Lectures 1 & 2, Math 804

Definition 0.1 f has a derivative at z if

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\equiv f'(z)$$

exists, independent of *arg* h. Alternatively, $f(z+h)-f(z)=h\,[f'(z)+\epsilon(z,h)]$, with $\epsilon\to 0$ as $h\to 0$.

Remark: Note continuity of f follows when f is differentiable.

Definition 0.2 f is analytic at z_0, if f' exists for $z\in\mathcal{B}_\delta(z_0)$ for some $\delta>0$.

Lemma 0.3 A necessary condition for $f(x+iy)=u(x,y)+iv(x,y)$ to be analytic is that the following Cauchy-Riemann (C-R) conditions are satisfied:

$$u_x=v_y \quad ; \quad u_y=-v_x$$

Proof If h is real,

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\left[\frac{u(x+h,y)-u(x,y)}{h}+i\frac{v(x+h,y)-v(x,y)}{h}\right]=u_x+iv_x$$

Similar steps for h imaginary shows

$$f'(z)=v_y-iu_y$$

Hence C-R conditions are satisfied.

Note: The converse of Lemma 0.3 is not generally true. However, if $u,v\in\mathbf{C^1}$, then the converse follows (See O. Costin notes, Thm. 5.8, page 3)

Definition We define

$$\int_C f(z)dz=\int_a^b f(\gamma(t))\gamma'(t)dt \quad , \quad \int_{-C} f(z)dz=\int_b^a f(\gamma(t))\gamma'(t)dt$$

where

$$C=\{z:\ z=\gamma(t), a\le t\le b\}$$

Definition A curve $C:\{z:\ z=\gamma(t), a\le t\le b\}$ is a *simple closed* curve if $\gamma(t_1)=\gamma(t_2)$ implies $t_1=t_2$, except when $t_1=a$, $t_2=b$ for which $\gamma(a)=\gamma(b)$.

Remark By writing $f(z)=u+iv$, $\gamma(t)=\alpha(t)+i\beta(t)$ and separating out real and imaginary parts, we can borrow from real variable theory to find conditions that guarantee existence of $\int_C f(z)dz$. Also note, from definition, $\int_{-C}=-\int_C$

Theorem 1.1, Cauchy's theorem

Let f be analytic in a region (open connected set) R in the complex plane $\mathbf{C}$. Consider any simple piecewise smooth closed curve $\mathcal{C}$, which together with its interior is contained in R (see Fig. 1). Then

$$\oint_{\mathcal{C}} f(z)\ dz\ =\ 0 \tag{1}$$

Proof: Note $f(z) = u(x,y) + iv(x,y)$, $dz = dx + idy$; so

$$\oint_C f(z)\, dz \;=\; \oint_C (udx - vdy) + i \oint_C (udy + vdx)$$

Use of Green's theorem and CR condition gives the desired result.
Remark: Theorem above holds for f is analytic in R and continuous in $\bar{R}$.

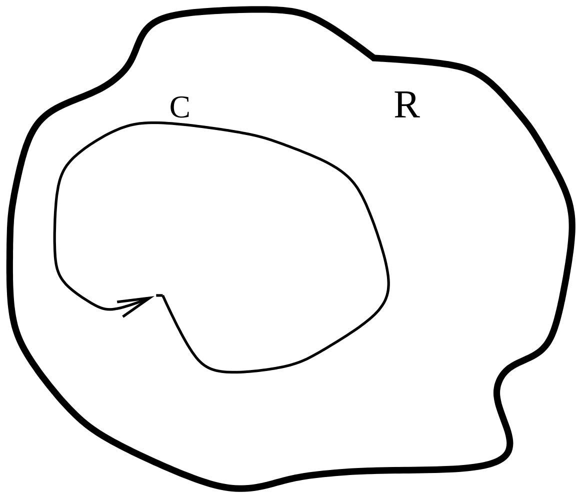

Figure 1: Closed contour C within R

Corollary 1.2: If $f(z)$ is analytic in a simply connected region R. Then for $z_0, z \in R$,

