Solution to Homework Set 2, Math 804

1. Let f have simple poles at $z = z_n$, $n = 1, 2, ..N$, with corresponding residues a_n, and analytic everywhere else. Show that

a.

$$\frac{1}{2\pi i}\oint_{C_N}\frac{f(z')}{z'-z} = f(z) + \sum_{n=1}^{N}\frac{a_n}{z_n - z},$$

where C_N is a large circle of radius R_N enclosing all the poles.

Solution: Since R_N is a large circle, it can be chosen to include all poles of the integrand $\frac{f(z')}{z'-z}$ in the complex z'-plane. Residue at $z' = z \neq z_n$ is clearly $f(z)$. Residue at $z = z_n$ is

$$\lim_{z'\to z_n}(z'-z_n)\frac{f(z')}{z'-z} = \frac{1}{z_n - z}\lim_{z'\to z_n}(z'-z_n)f(z') = \frac{a_n}{z_n - z}$$

Therefore, adding up all the residues, we get desired results.

b. Use result in **a.**, and assuming $z_n \neq 0$, show that

$$\frac{1}{2\pi i}\oint_{C_N}\frac{zf(z')}{z'(z'-z)} = f(z) - f(0) + \sum_{n=1}^{N} a_n\left(\frac{1}{z_n - z} - \frac{1}{z_n}\right) \tag{1}$$

Solution: If we put $z = 0$ in the solution in part **a.**,

$$\frac{1}{2\pi i}\oint_{C_N}\frac{f(z')}{z'} = f(0) + \sum_{n=1}^{N}\frac{a_n}{z_n}$$

Subtracting this from the result in part **a.**, we obtain

$$\frac{1}{2\pi i}\oint_{C_N}\frac{zf(z')}{z'(z'-z)} = f(z) - f(0) + \sum_{n=1}^{N}\left(\frac{a_n}{z_n - z} - \frac{a_n}{z_n}\right)$$

c. Assume f has simple poles at $\{z_n\}_{n=1}^{\infty}$ with corresponding residues $\{a_n\}_{n=1}^{\infty}$, with $z_n \neq 0$. Assume f is uniformly bounded on a sequence of contours $\{C_N\}_{n=N_0}^{\infty}$, with $R_N \to \infty$, and that the sum on the right of (1) converges for fixed z as $N \to \infty$. Then show that

$$f(z) = f(0) + \sum_{n=1}^{\infty} a_n\left(\frac{1}{z - z_n} + \frac{1}{z_n}\right)$$

Solution: Take a given z. We take a contour C_N so to contain z and m_N poles of f inside C_N. From the given conditions as $N \to \infty$, $m_N \to \infty$. So, using result from part **b.**,

$$\frac{1}{2\pi i}\oint_{C_N}\frac{zf(z')}{z'(z'-z)} = f(z) - f(0) + \sum_{n=1}^{m_N}\left(\frac{a_n}{z_n - z} - \frac{a_n}{z_n}\right)$$

On the otherhand, estimates on the circle give

$$|\frac{1}{2\pi i}\oint_{C_N}\frac{zf(z')}{z'(z'-z)}| \leq \frac{M|z|}{2\pi(R_N-|z|)} \to 0 \quad \text{as } N\to\infty$$

Therefore, result follows.

d. Use (c) for $f(z) = \pi\cot\pi z - \frac{1}{z}$ to show

$$\pi\cot\pi z = \frac{1}{z} + \sum\nolimits_{n=-\infty}^{\prime\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right),$$

where $\sum'$ means that $n=0$ term is missing in the summation.

Solution: We easily check using elementary means that $\lim_{z\to 0} f(z) = 0$; hence $f(0) = 0$. Take $z_{2m+1} = m$ for integer $m\geq 0$ and $z_{2m+1} = -m$ for $m\geq 1$. We note residue at z_{2m+1} is

$$\lim_{z\to m}(z-m)\left(\pi\cot\pi z - z^{-1}\right) = 1 \equiv a_{2m+1}$$

Residue at $z_{2m} = -m$ is given by

$$\lim_{z\to -m}(z+m)\left(\pi\cot\pi z - z^{-1}\right) = 1 \equiv a_{2m}$$

Further, since $\sin\pi z$ only has zeros for $z\in\mathbf{Z}$, by choosing contours of radius $R_N = (N+1/2)$, we are bounded away from poles of $f(z)$. Further,

