**Homework 4 Solution, Math 804**

**1.** Use Schwartz Reflection principle to show that an analytic function $f$ in a domain $D$ can be analytically continued across part of its boundary $\partial D$ where $\Im\ f$ (or $Re\ f$) is a constant provided the the boundary segment is analytic. (Recall an analytic curve is the graph of some analytic function $\Gamma(t)$ for $t$ real, $a \le t \le b$).

**Solution** First, there is no loss of generality taking $\Im f = 0$ on part of the boundary $\partial D$ in question. If $\Im f = c \neq 0$, we replace $f$ by $f - ic$ in the argument presented. If instead $\Re f = c$, we replace $f$ by $i(f - c)$ in the argument presented below since $\Im \{i(f - c)\} = 0$.

Further, the argument will be local in a neighborhood of arbitrary point $z_0$ on the boundary segment where $\Im f = 0$. We may represent a local boundary segment at $z_0$ to be $\gamma(t) = t + i\Gamma(t)$ for $t \in (-a, a)$ for $a$ sufficiently small, where $z_0 = \gamma(0)$. (If the boundary tangent at $z_0$ is vertical, repeat the same argument for $\gamma(t) = \Gamma(t) + it$). Since the boundary is analytic $\Gamma(t)$ is analytic in $(-a, a)$. Consider the image of the ball $\{t : |t| < a\}$ under the mapping $\gamma$. Since $\gamma'(0) = 1 + i\Gamma'(0) \neq 0$, it follows that there exists a ball $\mathcal{B}_\epsilon(z_0) \subset \gamma(\{t : |t| < a\})$ such that the mapping $\gamma$ is an analytic homeomorphism between $\gamma^{-1}\{\mathcal{B}_\epsilon(z_0)\}$ and $\mathcal{B}_\epsilon(z_0)$ as shown in the figure below (Recall the proof in class notes that show that a map $g$ is an 1-1 analytic map in a local neighborhood of a point where $g' \neq 0$). The subset

$$\mathcal{H} := \{\Im\ t > 0\} \cup \gamma^{-1}\{\mathcal{B}_\epsilon(z_0)\},$$

is mapped to $\gamma(\mathcal{H})$, adjacent to $\partial D$, each region shown in shade in Fig. 1. Therefore, $f \circ \gamma$ is a locally analytic function near $t = 0$ that maps $\mathcal{H}$ into a some region $f \circ \gamma(\mathcal{H})$ in the $\zeta$ plane adjacent to the real axis. From Schwarz reflection principle, $f \circ \gamma$ is analytically continuable to the the unshaded region inside $\gamma^{-1}(\mathcal{B}_\epsilon(z_0))$ including parts of the real $t$-axis. Since $\gamma$ is an analytic homeomorphism in this region, it follows that $f$ is analytic on $\partial D \cap \mathcal{B}_\epsilon(z_0)$.

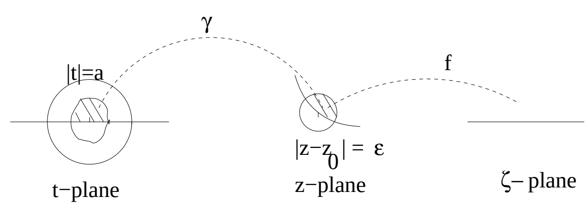


Figure 1: Analytic homeomorphism between shaded regions $\mathcal{H} = \{\Im t > 0\} \cup \gamma^{-1}(\mathcal{B}_\epsilon(z_0))$ and $\gamma(\mathcal{H})$. $f \circ \gamma$ maps shaded region adjacent to the real $t$ to the a region next to the real axis in the $\zeta$-plane.

**2.** Show that the function

$$w(z) = \frac{1}{2}\ (z + \frac{1}{z})$$

maps the exterior of a unit circle centered around the origin in the $z$ plane to the exterior of a straight line cut from -1 to 1 in the $w$ plane. What is the inverse of this mapping. Be specific on the choice of branch if the inverse function involves branches.