$$F(z) \;=\; \int_{z_0}^{z} f(z')\, dz' \tag{2}$$

is independent of the contour C in R, connecting z_0 and z, (see Fig. 2). Further, F is analytic in R with $F'(z) = f(z)$.
Proof: Consider two contours C_1 and C_2 (shown in bold and dashed lines respectively in Fig. 2 joining z_0 to z that are entirely within R. It is to be noted that $C_1 - C_2$ forms a closed contour that is entirely contained in R. Thus from theorem 1.1,

$$0 = \int_{C_1 - C_2} f(z)\, dz \;=\; \int_{C_1} f(z)\, dz \;-\; \int_{C_2} f(z)\, dz \tag{3}$$

and the corollary follows.
Remark: Since $F(z)$ is independent of the contour C, it can be seen readily that $F(z)$ has a derivative at each z in R, since

$$F'(z) = \lim_{\xi \to z} \frac{1}{\xi - z} \int_z^{\xi} f(z')\, dz' \;=\; f(z) \tag{4}$$

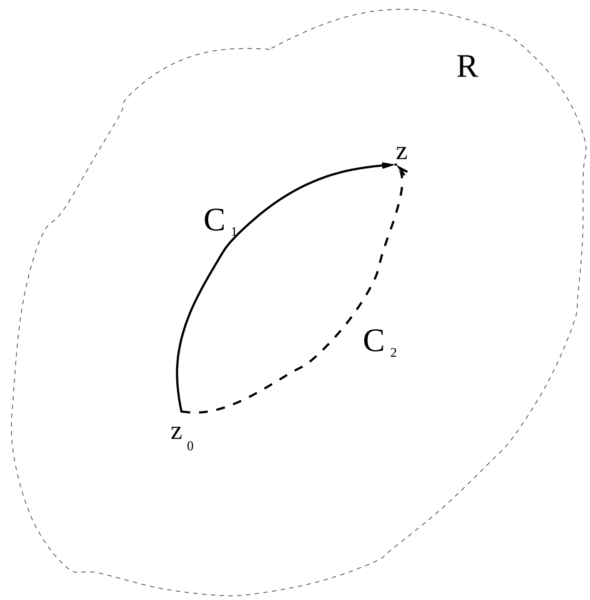

Figure 2: Two paths C_1 and C_2 connecting z_0 to z

Hence $F(z)$ is analytic in R.
Remark: If $G(z)$ is any other function so that $G'(z) \;=\; f(z)$, then

$$\frac{d}{dz}\left(F(z) - G(z)\right) \;=\; 0 \tag{5}$$

This implies, from application of the CR conditions, that $F(z) - G(z) \;=\; K$, where K is some complex constant. Actually, since $F(z_0) \;=\; 0$, it is seen that $K \;=\; -G(z_0)$. Thus

$$\int_{z_0}^{z} f(z')\, dz' \;=\; G(z) - G(z_0) \tag{6}$$

analogous to real functions. Because of this *Fundamental Theorem* all the usual integration formulae for elementary functions get carried over to complex variables.

Lemma 1.3 Let R be a simply connected region. Let $u(x, y)$ be a harmonic function of real variables x and y, with continuous second derivatives. Then $u(x, t) = Re\ F(x + iy)$ for some analytic function $F(z)$.

Proof: Define $f(x + iy) = u_x(x, y) - iu_y(x, y)$. The Cauchy Riemann conditions for this complex function $f(z)$ correspond to $u_{xx} = -u_{yy}$ and $u_{xy} = u_{yx}$, each of which are satisfied. From given condition on continuity of second derivatives of u, it follows that each of the real and imaginary parts of $f(z)$ are $\mathcal{C}^1$ functions of x and y. Therefore $f(z)$ is analytic. Define $G(z) = \int_{z_0}^{z} f(z')dz'$. From previous lemma, $G(z)$ is analytic with $G'(z) = f(z)$. Therefore, if the real and imaginary parts of $G(x + iy)$ are $g_1(x, y)$ and $g_2(x, y)$, it follows that

$$f(x + iy) = u_x(x, y) - iu_y(x, y) \qquad G'(x + iy) = g_{1_x}(x, y) - ig_{1_y}(x, y)$$

Therefore,

$$u(x, y) = g_1(x, y) + C$$

for some constant C and $u(x, y) = Re\ F(x + iy)$ where $F(z) = G(z) + C$.

