-2/3}d\zeta'\right]^{-1} \qquad (1)$$

Now, for $\zeta \in (0,1)$, we have with above restriction and noting $e^{\frac{4}{3}i\pi} = e^{-\frac{2}{3}i\pi}$,

$$z = f(\zeta) = -\frac{1}{2} + A\int_{0}^{\zeta} (1+\zeta')^{-2/3}(\zeta')^{-2/3}(1-\zeta')^{-2/3}d\zeta'$$

We note that the constraint relation on A ascertains $\zeta = 1$ automatically gives

$$z = f(1) = -\frac{1}{2} + A\int_{0}^{1} (1+\zeta')^{-2/3}(1+\zeta')^{-2/3}\zeta'^{-2/3}d\zeta' = \frac{1}{2}$$

Therefore, with A determined by (1),

$$z = -\frac{1}{2} + Ae^{\frac{2}{3}i\pi}\int_{0}^{\zeta} (\zeta'+1)^{-2/3}(\zeta')^{-2/3}(\zeta'-1)^{-2/3}d\zeta'$$

3. Solve the potential problem for $\phi(x,y)$

$$\Delta\phi = 0 \ \text{ in } \ y \ > \ 0$$

with

$$\phi(x,0) = 1 \ \text{ for } x > 1, \phi(x,0) = -1 \ \text{ for } x < -1 \ \text{ and } \ \frac{\partial\phi}{\partial y}(x,0) = 0 \ \text{ for} |x| < 1$$

State and prove conditions that will make the solution you find unique. **Hint:** Think of mapping to a semi-infinite strip and Schwarz Reflection Principle.
Solution: We want to map the upper-half $z = x + iy$ plane H into the semi-infinite half-strip

$$S := \{\zeta : \Im\zeta > 0 : \Re\zeta \in (-1, 1)\}$$

We already know from class that the map from H to the rectangular domain

$$S_c := \{\zeta : c > \Im\zeta > 0 : \Re\zeta \in (-1, 1)\}$$

with $z = \pm 1$ corresponding to $\zeta = \pm 1$ and $z = \pm 1 + ic$ corresponding to $\zeta = \pm \frac{1}{k}$ is given from Scwartz reflection principle to[1]

$$\zeta = A \int_0^z \frac{dz'}{\sqrt{1 - z'^2}\sqrt{1 - k^2 z'^2}}$$

and that in the limit $k \to 0^+$, $c \to \infty$ corresponding to the semi-infinite strip S. Therefore, for a strip. $\zeta = A \sin^{-1} z$. Or $z = \sin \frac{z}{A}$. In order for $z = \pm 1$ to correspond to $\zeta = \pm 1$, $A = \frac{2}{\pi}$, or $z = g(\zeta) = \sin \frac{\pi\zeta}{2}$ maps S into H. We can check in particular that real axis segement $[-1, 1]$ is mapped to real axis segment $[-1, 1]$ This map is conformal (*i.e.* preserves angles) except at $\zeta = \pm 1$ where $g' = 0$. Therefore, the boundary condition

$$\partial_y \phi(x, 0) = 0 \quad \text{for} \quad x \in (-1, 1)$$

becomes in the $\zeta = \xi + i\eta$ plane the following equation

$$\partial_\eta \Phi(\xi, 0) = 0 \quad \text{for} \quad \xi \in (-1, 1) \tag{1}$$

where $\Phi(\xi, \eta) = \phi(x(\xi, \eta), y(\xi, \eta))$ where $x(\xi, \eta)$ and $y(\xi, \eta)$ is defined by $x + iy = g(\xi + i\eta)$. Translating the conditions on ϕ on different parts of ∂H, the other boundary conditions on $\Phi(\xi, \eta)$ in S are

$$\Phi(-1, \eta) = -1 \quad , \quad \Phi(1, \eta) = 1 \quad \text{for } \eta > 0 \tag{2}$$

We note that $\Phi(\xi, \eta) = \xi$ solves $\Delta_{\xi,\eta}\Phi = 0$ as well as satisfy boundary conditions (1), (2). This implies

$$\phi(x, y) = \xi(x, y) = \Re g^{-1}(x + iy) = \frac{2}{\pi} \Re \left\{ \sin^{-1}(x + iy) \right\} \tag{3}$$