**Solution:** Easily checked that

$$\xi + i\eta \equiv w\left(re^{i\theta}\right) = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta + i\frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$$

Therefore, from above we can check circle $|z| = a$ for $a > 1$ is mapped to an ellipse

$$\frac{\xi^2}{c^2} + \frac{\eta^2}{d^2} = 1$$

where

$$c = \frac{1}{2}\left(a + \frac{1}{a}\right) \quad , \quad d = \frac{1}{2}\left(a - \frac{1}{a}\right)$$

with ellipse degenerating to the straight line segment [-1,1] as $a \to 1^+$. Solving for $w$ from the quadratic $2wz = z^2 + 1$, we obtain

$$z = w + \sqrt{w^2 - 1}$$

where

$$\sqrt{w^2-1} = |w-1|^{1/2}|w+1|^{1/2}\exp\left[\frac{\theta_1}{2} + \frac{\theta_2}{2}\right] \ , \ \text{where } \theta_{1,2} \equiv \arg(w \pm 1) \in (-\pi, \pi]$$

We notice that this is the right branch since as $w \to \infty$, $z \sim 2w \to \infty$, as necessary for exterior of [-1, 1] to map to exterior of $|z| = 1$. With the other choice of the branch, the exterior of $[-1, 1]$ would correspond to $|z| < 1$.

**3.** Think of solving $\Delta\phi = 0$ in the strip domain $\{x + iy, 0 < y < 1\}$, with boundary conditions $\phi = 1$ on $y = 1$ and $\phi = 0$ on $y = 0$. What *a priori* assumption do you need on the growth of $\phi(x, y)$ as $x \to \pm\infty$ to assure that the only solution to this problem is $\phi(x, y) = y$ ?

**Solution:** We know one solution is $\phi = y$ as it satisfies the boundary condition and solve $\Delta\phi = 0$. Consider the difference of two solutions $\tilde{\phi}$, which satisfies $\Delta\tilde{\phi} = 0$, with $\tilde{\phi}(x, 1) = 0 = \tilde{\phi}(x, 0)$. The solutions here are assumed to be continuous *a priori* upto the boundary, as would be deemed physically appropriate. Let $\tilde{\psi}$ be its harmonic conjugate, *i.e.* $w = \tilde{\phi} + i\tilde{\psi}$ is an analytic function of $z$ with $\Re w = 0$ on the boundary $\partial S$ of the strip domain $S$. Consider $f(z) = e^{w(z)}$. It satisfies $|f(z)| = 1$ on $\partial D$. Consider the map $\zeta = g(z) = -ie^{\pi z}$, which can be checked to map the strip $S$ into the upper half-plane $H := \{\zeta : \Re\zeta > 0\}$ in a 1-1 manner. It's inverse is the principal branch of

$$g^{-1}(\zeta) = \frac{1}{\pi}\ln[i\zeta]$$

Then, $f \circ g^{-1}$ is an analytic function in $H$, satisfying

$$|f \circ g^{-1}| = 1 \quad \text{on} \quad \partial H$$

From Phragmen-Lindeloff principle, we need to assume that for any $\zeta \in H$

$$|f \circ g^{-1}(\zeta)| \leq c_2 \exp\{c_1 |\zeta|^\alpha\} \quad \text{for} \quad \alpha < 1,$$

in order to apply maximum modulus Theorem which would imply $|f| \leq 1$. Therefore, we need

$$|f(z)| \leq c_2 \exp\left\{c_1 e^{\pi\alpha \Re z}\right\}$$

in order to apply the maximum modulus Theorem. This will give maximum principle on the harmonic function $\tilde{\phi}$ since from definition $|f(x+iy)| = e^{\tilde{\phi}(x,y)}$. To obtain the minimum principle for $\tilde{\phi}$ for this infinite strip, we take $f(z) = e^{-w(z)}$ and repeat the same discussion. Therefore, it is clear to apply Phragmen-Lindeloff to both $e^{w(z)}$ and $e^{-w(z)}$, we need the *apriori* assumption

$$|\tilde{\phi}(x,y)| \leq c_1 e^{\pi\alpha|x|} \quad \text{for some} \quad \alpha \in (0,1)$$

or

$$|\phi(x,y)| \leq c_1 e^{\pi\alpha|x|} \quad \text{for some} \quad \alpha \in (0,1)$$

to apply maximum principle on $\tilde{\phi}$ giving $\tilde{\phi} = 0$ everywhere, and $\phi(x,y) = y$ being the only solution.