$$|\coth\pi z| = |\frac{e^{2i\pi x}e^{-2\pi y}+1}{e^{2i\pi x}e^{-2\pi y}-1}|$$

So, on C_N where $|Im\ y|\geq 1/4$,

$$|\coth\pi z| \leq \frac{e^{2\pi|y|}+1}{e^{2\pi|y|}-1} \leq M$$

Hence f is bounded independent of N. And on the part of C_N where $|y|\leq\frac{1}{4}$, since $|x| = \sqrt{(N+1/2)^2-|y|^2}$, $\frac{1}{4}\leq|x|-N\leq 1/2$ for N large enough. Since

$$|e^{i2\pi x}e^{-2\pi y}-1| = \sqrt{(1-e^{-2\pi y})^2+2e^{-2\pi y}(1-\cos 2\pi x)} \geq 2|\sin\pi x| \geq \sqrt{2}$$

It follows that for $|y|\leq\frac{1}{4}$, f is also bounded independent of N. So, using result from part **c.**, we obtain

$$\pi\cot\pi z - \frac{1}{z} = \sum_{n=1}^{\infty} a_n\left(\frac{1}{z-z_n}+\frac{1}{z_n}\right)$$

$$= \sum_{m=0}^{\infty} a_{2m+1}\left(\frac{1}{z-z_{2m+1}}+\frac{1}{z_{2m+1}}\right) + \sum_{m=1}^{\infty} a_{2m}\left(\frac{1}{z-z_{2m}}+\frac{1}{z_{2m}}\right)$$

$$= \sum_{m=0}^{\infty}(-1)^m\left(\frac{1}{z-m}+\frac{1}{m}\right) + \sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{z+m}-\frac{1}{m}\right),$$

which gives the desired result if we replace m by $-m$ in the second sum.

2. Consider the analytic function $\Gamma(z)$ defined in $Re\ z\ >\ 0$ through

$$\Gamma(z) = \int_0^\infty t^{z-1}\ e^{-t}\ dt$$

(This is analytic because $P(t,z) =\ t^{z-1}e^t$ is analytic in z and $\mathcal{L}_1$ in t; hence $\int P(t,z)dt$ is analytic from a theorem in complex analysis). Show that

$$\Gamma(z) = \frac{1}{2i\sin\ (\pi z)}\ \int_C t^{z-1}e^t dt,$$

with C given in Fig. 1, provides for the analytic continuation to $Re\ z\ \leq\ 0$, and that $\Gamma(z)$ has only simple pole singularities when z is a non-positive integer.

Solution: In order to show that latter expression for $\Gamma(z)$ is an analytic continuation of the original formula, it suffices to show that they are identical when $\Re z\ >\ 0$. For $\Re z\ >\ 0$, we notice that on the top straight line segment L_1 of contour C, $t = re^{i\pi}$, while at the bottom straight line segment L_2, $t = re^{-i\pi}$, with $r \geq \epsilon$ So,

$$\int_{L_1} + \int_{L_2} = \int_\epsilon^\infty e^{-r} r^{z-1} e^{i\pi(z-1)} e^{i\pi} dr + \int_\infty^\epsilon e^{-r} r^{z-1} e^{-i\pi(z-1)} e^{-i\pi} dr$$

$$= 2i \sin \pi z \int_\epsilon^\infty e^{-r} r^{z-1} dr$$

On C_ϵ circling the origin $t = \epsilon e^{i\theta}$, with $\theta \in (-\pi, \pi)$; so for $Re z > 0$,

$$|t^{z-1}dt| = |\exp\left\{[\ln\epsilon + i\theta][\Re z + i\Im z]\right\} d\theta| \leq \epsilon^{\Re z} \exp[-\theta\Im z] \leq e^{\pi|\Im z|}\epsilon^{\Re z}$$

$$|\int_{C_\epsilon} t^{z-1} e^t dt| \leq \epsilon^{\Re z} e^{\pi|\Im z|} \int_{-\pi}^{\pi} e^{\epsilon\cos\theta} d\theta \leq \epsilon^{\Re z} e^{\pi|\Im z|} e^\epsilon 2\pi \to 0 \quad \text{as } \epsilon \to 0$$

So, we obtain for $\Re z > 0$, by taking limit $\epsilon \to 0$ in the above, we obtain

$$\frac{1}{2i\sin\pi z} \int_C e^t t^{z-1} dt = \int_0^\infty e^{-r} r^{z-1} dr$$

