

Homework 3 Solution, Math 804

1. Determine $\oint_{|z|=2} \frac{z}{\sin^3 z} dz$.

Solution: The singularities of $f(z) = \frac{z}{\sin^3 z}$ are only at $z = 0$ since other zeros of $\sin z$ are outside $|z| = 2$. We determine the residue through a local expansion:

$$f(z) = \frac{z}{(z - z^3/3! + O(z^5))^3} = \frac{1}{z^2} [1 - z^2/6 + O(z^4)]^{-3} = \frac{1}{z^2} [1 + 3z^2/6 + O(z^4)]$$

So residue of f at $z = 0$ is zero. Hence $\oint_{|z|=2} \frac{z}{\sin^3 z} dz = 0$.

2. Use contour integration to evaluate $I = \int_0^\infty \frac{\ln x}{1+x^2} dx$.

Solution: (Note: the method shown here is applicable for any Polynomial of order bigger than 1 in the denominator, not just $1 + x^2$, for which easier methods exist) Consider $\oint_C \frac{\ln^2 z}{1+z^2} dz$ for C shown in fig-

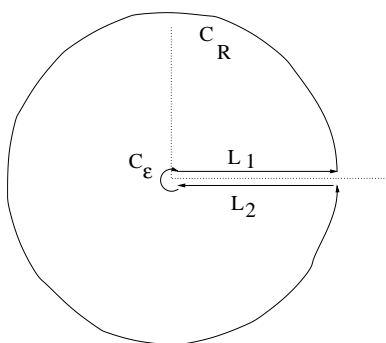


Figure 1: Integration path C for Problem 2.

ure 1, with $\ln z = \ln |z| + i \arg z$, $\arg z \in [0, 2\pi)$. Residue at $z = i$ is simply $\frac{1}{2i} \left(\frac{i\pi}{2}\right)^2 = \frac{i\pi^2}{8}$. At $z = -i$ residue is $\frac{1}{-2i} \left(\frac{3i\pi}{2}\right)^2 = \frac{-9i\pi^2}{8}$. So, for $R > 1$,

$$\oint_C = 2\pi i \sum [\text{Residues}] = \pi^3 \left(-\frac{1}{4} + \frac{9}{4} \right) = 2\pi^3$$

On L_1 , $z = r$ with $\epsilon < r < \infty$ On L_2 , $z = re^{i2\pi}$. So,

$$\begin{aligned} \int_{L_1} + \int_{L_2} \frac{\ln^2 z}{1+z^2} dz &= \int_\epsilon^\infty \frac{\ln^2 r}{1+r^2} dr + \int_\infty^\epsilon \frac{\ln^2 r + 2i\pi \ln r - 4\pi^2}{1+r^2} dr \\ &= -2i\pi \int_\epsilon^\infty \frac{\ln r}{1+r^2} + 4\pi^2 \int_\epsilon^\infty \frac{1}{1+r^2} \end{aligned}$$

Now, on C_ϵ , $z = \epsilon e^{i\theta}$, $\theta \in [0, 2\pi)$. Therefore,

$$\left| \int_{C_\epsilon} \right| \leq \int_0^{2\pi} \frac{|\ln \epsilon + i\theta|^2}{1-\epsilon^2} \epsilon d\theta \leq \frac{\epsilon |\ln \epsilon + i\pi|^2}{1-\epsilon^2} 2\pi \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \quad (1)$$

On C_R , we have

$$\left| \int_{C_R} \right| \leq \int_0^{2\pi} \frac{|\ln R + i\theta|^2}{R^2 - 1} R d\theta \leq \frac{R |\ln R + i\pi|^2}{R^2 - 1} 2\pi \rightarrow 0 \text{ as } R \rightarrow \infty \quad (2)$$

Therefore, on taking $\epsilon \rightarrow 0$, $R \rightarrow \infty$, we obtain in the limit

$$2\pi^3 = \oint_C = \int_0^\infty = -2i\pi I + 4\pi^2 \int_0^\infty \frac{1}{1+r^2} = -2i\pi I + 2\pi^3$$

So, $I = 0$.

3. Use contour integration to determine the Taylor series coefficient about $z = 0$ of $J(z)$ defined by

$$J(z) = \int_0^\pi e^{z \cos \theta} d\theta$$

Solution: Note that

$$J(z) = \frac{1}{2} \int_{-\pi}^\pi e^{z \cos \theta} d\theta = \frac{1}{2i} \oint_{|t|=1} \exp \left\{ \frac{z}{2} \left(t + \frac{1}{t} \right) \right\} \frac{dt}{t}$$

Residue of

$$f(t) = \frac{1}{t} \exp \left\{ \frac{z}{2} \left(t + \frac{1}{t} \right) \right\}$$

is the coefficient of the constant term in the Laurent series in t of $e^{\frac{z}{2}t} e^{\frac{z}{2t}}$ at the origin. Using series for exponential, it is easily seen that the coefficient of the constant term is

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n} n!^2}$$

So,

$$J(z) = \pi \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n} n!^2}$$

4. For $a > b > 0$, use contour integration to evaluate

$$I(b) := \int_0^{2\pi} \ln [a + b \cos \theta] d\theta$$

One way of evaluating this is simply to note that

$$\begin{aligned} I'(b) &= \int_0^{2\pi} \frac{\cos \theta}{a + b \cos \theta} d\theta = \frac{1}{b} \int_0^{2\pi} \left\{ 1 - \frac{a}{a + b \cos \theta} \right\} d\theta \\ &= \frac{2\pi}{b} - \frac{1}{b} \int_0^{2\pi} \frac{1}{1 + \epsilon \cos \theta} d\theta, \text{ where } \epsilon = \frac{b}{a} < 1 \end{aligned}$$