**4.** Solve the potential problem for $\phi(x,y)$

$$\Delta\phi = 0 \quad \text{in} \quad y \; > \; 0$$

with

$$\phi(x,0) = 1 \quad \text{for } x > 1, \phi(x,0) = -1 \quad \text{for } x < -1 \quad \text{and} \quad \frac{\partial\phi}{\partial y}(x,0) = 0 \quad \text{for} |x| < 1$$

State and prove conditions that will make the solution you find unique.

**Solution:** We want to map the upper-half $z = x + iy$ plane $H$ into the semi-infinite half-strip

$$S := \{\zeta : \Im\zeta > 0 : \Re\zeta \in (-1,1)\}$$

We already know from class that the map from $H$ to the rectangular domain

$$S_c := \{\zeta : c > \Im\zeta > 0 : \Re\zeta \in (-1,1)\}$$

with $z = \pm 1$ corresponding to $\zeta = \pm 1$ and $z = \pm 1 + ic$ corresponding to $\zeta = \pm\frac{1}{k}$ is given from Scwartz reflection principle to[1]

$$\zeta = A \int_0^z \frac{dz'}{\sqrt{1 - z'^2}\sqrt{1 - k^2 z'^2}}$$

and that in the limit $k \to 0^+$, $c \to \infty$ corresponding to the semi-infinite strip $S$. Therefore, for a strip. $\zeta = A \sin^{-1} z$. Or $z = \sin\frac{z}{A}$. In order for $z = \pm 1$ to correspond to $\zeta = \pm 1$, $A = \frac{2}{\pi}$, or $z = g(\zeta) = \sin\frac{\pi\zeta}{2}$ maps $S$ into $H$. We

[1] Variables $z$ and $\zeta$ are switched here from class notes

can check in particular that real axis segement $[-1,1]$ is mapped to real axis segment $[-1,1]$ This map is conformal (*i.e.* preserves angles) except at $\zeta = \pm 1$ where $g' = 0$. Therefore, the boundary condition

$$\partial_y \phi(x,0) = 0 \ \text{ for } \ x \in (-1,1)$$

becomes in the $\zeta = \xi + i\eta$ plane the following equation

$$\partial_\eta \Phi(\xi,0) = 0 \ \text{ for } \ \xi \in (-1,1) \tag{1}$$

where $\Phi(\xi,\eta) = \phi\left(x(\xi,\eta), y(\xi,\eta)\right)$ where $x(\xi,\eta)$ and $y(\xi,\eta)$ is defined by $x+iy = g(\xi+i\eta)$. Translating the conditions on $\phi$ on different parts of $\partial H$, the other boundary conditions on $\Phi(\xi,\eta)$ in $S$ are

$$\Phi(-1,\eta) = -1 \ \ , \ \ \Phi(1,\eta) = 1 \ \text{ for } \eta > 0 \tag{2}$$

We note that $\Phi(\xi,\eta) = \xi$ solves $\Delta_{\xi,\eta}\Phi = 0$ as well as satisfy boundary conditions (1), (2). This implies

$$\phi(x,y) = \xi(x,y) = \Re g^{-1}(x+iy) = \frac{2}{\pi}\Re\left\{\sin^{-1}(x+iy)\right\} \tag{3}$$