Thus, the two expressions agree for $\Re z > 0$. Since each defines an analytic function. (See Rudin's real-complex analysis proof that $\int P(t,z)dt$ is analytic in z when $P(t,z)$ is L^1 in t and analytic in z in some region, as is true for $e^t t^{z-1}$). When $z = -n$, with integer $n \geq 0$, the integral

$$\int_C e^t t^{z-1} dt = \oint_C \frac{e^t}{t^{n+1}} dt = 2\pi i n!$$

on collecting residue at $t = 0$. So, as $z \to -n$,

$$[2i\sin\pi z]\ \Gamma(z) \to 2\pi i n!$$

Therefore, $z = -n$ is a pole, since $\sin \pi z$ has a zero at that point. On the other hand if $z = n$, is no singularity as is clear from the representation

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$

even though the analytically continued formula may suggest that there is (Actually zero in the denominator $\sin \pi z$ is cancelled by a simple zero of the numerator). Thus, the only poles of $\Gamma(z)$ is where $z = -n$ for integer $n \geq 0$.

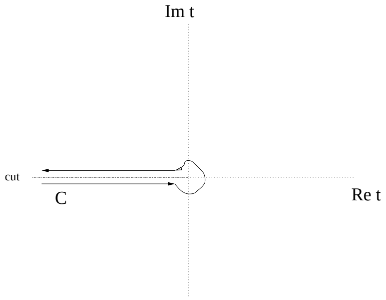

Figure 1: Integration path C in t-plane in defining $\Gamma(z)$.

3. Specify cuts and branches for ln so that

$$\int_1^z \frac{1}{z'} dz' = \ln\ z$$

for each of the three integration contours (a)-(c) in Figures 2-4:

Solution: The main point here is that branch cut chosen for $\ln z$ so that it is consistent with choice of integration paths in z' plane. $\int_1^z \frac{1}{z'} dz'$ clearly continuous in z for all paths in z'-plane chosen to avoid the origin from the top by circling around, if necessary, along the anti-clockwise manner. Also, it is continous again for all paths chosen to avoid the origin in the z'-plane by going below it.

Therefore, in Fig. 2, a consistent choice of cut is $\arg z \in [0, 2\pi)$. We choose the branch so that $\ln z = \ln |z| + i \arg z$, with $\arg z$ restricted as above.

From similar consideration, for Fig. 3, $\ln z = \ln |z| + i \arg z$, with $\arg z \in (-\pi, \pi]$.

From similar consideration, for Fig. 4, $\ln z = \ln |z| + i \arg z$, with $\arg z \in \left[-\pi/2, \frac{3}{2}\pi\right)$.

4. Recall $\sin\ z = \frac{e^{iz} - e^{-iz}}{2i}$. Using this, determine $\sin^{-1}\ z$ in terms of log and specify different choice of branch cuts and branches. On the Riemann surface, describe how

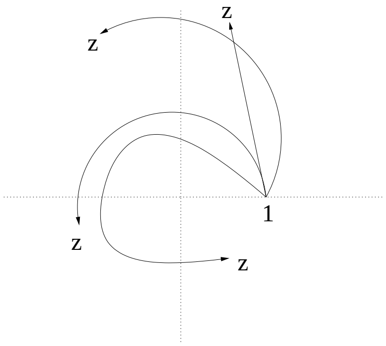

Figure 2: Integration paths joining 1 to z in different quadrants for case (a)

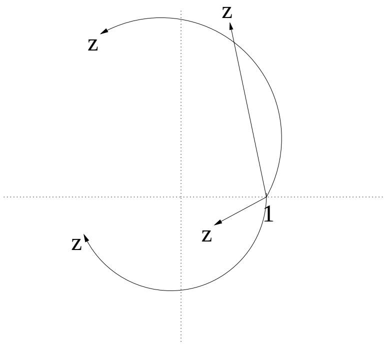

Figure 3: Integration paths joining 1 to z in different quadrants for case (b)

branches are connected to each other as we cross branch cuts. Determine cuts and branch so that $\sin^{-1}(1) = \frac{\pi}{2}$.