Remark: From PDE theory, harmonic functions are in $\mathcal{C}^\infty$; so continuity assumptions on second derivatives in Lemma 1.3 is redundant.

Lemma 1.4 Suppose R is a *multiply connected* region *i.e.* not all closed contour in R contain points **exclusively** in R. Suppose C_1, C_2, C_3, C_4,....C_n are each simple closed curve such that the region inside C_1 and outside all the inner curves C_j, $j = 2, ...n$ is entirely within R, including the curves themselves (see Fig. 3). If f is analytic and single valued in R, then

$$\sum_{j=1}^{n} \int_{C_j} f(z)\ dz \ = \ 0 \tag{7}$$

where C_1 is traversed counter-clockwise (positive sense), while all other contours are traversed in the negative sense.

Proof: We consider a singly connected domain formed by connecting the inner contours C_2, ..C_n with the outer contour C_1 by nearly coincident lines, as shown in Fig. 3 for $n = 3$. The contribution $\int\ f(z)dz$ from nearly coincident lines vanish when they come close to each other since they are traversed in opposite directions, while f is continuous. The contour integral for this singly connected domain in the limit of coinciding lines approaches the the left of equation (7). Therefore, from Cauchy's theorem (Thm. 2.1), (7) follows.

Remark: In the special case of $n = 2$, if C_2 is also traversed counter-clockwise like C_1, then (7) implies:

$$\oint_{C_1} f(z)\ dz \ = \ \oint_{C_2} f(z)\ dz \tag{8}$$

i.e. contours can be shrunk or expanded with no change of the contour integral, provided in the process we do not cross the region R of analyticity of $f(z)$.

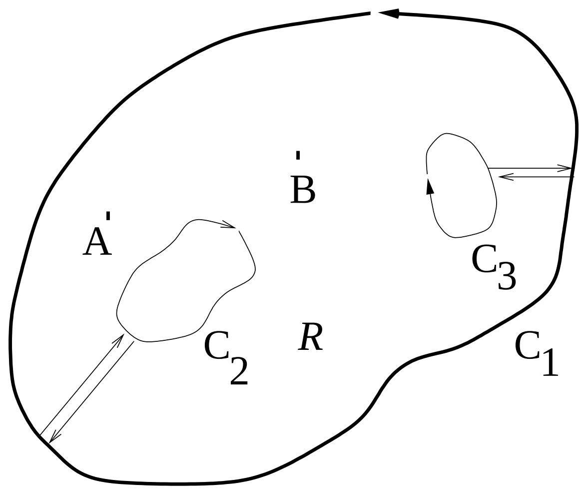

Figure 3: Contours C_1, C_2 and C_3

Lemma 1.5 Let z be a point inside a contour C. Then

$$\oint_C \frac{1}{\zeta - z}\, d\zeta \;=\; 2\,\pi\, i \quad \text{and} \quad \oint_C \frac{1}{(\zeta - z)^n}\, d\zeta \;=\; 0 \;\text{ for integer }\; n \;\neq\; 1 \tag{9}$$

where the contour C is understood in the positive (counter-clockwise) sense.

Proof: According to (8),

$$\oint_C \frac{1}{(\zeta - z)^n}\, d\zeta \;=\; \oint_{C_\epsilon} d\zeta \frac{1}{(\zeta - z)^n} \tag{10}$$

where C_ϵ is a circle of radius ϵ in the ζ plane around $\zeta = z$, traversed in the positive sense. On C_ϵ, $\zeta - z \;=\; \epsilon\, e^{i\phi}$ is a parametrization of the curve. So $d\zeta = i\epsilon\; e^{i\phi}\; d\phi$. Thus

$$\oint_{C_\epsilon} \frac{1}{(\zeta - z)^n}\, d\zeta \;=\; \int_0^{2\pi} i\, \epsilon^{1-n}\; e^{i(1-n)\phi}\, d\phi \;=\; 0 \;\text{ for } n \;\neq\; 1 \;\text{ and } = 2\pi i \text{ for } n = 1$$

and (9) follows.

