Homework 5, Math 804 Tanveer

1. Consider the Fourier Transform of $f(x)$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

For $f(x) = \frac{1}{\sqrt{(x+1)^2+1}}$, determine the asymptotic series as $k \to \pm\infty$ and explicitly calculate the first two terms.

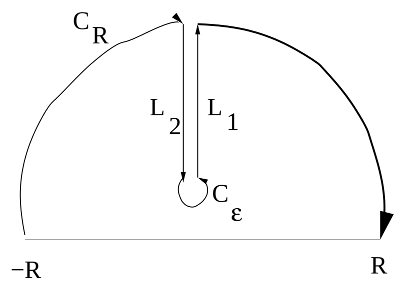

FIGURE 1. Integration path C for Problem 1 in the y-plane.

Solution: It is convenient to replace $y = x + 1$ as the integration variable to obtain

$$F(k) = e^{-ik} G(k) \ , \ \text{where } G(k) = \int_{-\infty}^{\infty} g(y) e^{iky} dy \ , \ g(y) = \frac{1}{\sqrt{1+y^2}}$$

To asymptotically evaluate $G(k)$ For $k > 0$, we deform the contour as shown in Fig. 1 From Jordan's Lemma there will be no contribution from circular arcs of radius R as $R \to \infty$. On L_1, $\arg(y - i) = \frac{\pi}{2}$, $\arg(y + i) = \frac{\pi}{2}$; so if $y = ir$ on L_1, we have $\sqrt{1+y^2} = e^{i\pi/2}\sqrt{r^2-1}$ and on L_2, since $\arg(y - i) = -\frac{3}{2}\pi$, and $\arg(y + i) = \frac{\pi}{2}$, we have $\sqrt{1+y^2} = e^{-i\pi/2}\sqrt{r^2-1}$. So,

$$\left(\int_{L_1} + \int_{L_2} \right) = 2 \int_{1+\epsilon}^{R} \frac{e^{-kr}}{\sqrt{r^2-1}} dr$$

Further on C_ϵ, since $y = i + \epsilon e^{i\theta}$,

$$| \int_{C_\epsilon} | \le \int_{-\frac{3}{2}\pi}^{\pi/2} e^{-k} \left| \exp\left[ik\epsilon e^{i\theta}\right] \right| \frac{1}{\sqrt{\epsilon(2-\epsilon)}} \epsilon d\theta \le C\sqrt{\epsilon}$$

So, taking limit $\epsilon \to 0$, $R \to \infty$, we obtain

$$G(k) = 2 \int_1^{\infty} \frac{e^{-kr}}{\sqrt{r^2-1}} dr = 2e^{-k} \int_0^{\infty} \frac{e^{-kp}}{\sqrt{p(2+p)}} dp$$

Since

$$\frac{2}{\sqrt{p(2+p)}} = \sqrt{2}p^{-1/2}(1+p/2)^{-1/2} = \sqrt{2}p^{-1/2}(1-p/4+O(p^2))$$

From Watson's Lemma, as $k \to +\infty$

$$e^k G(k) \sim \sqrt{2}\left[\Gamma(1/2)k^{-1/2} - \frac{\Gamma(3/2)}{4}k^{-3/2} \ + \ O(k^{-5/2})\right]$$

Therefore, using $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$ we have as $k \to +\infty$

$$F(k) \sim e^{-k-ik}\sqrt{2\pi}\left[k^{-1/2} - \frac{1}{8}k^{-3/2} + O(k^{-5/2})\right]$$

Since $f(x)$ is real for real x, for $k<0$, we may use $F(k) = [F(-k)]^*$ to obtain that as $k \to -\infty$,

$$F(k) \sim e^{k-ik}\sqrt{2\pi}\left[(-k)^{-1/2} - \frac{1}{8}(-k)^{-3/2} + O(k^{-5/2})\right]$$

Note the exponential decay in $|k|$ with exponent depending on the analyticity width of f. This is a general result as seen in the next exercise.