From example in class notes we know therefore that

$$I'(b) = \frac{2\pi}{b} - \frac{2\pi}{b\sqrt{1-\epsilon^2}} = \frac{2\pi}{b} \left\{ 1 - \frac{a}{\sqrt{a^2-b^2}} \right\}$$

Integrating with respect to b , and using $I(0) = 2\pi \ln a$, it follows that

$$I(b) = 2\pi \ln \frac{1}{2} \left[a + \sqrt{a^2 - b^2} \right]$$

Alternately:

$$\int_0^{2\pi} \ln [a + b \cos \theta] d\theta = 2\pi \ln a + \int_0^{2\pi} \ln [1 + \epsilon \cos \theta] d\theta,$$

for $\epsilon = \frac{b}{a}$. Also, note that for $1 > \delta > 0$, for principal branch of the \ln function,

$$\Re \ln(1 + \delta e^{i\theta}) = \ln |1 + \delta e^{i\theta}| = \ln \sqrt{1 + \delta^2 + 2\delta \cos \theta} = \frac{1}{2} \ln [1 + \delta^2] + \frac{1}{2} \ln \left[1 + \frac{2\delta}{1 + \delta^2} \cos \theta \right]$$

Associating $\epsilon = \frac{2\delta}{1 + \delta^2}$, we obtain

$$\int_0^{2\pi} \ln [1 + \epsilon \cos \theta] d\theta = -\pi \ln [1 + \delta^2] + \Re \left\{ \oint_{|\zeta|=1} \ln [1 + \delta\zeta] \frac{d\zeta}{i\zeta} \right\}$$

The latter integral is zero since $\frac{1}{\zeta} \ln [1 + \delta\zeta]$ is analytic on and inside $|\zeta| = 1$, with removable singularity at $\zeta = 0$. Therefore,

$$\int_0^{2\pi} \ln [a + b \cos \theta] d\theta = 2\pi \ln a - \pi \ln [1 + \delta^2]$$

where δ is the root of the quadratic $(1 + \delta^2)\frac{b}{a} - 2\delta = 0$ inside the unit circle, which is $\delta = \frac{a - \sqrt{a^2 - b^2}}{b}$ we get the desired result above.

5. For $-\pi < \alpha < \pi$, evaluate $I = \int_0^\infty \frac{\sin \alpha x}{\sinh \pi x} dx$.

Solution: We note that

$$I = \Im \int_{-\infty}^\infty \frac{e^{i\alpha x}}{e^{\pi x} - e^{-\pi x}} dx \equiv \Im I_1$$

Consider

$$\oint_C \frac{e^{i\alpha z}}{e^{\pi z} - e^{-\pi z}} dz$$

with C as shown in Figure below.

We note that the closed path contains no singularities of the integrand; hence

$$0 = \oint_C = \int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4} + \int_{L_5} + \int_{L_6} + \int_{C_{0,\epsilon}} + \int_{C_{i,\epsilon}}$$

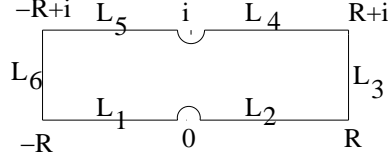


Figure 2: Integration path C for Problem 4.

where $C_{0,\epsilon}$ and $C_{i,\epsilon}$ are semi-circular arcs centered at 0 and i respectively. Clearly, we want

$$I_1 = \lim_{\epsilon \rightarrow 0, R \rightarrow \infty} \left\{ \int_{L_1} + \int_{L_2} \right\}$$

However, we note that on L_4 and L_5 , $z = x + i$ and

$$f(x + i) = \frac{e^{-\alpha} e^{i\alpha x}}{-e^{\pi x} + e^{-\pi x}} = -e^{-\alpha} f(x)$$

So,

$$\int_{L_4} + \int_{L_5} = e^{-\alpha} \left\{ \int_{L_1} + \int_{L_2} \right\}$$

On L_3 ,

$$\begin{aligned} \left| \int_{L_3} \right| &= \left| \int_0^1 \frac{e^{i\alpha R} e^{-\alpha y}}{e^{\pi R} e^{i\pi y} - e^{-\pi R} e^{-i\pi y}} i dy \right| \\ &\leq \frac{1}{e^{\pi R} - 1} \int_0^1 e^{-\alpha y} dy \rightarrow 0 \text{ as } R \rightarrow \infty \end{aligned}$$

Almost identical arguments show $\int_{L_6} \rightarrow 0$, as $R \rightarrow 0$. From $C_{0,\epsilon}$, since we go clockwise, we collect $-\pi i$ (residue of integrand at 0) in the limit $\epsilon \rightarrow 0$; this is easily checked to be $-\pi i \left(\frac{1}{2\pi} \right) = -\frac{i}{2}$. Since $C_{i,\epsilon}$ is a semi-circular contour which is traversed clockwise, we get in the limit $\epsilon \rightarrow 0$,

$$\int_{C_{i,\epsilon}} = \pi i (\text{residue of integrand at } i) = -\pi i \left(\frac{e^{-\alpha}}{-2\pi} \right) = \frac{i}{2} e^{-\alpha}$$

Therefore, combining all the relations, in the limit $\epsilon \rightarrow 0$, $R \rightarrow \infty$, we obtain

$$0 = I_1 (1 + e^{-\alpha}) - \frac{i}{2} (1 - e^{-\alpha})$$

Therefore,

$$I = \Im I_1 = \frac{1}{2} \left(\frac{1 - e^{-\alpha}}{1 + e^{-\alpha}} \right)$$