Lemma 1.6 (Cauchy's integral formula) Let $f(z)$ be analytic in a simply connected region R, and $\mathcal{C}$ be any closed contour entirely within R. Let z be inside $\mathcal{C}$ (See Fig. 4). Then

$$\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \;=\; 2\,\pi\, i\; f(z) \tag{11}$$

Proof: Using lemma 1.5, (1) is equivalent to

$$\oint_C \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \;=\; 0 \tag{12}$$

We will now prove (12). For any $\epsilon \;>\; 0$, choose δ so that

$$|f(\zeta) - \; f(z)| \;\leq\; \epsilon \tag{13}$$

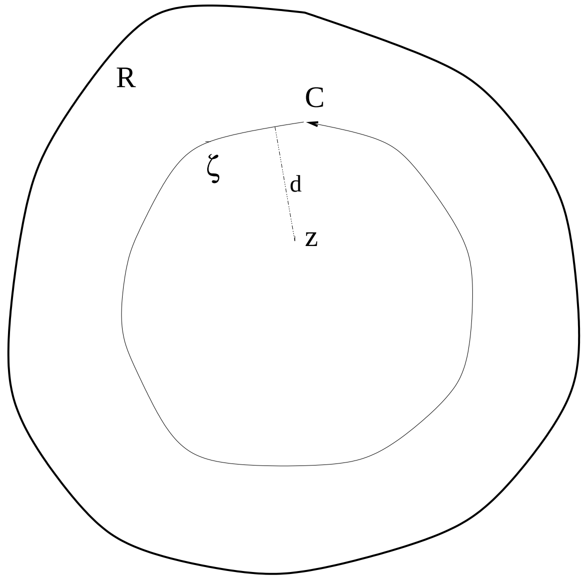

Figure 4: z within closed contour C

when ζ is on a circular contour $\mathcal{C}_\delta$ of radius δ around z. On $\mathcal{C}_\delta$, $\zeta \;=\; z \;+\; \delta\; e^{i\phi}$ and $d\zeta \;=\; i\;\delta\; e^{i\phi}\; d\phi$. Therefore, applying triangular inequality,

$$|\oint_{C_\delta} \frac{f(\zeta)-f(z)}{\zeta - z}\, d\zeta| \;\leq\; \int_0^{2\pi} d\phi\;\epsilon \;=\; 2\pi\;\epsilon \tag{14}$$

and this be made arbitrarily small. Thus the relation (12) is proved since integral $\oint_{C_\delta} = \oint_C$.
Remark: If z is outside the contour C, the expression on the left of (11) will be zero, because the function $f(\zeta)/(\zeta - z)$ is an analytic function of ζ on and within the contour C.
Remark: If z is on the contour C, the integral in (11) is not defined in the traditional sense. If z is a point where the contour C has a continous tangent, then we can draw an approximately semi-circular detour of radius ϵ so that the closed contour containing this small detour (as shown in Fig. 5) does not contain z. In that case, it is clear that

$$\oint_{C_{detour}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \;=\; 0 \tag{15}$$

But C_{detour} consists of an open contour C' and a semi-circle C_ϵ. On the semi-circle $\zeta - z \;=\; \epsilon\; e^{i\phi}$. So, in the limit $\epsilon \;\to\; 0$, using arguments similar to the proof of Lemma 2.5, one can prove that

$$\frac{1}{2\pi i} \int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z} \;\to\; -\frac{1}{2}\; f(z) \tag{16}$$

On using (15), $\int_{C'} = -\int_{C_\epsilon}$. Therefore, as $\epsilon \;\to\; 0$,

$$\frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{\zeta - z} \;\to\; \frac{1}{2}\; f(z) \tag{17}$$

The integral above, in the limit $\epsilon \to 0$ is denoted by the symbol $⨍$. From the above, we get

$$\frac{1}{2\pi i} ⨍_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \;=\; \frac{1}{2}\, f(z) \tag{18}$$