2. Determine solution to the following singular integral equation for $x \in (-1,1)$:

$$f(x) + \fint_{-1}^{1} \frac{f(s)}{s-x} ds = x$$

Solution: Look for solution f within the class of Holder-continuous functions in (-1, 1), except possibly at end points where it is allowed to have weak singularity. Accordingly, define

$$F(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{f(s)}{s-x}$$

which is well-defined if indeed there is solution f within the class considered. So, from Plemelj formula, the integral equation becomes

$$F_+(x) - F_-(x) + \pi i\left[F_+(x) + F_-(x)\right] = x$$

So,

$$F_+(x) = \frac{1-\pi i}{1+\pi i}F_-(x) + \frac{x}{1+i\pi}$$

We define sectionally holomorphic $W(z)$ so that

$$W_+(x) = \frac{1-\pi i}{1+\pi i}W_-(x)$$

Since $\ln\left(\frac{1-\pi i}{1+\pi i}\right) = e^{-2i\theta_0}$, where $\theta_0 = \tan^{-1}\pi$ (principal branch), We have

$$\Psi(z) = -\frac{\theta_0}{\pi}\int_{-1}^{1}\frac{ds}{s-z} = -\frac{\theta_0}{\pi}\log\left[\frac{z-1}{z+1}\right]$$

Therefore, one solution for $W(z)$ is

$$\tilde{W}(z) = \left(\frac{z-1}{z+1}\right)^{-\theta_0/\pi}$$

However, this has weak singularity at $z = 1$, but not at $z = -1$. So we multiply $\tilde{W}$ by a factor of $(z+1)^{-1}$ so that

$$W(z) = (z-1)^{-\theta_0/\pi}(z+1)^{-1+\theta_0/\pi}$$

has weak singularities at the end point. We note that for $x \in (-1, 1)$,

$$W_+(x) = e^{-i\theta_0}(1-x)^{-\theta_0/\pi}(1+x)^{-1+\theta_0/\pi}$$

$$W_-(x) = e^{i\theta_0}(1-x)^{-\theta_0/\pi}(1+x)^{-1+\theta_0/\pi}$$

Therefore, the Riemann Hilbert Problem for F becomes

$$\frac{F_+(x)}{W_+(x)} - \frac{F_-(x)}{W_-(x)} = \frac{x}{W_+(x)}$$

$$= xe^{i\theta_0}(1-x)^{\theta_0/\pi}(1+x)^{1-\theta_0/\pi} \equiv \phi(x)$$

Therefore,

$$F(z)/W(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{\phi(s)}{(s-z)}ds + E(z)$$

with $E_+(x) = E_-(x)$. Since ϕ has at best a weak singularity at $s = \pm 1$, it follows that $\frac{1}{2\pi i}\int_{-1}^{1}\frac{\phi(s)}{(s-z)}ds$ can have atmost weak singularity at the end point. $F(z)$ can only have a weak singularity at the end point from lemma mentioned in class. Therefore, since $W(z)$ evidently is not zero at the end points $z = \pm 1$ from the given expression, it follows that $\frac{F(z)}{W(z)}$ can only have weak singularity at the end points. Therefore, $E(z)$ can have atmost weak singularities at $z = \pm 1$. Since the jump condition implies E is single valued, $E(z)$ must be entire. Further, from expression for $W(z)$, we have $W(z) \sim \frac{1}{z}$ as $z \to \infty$ and from expression of $F(z)$, $F(z) \sim -\frac{1}{2\pi i z}\int_{-1}^{1} f(s)ds$. Therefore, as $z \to \infty$, $\frac{F(z)}{W(z)} \to -\frac{1}{2\pi i}\int_{-1}^{1} f(s)ds \equiv k$. Further, since $\frac{1}{2\pi i}\int_{-1}^{1}\frac{\phi(s)}{(s-z)}ds \to 0$ as $z \to \infty$, it follows that $E(z) \sim k$. as $z \to \infty$, implying $E(z) = k$. Therefore, it follows that

$$F(z) = \frac{W(z)}{2\pi i}\int_{-1}^{1}\frac{\phi(s)}{(s-z)}ds + kW(z)$$

Therefore, it follows from Plemelj formula

$$f(x) = F_+(x) - F_-(x) = \frac{W_+(x)}{2\pi i}\left\{ ⨍_{-1}^{1} \frac{\phi(s)}{(s-x)} ds + \pi i \phi(x) \right\}$$
$$- \frac{W_-(x)}{2\pi i}\left\{ ⨍_{-1}^{1} \frac{\phi(s)}{(s-x)} ds - \pi i \phi(x) \right\} + k\left[W_+(x) - W_-(x)\right]$$

where $\phi(x)$ and $W_\pm(x)$ are as defined above.

3. Obtain f to the singular integral equation

$$a(z)f(z) = \lambda ⨍_C \frac{f(t)dt}{t-z} + g(z)$$

where C is a smooth simple closed curve. State conditions on λ, $a(z)$ and $g(z)$ for solution to exist and for solution to be unique.