To explore whether or not this is the only solution take the difference between two solutions $\tilde{\Phi}$ in S. Then, define analytic function W so that

$$W(\xi + i\eta) = \tilde{\Phi}(\xi, \eta) + i\tilde{\Psi}(\xi, \eta)$$

Therefore, if we assume that the solution we are seeking has continuous first derivatives as $\eta \to 0^+$ for $\xi \in (-1, 1)$, then from Cauchy Riemann conditions for $\xi \in (-1, 1)$

$$\partial_\eta \Phi(\xi, 0) = \partial_\xi \Psi(\xi, 0)$$

[1]Variables z and ζ are switched here from class notes

implying

$$\tilde{\Psi}(\xi, 0) = 0 \text{ without loss of generality}$$

From Schwartz reflection principle $W(\zeta)$ extends to an analytic function to the entire-strip

$$\mathcal{S} := \{\zeta : -1 < \Re\zeta < 1\}$$

Further, since the solutions are assumed continuous upto the boundary, then repeated reflection using Schwartz reflection principle gives W to be an entire function of ζ, and therefore using Phragmen-Lindeloff principle on e^W and e^{-W}, it follows that if we assume $|\Phi(\xi, \eta)| \leq c_1 + c_2|\xi^2 + \eta^2|^{\alpha/2}$, *i.e.* $\alpha \in [0, 1)$, then the difference of two solutions $\tilde{\Phi}(\xi, \eta) = 0$, and we have uniqueness.

4. Complete the proof of Lemma 11.1 in Week 5 notes, page 1.

Lemma: The most general analytic function f that maps the domain bounded by a circle/straight line to another bounded by a circle/straight line in a 1-1 manner is a fractional linear map.

Proof: Since the composition and inversions of fractional linear maps also belong to the same class, it it is enough to show that this is true when one of the domain (say the ζ-plane) is a unit circle around the origin. Take a general circular domain $D := \{z : |z - z_0| = a\}$. Clearly the explicit map $\zeta = S_0(z) = \frac{1}{a}(z - z_0)$ maps D into the unit circle D_1 centered at the origin. Let $g : D_1 \to D$ be any 1-1 analytic map. Then, it is clear that $S_0 \circ g : D_1 \to D_1$ is an analytic homeomorphism. We know from class that the most general map from 1-1 analytic map from unit circle to a unit circle is a fractional linear map S. Hence $S_0 \circ g = S$, implying $g = S_0^{-1} \circ S$, which is a fractional linear map. For a straight line, we apply first a constant rotation $e^{i\theta_0}$ and translation a so that $\left\{e^{i\theta_0} - a\right\} D$ is the the upper-half plane. So, we may as well assume D to be the upper-half plane itself. We readily check that for

$$S_0(z) = \frac{z - i}{z + i}$$

$S_0 : D \to D_1$. With this variation, the rest of the proof is identical.

5. a. Find a Mobius transformation that maps the unit disk centered at 0 to the upper-half plane. **b.** Find a 1-1 analytic mapping of the interior of $|z - 1 - 2i| = 2$ to the interior of the unit disk centered at $i/2$.

Solution: Consider $\zeta = f(z) = -i\frac{z+1}{z-1}$. It is readily checked that $f(e^{i\theta}) = -\cot\frac{\theta}{2}$ maps the unit circle to the real line with $z = \pm 1$ corresponding to $\zeta = \infty$ and $\zeta = 0$ respectively. This is a fractional linear map and therefore 1-1 on inspection. Further, we may check that $f(0) = i$ confirming that $f : D_1 \to H$, where H is the upper-half plane.

b. Find a 1-1 analytic mapping of the interior of $|z - 1 - 2i| = 2$ to the interior of the unit disk centered at $i/2$.

Solution Note $S_0(z) = \frac{1}{2}(z - 1 - 2i)$ maps D, which is the interior of the circle $|z - 1 - 2i|$ of radius 2 into the D_1, the unit circle centered at the origin. Further note that $S_1(w) = w - \frac{i}{2}$ also maps the unit circle centered at $i/2$ into D_1. Therefore, it is clear that

$$f \equiv S_1^{-1} \circ S_0 : \mathcal{D} \to D_1$$

in a 1-1 analytic manner. So

$$f(z) = \frac{i}{2} + \frac{1}{2}(z - 1 - 2i)$$