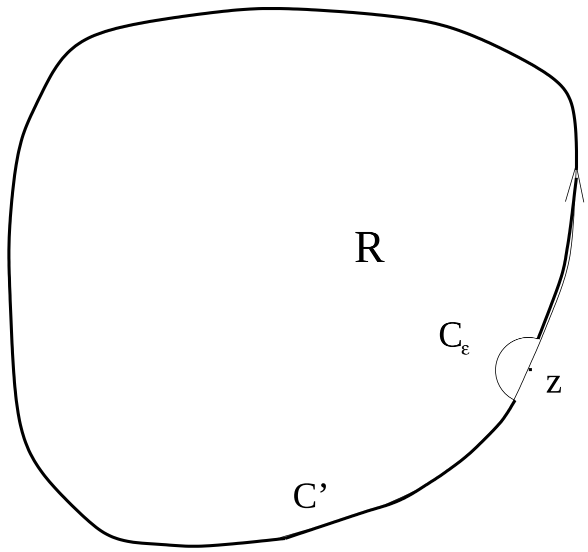

Figure 5: Deformed contour C_{detour} consisting of C' and C_ϵ

Remark: If z is a point on the boundary of C, where the boundary does not have a smooth tangent, but instead makes an angle θ_0 (see Fig. 6), then it is possible to make a slight modification of the arguments above to show that

$$\frac{1}{2\pi i} ⨍_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \;=\; \frac{\theta_0}{2\pi}\, f(z) \tag{19}$$

Exercise: Prove relation (19).

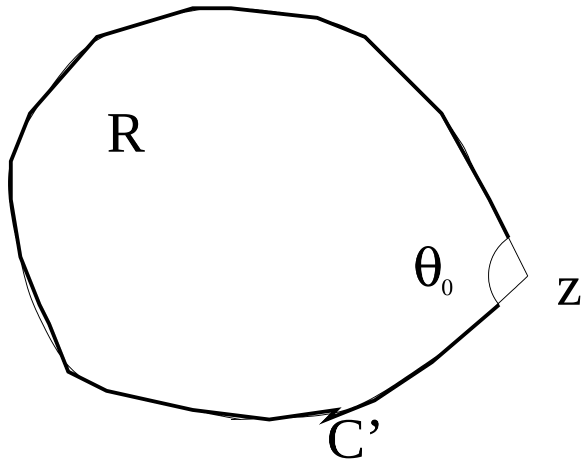

Figure 6: Corner point z, where (9) holds

Formulae for higher derivatives

Lemma 1.7 Let $f(z)$ be analytic in a simply connected region R, and $\mathcal{C}$ be any closed contour in the positive sense that is *entirely within* R. Let z be inside $\mathcal{C}$. Then

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \;=\; f'(z) \tag{20}$$

Proof: Using (11), it is clear that

$$\frac{f(\xi)-f(z)}{\xi - z} \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}} \left[\frac{f(\zeta)}{\zeta-\xi} - \frac{f(\zeta)}{\zeta - z}\right] \frac{d\zeta}{\xi - z} \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta-\xi)(\zeta-z)}$$

$$= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta-z)^2}\left(1+\frac{\xi-z}{\zeta-\xi}\right) \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta-z)^2} \;+\; \oint_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta-z)^2}\left(\frac{\xi-z}{\zeta-\xi}\right) \tag{21}$$

Now, let L be the length of the contour, $d = dist(z, \mathcal{C})$ (see Fig. 4) and $M = Max\ |f(\zeta)|$ on and inside $\mathcal{C}$. Then for any $\epsilon > 0$, choose $\delta = Min \left\{\frac{d}{2}, \frac{\pi\ \epsilon d^3}{M\ L}\right\}$. Then for $|\xi - z| < \delta$ it is clear that

$$\left|\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta-z)^2(\zeta-\xi)}\left(\frac{\xi-z}{\zeta-\xi}\right)\right| \;\leq\; \frac{M\ L\ \delta}{2\ \pi\ d^3/2} \;\leq\; \epsilon \tag{22}$$

Examining (21), in light of (22), we get

$$\lim_{\xi - z\ \to\ 0} \frac{f(\xi)-f(z)}{\xi - z} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta-z)^2} \tag{23}$$

Hence (20) follows and the lemma is proved.