Solution: Define as usual

$$F_\pm(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z} ds,$$

for z inside and outside respectively of the positively oriented curve simple closed curve C. From Plemelj-Formulae, the integral equation reads as

$$a(t)\left[F_+(t) - F_-(t)\right] = \lambda \pi i \left[F_+(t) + F_-(t)\right] + g(t)$$

for $t \in C$. Thus provided

$$a(t) - \lambda\pi i \neq 0 \text{ , for } t \in C$$

we obtain the inhomogeneous RH problem:

$$F_+(t) = \frac{a(t) + \lambda\pi i}{a(t) - \lambda\pi i} F_-(t) + \frac{g(t)}{a(t) - \lambda\pi i}$$

We assume also that

$$a(t) + \lambda\pi i \neq 0 \text{ , for } t \in C$$

Let $ind_C \left\{ \frac{a(t)+\lambda\pi i}{a(t)-\lambda\pi i} \right\} = k$. Then, for z_0 inside C, we define two functions

$$W_+(z) = \exp\left[\frac{1}{2\pi i}\oint_C \ln\left\{ \frac{a(s)+\lambda\pi i}{(a(s)-\lambda\pi i)(s-z_0)^k} \right\} \frac{ds}{s-z} \right]$$

$$W_-(z) = (z-z_0)^{-k} \exp\left[\frac{1}{2\pi i}\oint_C \ln\left\{ \frac{a(s)+\lambda\pi i}{(a(s)-\lambda\pi i)(s-z_0)^k} \right\} \frac{ds}{s-z} \right]$$

Here we assume $a(t)$ to be Holder continuous on C, which ascertains use of Plemelj formula to obtian $W_\pm$ on the boundary: Further, from the explicit expression above, it is clear from Plemelj formulae that

(1)

$$W_+(t) = \exp\left[\frac{1}{2\pi i}\fint_C \ln\left\{\frac{a(s)+\lambda\pi i}{(a(s)-\lambda\pi i)(s-z_0)^k}\right\}\frac{ds}{s-z} + \frac{1}{2}\ln\left\{\frac{a(t)+\lambda\pi i}{(a(t)-\lambda\pi i)(t-z_0)^k}\right\}\right]$$

$$= \left\{\frac{a(t)+\lambda\pi i}{(a(t)-\lambda\pi i)(t-z_0)^k}\right\}^{1/2} \exp\left[\frac{1}{2\pi i}\fint_C \ln\left\{\frac{a(s)+\lambda\pi i}{(a(s)-\lambda\pi i)(s-z_0)^k}\right\}\frac{ds}{s-z}\right]$$

Similarly,

$$(2)\quad W_-(t) = (t-z_0)^{-k}\left\{\frac{a(t)+\lambda\pi i}{(a(t)-\lambda\pi i)(t-z_0)^k}\right\}^{-1/2} \times$$
$$\exp\left[\frac{1}{2\pi i}\fint_C \ln\left\{\frac{a(s)+\lambda\pi i}{(a(s)-\lambda\pi i)(s-z_0)^k}\right\}\frac{ds}{s-z}\right]$$

It is explicitly clear from the above expressions that $W_+(t) = \frac{a(t)+\lambda\pi i}{a(t)-\lambda\pi i}W_-(t)$ and that $W_+ \neq 0$ on C. Therefore, we may write the inhomogeneous RH problem as

$$\frac{F_+(t)}{W_+(t)} = \frac{F_-(t)}{W_-(t)} + \frac{g(t)}{(a(t)-\lambda\pi i)W_+(t)}$$

If we make another assumption that $g(t)$ is Holder continuous on C, then for some entire function

$$\frac{F_+(z)}{W_+(z)} = \frac{1}{2\pi i}\oint_C \left[\frac{g(s)}{(a(s)-\lambda\pi i)W_+(s)}\right]\frac{ds}{s-z} + E(z)$$
$$\frac{F_-(z)}{W_-(z)} = \frac{1}{2\pi i}\oint_C \left[\frac{g(s)}{(a(s)-\lambda\pi i)W_+(s)}\right]\frac{ds}{s-z} + E(z)$$