Solution: Take $e^{iz} = \zeta$. Then $\zeta - \frac{1}{\zeta} = 2i \sin z \equiv 2iw$. So, $\zeta^2 - 2iw\zeta - 1 = 0$. Therefore, solving for ζ,

$$\zeta = i\left[w + (w^2-1)^{1/2}\right]$$

Hence

$$z = -i \ln \zeta = \frac{\pi}{2} - i \ln\left[w + (w^2-1)^{1/2}\right] = \sin^{-1} w$$

For the function $\zeta = w + (w^2-1)^{1/2}$, one choice of cut is to join $[-1,1]$ in the w plane. This is accomplished by restricting each of $\arg(w \pm 1) \in (-\pi, \pi]$. Then, we can define two branches

$$\zeta_1(w) = w + |w^2-1|^{1/2} \exp\left[\frac{i}{2}\arg(w-1) + \frac{i}{2}\arg(w+1)\right]$$

$$\zeta_2(w) = w - |w^2-1|^{1/2} \exp\left[\frac{i}{2}\arg(w-1) + \frac{i}{2}\arg(w+1)\right]$$

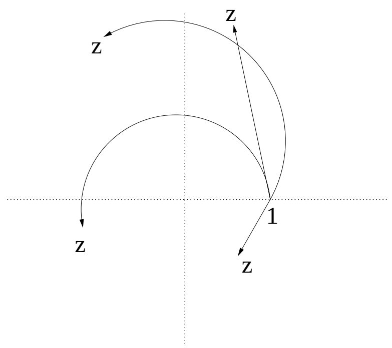

Figure 4: Integration paths joining 1 to z in different quadrants for case (c)

Now, we want to determine the branch cut for the composite function $-i\ln\zeta(w)$. We choose in the ζ plane branch cut for $\ln\zeta$ along negative ζ axis. The most general formula for $\ln\zeta$ on a Riemann sheet corresponding to this cut is therefore given by

$$\ln_m \zeta = \ln|\zeta| + i\arg\zeta + 2im\pi \quad \text{with } \arg z \in (-\pi, \pi]$$

To find the corresponding cut location in the w-plane, we have to look at the preimage of $[-\infty, 0)$ in the w plane under the mappings $\zeta = \zeta_1(w)$ and $\zeta = \zeta_2(w)$. We note that if $\zeta = -r = \zeta_{1,2}(w)$, then $-(r+w) = \sqrt{w^2-1}$, or $w^2 + 2wr + r^2 = w^2 - 1$ Therefore, $w = -\frac{1+r^2}{2r}$. For $r \in [1, \infty)$, *i.e.* $\zeta \in (-\infty, -1]$, we have corresponding $w \in (-\infty, -1]$ under the mapping $\zeta = \zeta_1(w)$. On the otherhand, $r \in (0, 1]$, *i.e.* $\zeta \in [-1, 0)$, is seen to correspond to $w \in (-\infty, -1]$ for the mapping $\zeta = \zeta_2(w)$.

Therefore, the cuts in the w-plane for the function

$$\sin^{-1} w = \frac{\pi}{2} - i\ln\left[w + (w^2-1)^{1/2}\right]$$

are as shown in the figure. So, all different Riemann sheets is desribed by

$$\frac{\pi}{2} - i\ln_m \zeta_1(w), \quad \text{or } \frac{\pi}{2} - i\ln_m \zeta_2(w)$$

If we choose $m = 0$, then on either branch ζ_1 or ζ_2, we get $\sin^{-1} 1 = \frac{\pi}{2}$. To see how the sheets are connected on the Riemann surface, note that if we are originally on a sheet where

$$\sin^{-1} w = \frac{\pi}{2} - i\ln_m \zeta_1(w), \tag{1}$$

then crossing the cut $[-1, 1]$ from the top will result in going over to the sheet where

$$\sin^{-1} w = \frac{\pi}{2} - i\ln_m \zeta_2(w)$$

On the other hand if we were to start with the sheet where (1) is valid and cross $(-\infty, -1]$ from the top in the w plane, then we would access the Riemann sheet where

$$\sin^{-1} w = \frac{\pi}{2} - i \ln_{m+1} \zeta_1(w)$$

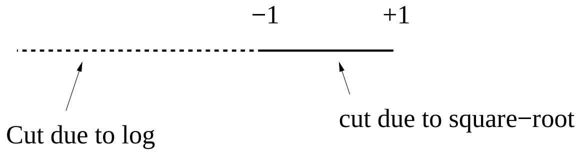

Figure 5: Cuts for $\sin^{-1} w$ in the w-plane.