To explore whether or not this is the only solution take the difference between two solutions $\tilde{\Phi}$ in $S$. Then, define analytic function $W$ so that

$$W(\xi+i\eta) = \tilde{\Phi}(\xi,\eta) + i\tilde{\Psi}(\xi,\eta)$$

Therefore, if we assume that the solution we are seeking has continuous first derivatives as $\eta \to 0^+$ for $\xi \in (-1,1)$, then from Cauchy Riemann conditions for $\xi \in (-1,1)$

$$\partial_\eta \Phi(\xi,0) = \partial_\xi \Psi(\xi,0)$$

implying

$$\tilde{\Psi}(\xi,0) = 0 \ \text{ without loss of generality}$$

From Schwartz reflection principle $W(\zeta)$ extends to an analytic function to the entire-strip

$$\mathcal{S} := \{\zeta : -1 < \Re\zeta < 1\}$$

So, from last problem (with trivial 90 degree rotation), it follows that we will get the unique soloution

$$\tilde{\Phi} = 0$$

if we assumed *a priori* that the solution we are looking for, aside from satisfying continuity condition upto the boundary satisfied

$$|\Phi(\xi,\eta)| \le c_1 e^{\frac{\pi}{2}\alpha|\eta|} \ \text{ for some } \ \alpha \in (0,1)$$

where factor 2 in the exponent accounts for the strip width being twice what it is in previous problem. We can clearly translate this condition to

$$|\phi(x,y)| \le c_1 \exp\left\{\frac{\pi}{2}\alpha|\eta(x,y)|\right\} \ \text{ for some } \ \alpha \in (0,1)$$

as an *a priori* condition, which together with conditions on continuity of $\phi$ as boundary is approached gives (3) as the unique solution to the problem.

**5.** Complete the proof of Lemma 11.1 in Week 5 notes, page 1.
**Lemma:** The most general analytic function $f$ that maps the domain bounded by a circle/straight line to another bounded by a circle/straight line in a 1-1 manner is a fractional linear map.
**Proof:** Since the composition and inversions of fractional linear maps also belong to the same class, it it is enough to show that this is true when one of the domain (say the $\zeta$-plane) is a unit circle around the origin. Take a general circular domain $D := \{z : |z - z_0| = a\}$. Clearly the explicit map $\zeta = S_0(z) = \frac{1}{a}(z - z_0)$ maps $D$ into the unit circle $D_1$ centered at the origin. Let $g : D_1 \to D$ be any 1-1 analytic map. Then, it is clear that $S_0 \circ g : D_1 \to D_1$ is an analytic homeomorphism. We know from class that the most general map from 1-1 analytic map from unit circle to a unit circle is a fractional linear map $S$. Hence $S_0 \circ g = S$, implying $g = S_0^{-1} \circ S$, which is a fractional linear map. For a straight line, we apply first a constant rotation $e^{i\theta_0}$ and translation $a$ so that $\left\{e^{i\theta_0} - a\right\} D$ is the the upper-half plane. So, we may as well assume $D$ to be the upper-half plane itself. We readily check that for

$$S_0(z) = \frac{z - i}{z + i}$$

$S_0 : D \to D_1$. With this variation, the rest of the proof is identical.

**6. a.** Find a linear fractional transformation that maps the unit disk centered at 0 to the upper-half plane.
**Solution:** Consider $\zeta = f(z) = -i\frac{z+1}{z-1}$. It is readily checked that $f(e^{i\theta}) = -\cot\frac{\theta}{2}$ maps the unit circle to the real line with $z = \pm 1$ corresponding to $\zeta = \infty$ and $\zeta = 0$ respectively. This is a fractional linear map and therefore 1-1 on inspection. Further, we may check that $f(0) = i$ confirming that $f : D_1 \to H$, where $H$ is the upper-half plane.

**b.** Find a 1-1 analytic mapping of the interior of $|z - 1 - 2i| = 2$ to the interior of the unit disk centered at $i/2$.
**Solution** Note $S_0(z) = \frac{1}{2}(z - 1 - 2i)$ maps $D$, which is the interior of the circle $|z - 1 - 2i|$ of radius 2 into the $D_1$, the unit circle centered at the origin. Further note that $S_1(w) = w - \frac{i}{2}$ also maps the unit circle centered at $i/2$ into $D_1$. Therefore, it is clear that

$$f \equiv S_1^{-1} \circ S_0 : \mathcal{D} \to D_1$$

in a 1-1 analytic manner. So

$$f(z) = \frac{i}{2} + \frac{1}{2}(z - 1 - 2i)$$