However, we note from explicit expression of $W_-(z)$ above, it is seen that as $z\to\infty$, $W_-(z) = O(z^{-k})$, while from expression for $F_-(z)$,

$$F_-(z) \sim -\frac{1}{2\pi i z}\oint_C f(s)ds$$

Therefore, $\frac{F_-(z)}{W_-(z)} = O(z^{k-1})$ as $z\to\infty$, while

$$\frac{1}{2\pi i}\oint_C \left[\frac{g(s)}{(a(s)-\lambda\pi i)W_+(s)}\right]\frac{ds}{s-z} = O(1/z)$$

as $z\to\infty$. So, if $k\geq 1$, then $E(z) = P_{k-1}(z)$ for some $(k-1)$-st order polynomial. If $k\leq 0$, it is clear that $E(z)=0$, and we obtain the unique solution

$$F_+(z) = \frac{W_+(z)}{2\pi i}\oint_C \left[\frac{g(s)}{(a(s)-\lambda\pi i)W_+(s)}\right]\frac{ds}{s-z}$$

$$F_-(z) = \frac{W_-(z)}{2\pi i} \oint_C \left[\frac{g(s)}{(a(s)-\lambda\pi i)W_+(s)} \right] \frac{ds}{s-z}$$

So, approaching the boundary C,

$$F_+(t) = \frac{W_+(t)}{2\pi i} \left\{ \fint_C \left[\frac{g(s)}{(a(s)-\lambda\pi i)W_+(s)} \right] \frac{ds}{s-t} + \pi i \frac{g(t)}{(a(t)-\lambda\pi i)W_+(t)} \right\}$$

$$F_-(t) = \frac{W_-(t)}{2\pi i} \left\{ \fint_C \left[\frac{g(s)}{(a(s)-\lambda\pi i)W_+(s)} \right] \frac{ds}{s-t} - \pi i \frac{g(t)}{(a(t)-\lambda\pi i)W_+(t)} \right\}$$

with solution to integral equation

$$f(t) = F_+(t) - F_-(t),$$

as given above. So, recapitulating, condition for existence of solution are

$$a(t) \neq \pm\lambda\pi i \text{ , and } a(t), g(t) \text{ Holder continuous on } C$$

and further condition

$$k \equiv ind_C \frac{a(t)+\lambda\pi i}{a(t)-\lambda\pi i} \leq 0$$

guarantees uniqueness.

4. Suppose f is Holder continuous in (0, 1). Assume also that near 0 and 1, $t^\beta f(t)$ and $(1-t)^\beta f(t)$ are respectively Holder continuous for $\beta < 1$. Prove that $\Phi(z) = \frac{1}{2\pi i} \int_0^1 \frac{\phi(t)dt}{t-z}$ can have at most weak singularity. at $z = 0$ and $z = 1$. (**Note:** The result is true for generally smooth simple open contour C. After you are done, think how you would produce more generally)

Solution: Consider neighborhood of $z = 0$. First consider the case when $|\arg z| \equiv \theta \geq \frac{\pi}{4}$. Then, if follows that

$$\left| \frac{1}{2\pi i} \int_0^1 \frac{\phi(s)ds}{s-z} ds \right| \leq \frac{A|z|^{-\beta_1}}{2\pi} \int_0^{1/|z|} \tau^{-\beta_1} |\tau - e^{i\theta}| d\tau$$
$$\leq \frac{A|z|^{-\beta_1}}{2\pi} \int_0^{\infty} \tau^{-\beta_1} |\tau - e^{i\pi/4}| d\tau \leq C|z|^{-\beta_1}$$

Now, take $|\arg z| < \frac{\pi}{4}$. Then, define $x = \Re z$. We note from geometry that $|x| \geq \frac{|z|}{\sqrt{2}}$. We write

(3)
$$\frac{1}{2\pi i} \int_0^1 \frac{\phi(s)}{s-z} ds = \frac{1}{2\pi i} \int_{2x}^1 \frac{\phi(s)}{s-z} ds + \frac{1}{2\pi i} \int_0^{2x} \frac{s^{\beta_1}\phi(s) - x^{\beta_1}\phi(x)}{s^{\beta_1}(s-z)} ds$$
$$+ \frac{x^{\beta_1}\phi(x)}{2\pi i} \int_0^{2x} \frac{ds}{s^{\beta_1}(s-z)}$$

In the first integral in (3), we further break it up as

$$\frac{1}{2\pi i}\int_{2x}^{1}\frac{\phi(s)}{s-z}ds = \frac{1}{2\pi i}\int_{2x}^{\epsilon}\frac{\phi(s)}{s-z}ds + \frac{1}{2\pi i}\int_{\epsilon}^{1}\frac{\phi(s)}{s-z}ds \tag{4}$$