Remark: Using similar arguments, with (20), as the starting point, we can write f'' as a divided difference of the first derivative and prove (similar to above) that

$$f''(z) \;=\; \frac{2}{2\pi i}\oint_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta-z)^3} \tag{24}$$

Through routine inductive procedure, it follows that

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta - z)^{(n+1)}} \tag{25}$$

Remark: An function analytic at z has derivatives at that point to arbitrary order.

Remark: The Cauchy integral formula (20) and its variants are very useful in obtaining definite integrals that could be related to a closed path integral. We will see many examples of this when we do contour integration exercises.

Lemma 1.8 (Mean value theorem on a circle): If $f(\zeta)$ is analytic on and inside a circle of radius r about $\zeta = z$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; f(z \;+\; r\; e^{i\theta}) \tag{26}$$

i.e the average value on a circle equals the functional value at the center.

Proof: Choose $\mathcal{C}$ to be a circle of radius r in the ζ-plane around $\zeta = z$. Then on substituting $\zeta \;=\; z \;+\; re^{i\theta}$, $d\zeta \;=\; i\; re^{i\theta}$ back into the Cauchy integral formulae:

$$f(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{d\zeta\; f(\zeta)}{\zeta - z}$$

the conclusion (26) follows.

Corollary 1.9: Let A denotes the interior of a circle of radius R centered around $\zeta = z$. Let $f(\zeta)$ be analytic on A and its closure, then

$$f(z) \;=\; \frac{1}{\pi R^2}\int_A f(z \;+\; r\; e^{i\theta})\; dA \tag{27}$$

This means that the mean value inside the circle is the same as at its center.

Proof: Simply multiply (26) by r and integrate in r from 0 to R, and divide the resulting expression by $R^2/2$ to obtain (27).

Theorem 1.10 (Maximum Modulus Theorem: Let $f(z)$ be analytic inside and on a closed contour $\mathcal{C}$. Then $|f(z)|$ attains its maximum value M on $\mathcal{C}$. Further if z_0 is an interior point where $|f(z_0)| = M$, then $f(z)$ is identically a constant.

Proof: Suppose, z_0 is an interior point where $|f(z_0)| = M$. Choose R small enough so that a circle of radius R is entirely within $\mathcal{C}$. Applying triangular inequality to (27), it follows that

$$M = |f(z_0)| \;\leq\; \frac{1}{\pi\; R^2}\int_A |f(z_0 \;+\; r\; e^{i\theta})| \;\leq\; M \tag{28}$$

Now, claim that the equality in (28) can only hold if $|f(z_0 \;+\; r\; e^{i\theta})| \;=\; M$ for any r between 0 and R. Note that equality in (28) implies

$$\frac{1}{\pi\; R^2}\int_A [M - |f(z_0 \;+\; r\; e^{i\theta})|]\; dA \;=\; 0 \tag{29}$$

The integrand in (29) is non-negative. If it is positive for some $z_0 + re^{i\theta}$, from continuity, it must be so in a neighborhood of that point. In that case, the integral on the left of (29) must be positive, contradicting the equation. Thus $|f(z_0 + r\ e^{i\theta})| = M$ for $0 \leq r \leq R$, i.e. everywhere inside a circle of radius R. We can now choose a point $z_1 \neq z_0$, where $|f(z_1)| = M$. We take a circle of radius R_1 that is entirely contained within $\mathcal{C}$ (see Fig. 7), and then establish $|f|$ to be constant M. Continuing this procedure, at every point inside $\mathcal{C}$ $|f|$ will be a constant. If $M = 0$, then f equals constant 0 everywhere. If $M > 0$, then $Re\ ln\ f = ln\ M =$ Constant. Applying the C-R conditions, $Im\ ln\ f = Constant$. Hence $ln\ f$ and therefor f is a constant.