The second term in (4) is clearly analytic at $z = 0$ since for $|z|$ sufficiently small, $dist(s-z) \geq \epsilon/2 > 1$. In the first integral in (4), we use local bounds on ϕ and rescaling to obtain

$$\left|\frac{1}{2\pi i}\int_{2x}^{\epsilon}\frac{\phi(s)}{s-z}ds\right| \leq \frac{B}{2\pi}\int_{2x}^{\epsilon} s^{-\beta_1}|s-x|^{-1}ds \leq \frac{B}{2\pi}\int_{2x}^{\epsilon} s^{-\beta_1}|s-x|^{-1}ds$$

$$\leq \frac{Bx^{-\beta_1}}{2\pi}\int_{2}^{\infty}\tau^{-\beta_1}|\tau-1|^{-1}d\tau$$

For the second integral in (3), we use Holder-Continuity and rescaling, we obtain

$$\left|\frac{1}{2\pi i}\int_{0}^{2x}\frac{s^{\beta_1}\phi(s) - x^{\beta_1}\phi(x)}{s^{\beta_1}(s-z)}ds\right|$$

$$\leq \frac{A}{2\pi}\int_{0}^{2x}\frac{|s-x|^{\lambda-1}}{s^{\beta_1}}ds \leq \frac{A}{2\pi}x^{\lambda_1-\beta_1}\int_{0}^{2}|\tau-1|^{\lambda-1}\tau^{-\beta_1}d\tau = o(x^{-\beta_1})$$

The third integral on rescaling $s = x\tau$ becomes

$$\frac{x^{\beta_1}\phi(x)}{2\pi i}\int_{0}^{2x}\frac{ds}{s^{\beta_1}(s-z)} = \frac{x^{\beta_1}\phi(x)}{2\pi i}\int_{0}^{2}\frac{d\tau}{\tau^{\beta_1}(\tau-w)} \text{ where } w = z/x$$

We know that the latter integral exists for any w and approaches a finite value as $w \to 1$ from the top or bottom because of Plemelj formula. Therefore, the last integral above is $O(1)$. Combining, we have

$$\Phi(z) = O(z^{-\beta_1})$$

and therefore a weak singularity is assured at $z = 0$.

5. Use results in the last exercise to characterize the most general solution to the Riemann Hilbert problem

$$\Phi_+(t) - \Phi_-(t) = \phi(t) \text{ for } t \in (a, b)$$

where ϕ satisfies conditions of the last exercise with growth condition

$$\lim_{z\to\infty} z^{-n}\Phi(z) = 0$$

for integer $n > 0$, $i.e.$ $n \in \mathbb{Z}^+$.

Solution: Consider

$$\tilde{\Phi}(z) = \frac{1}{2\pi i}\int_{a}^{b}\frac{\phi(s)}{s-z}ds$$

We know from from Plemelj formula that $\tilde{\Phi}$ is a solution to the RH problem posed above. Further, since ϕ has only weak singularity at the end points, from last exercise $\tilde{\Phi}(z)$ (with appropriate rescaling so that $[0,1]$ is mapped to $[a,b]$), we have $\tilde{\Phi}$ at most weak singularity at $z=a$ and b. Further, as $z \to \infty$ we obtain $\tilde{\Phi}(z) = O(1/z)$. Consider

$$E(z) = \Phi(z) - \tilde{\Phi}(z)$$

Since each of Φ and $\tilde{\Phi}$ separately satisfy the jump condition, $E(z)$ is single valued across at $z=a$ and $z=b$. Therefore, it can at best have isolated singularities only at those points. Further, since $\tilde{\Phi}(z) = O(1/z)$ as $z \to \infty$, we may have in general any asymptotic behavior as $z \to \infty$. Thus

$$\Phi(z) = \frac{1}{2\pi i}\int_a^b \frac{\phi(s)}{s-z} ds + E(z)$$

where $E(z)$ is any analytic function which has isolated singularities at $z=a$, $z=b$ with arbitrary behavior at ∞. If we specified Φ to have no worse than a weak singularity at $z=a$ and b, then $E(z)$ is an arbitrary entire function. Further growth condition at ∞ can restrict $E(z)$. Requirement that $\lim_{z\to\infty} z^{-n}\Phi(z) = 0$, implies that $E(z) = P_{n-1}(z)$, a polynomial of degree $n-1$, since $\frac{1}{2\pi i}\int_a^b \frac{\phi(s)ds}{s-z} \to 0$ as $z \to \infty$.