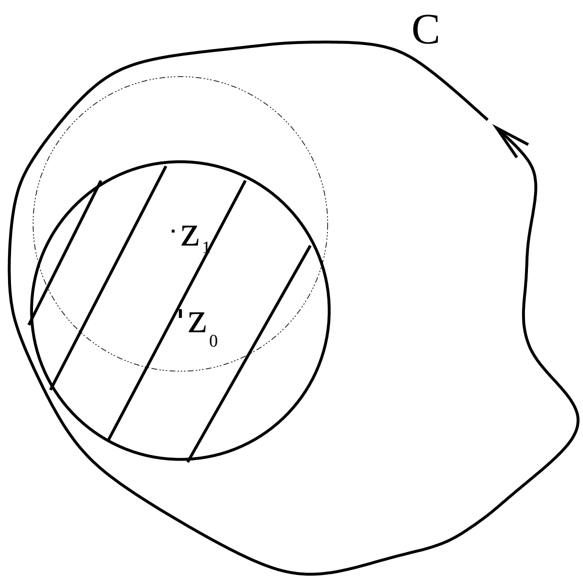

Figure 7: Circles within which $|f| = M$

Exercise: Determine maximum value of $\sin\ z$ in $|z| \leq 1$

Corollary 1.11: If $u(x, y)$ is harmonic on and inside a closed contour $\mathcal{C}$, then $u(x, y)$ attains its maximum and minimum on its boundary, unless it is identically a constant.

Proof: We know there exists analytic function $f(z)$ on and within $\mathcal{C}$, so that $Re\ f(x + iy) = u(x, y)$. Now, define $g(z) = e^{f(z)}$. Then from maximum modulus theorem, $g(z)$ attains its maximum on its boundary, unless it is a constant. Thus, $|e^f| = exp(Re\ f) = e^u$ attains its maximum on the boundary, i.e. $u(x, y)$ attains the maximum on the boundary.

For proving the minimum, choose $g(z) = e^{-f(z)}$ and repeat the same argument as above, noting that the maximum of e^{-u} corresponds to the minimum of u.

Remark: Actually the assumption in Corollary 1.11 on $u(x, y)$ being harmonic on the boundary $\mathcal{C}$ can be relaxed. We only need continuity upto the boundary.

Theorem 1.12 (Liouville's theorem: A bounded analytic function in all of the complex plane must be a constant.

Proof: Let z be an arbitrary point of the complex plane. Take a circle of radius R around $\zeta = z$. as the contour $\mathcal{C}$ in the complex ζ-plane Then, on $\mathcal{C}$, $\zeta \ = \ z + \ R\, e^{i\theta}$, $d\zeta \ = \ i\ R\, e^{i\theta} d\theta$. So, if we substitute these expressions into:

$$f'(z) \ = \ \frac{1}{2\pi i} \oint_{C} \frac{d\zeta \ f(\zeta)}{(\zeta - z)^2}$$

we obtain from triangular inequality

$$|f'(z)| \ \leq \ \frac{M}{R} \tag{30}$$

where M is a finite upper-bound of $|f|$ in the complex plane. Taking the limit of $R \ \rightarrow \ \infty$, it follows that $f'(z) = 0$. This is true for any z, therefore $f(z)$ must be a constant.

Remarks: Liouville's theorem is very useful in a number of context. For instance, it can be used to prove that a polynomial of degree n has exactly one n generally complex roots (including multiplicity). First, it is shown that the polynomial has at least one root. This is done by noting that otherwise $1/p(z)$ is a bounded analytic function and therefore a constant. But, for a nontrivial polynomial, this is not the case. Hence $p(z)$ must have one root.

Remarks: Liouville's theorem is also useful in completely characterizing functions, once their singularities are specified.

Eg. Determine the most general form of the single valued f, bounded at ∞ and analytic everywhere except at $z = z_0$ where

$$\lim_{z \ \rightarrow \ z_0} \left(f(z) - \frac{1}{z - z_0} \right) \ = \ c$$

for some constant c.

Answer: Define

$$g(z) = \left(f(z) - \frac{1}{z - z_0} \right)$$

From the given conditions $g(z)$ is a bounded entire (analytic everywhere in $\mathbf{C}$) function. From Liouville's theorem, $g(z)$ = constant. However, since $g(z_0) = c$, $g(z) = c$. Hence $f(z) = \frac{1}{z - z_0} \ + \ c$