Week 10 lectures, Math 804

1. Elliptic functions, Treatment as in Whittaker-Watson

Definition 1.1. *A function f is doubly periodic if there exists two nonzero $\omega_1, \omega_2 \in \mathbb{C}$, whose ratio is not purely real, such that*

$$f(z + 2\,\omega_1) = f(z)\ ; \ f(z + 2\,\omega_2) = f(z) \tag{1.1}$$

for all values of z where $f(z)$ exists.

Definition 1.2. *A doubly-periodic meromorphic function (functions with only pole singularities) is called an elliptic function.*

Remark 1.3. *The periods $2\,\omega_1$ and $2\,\omega_2$ play the same role in the elliptic function theory as a single period does in theory of circular (trigonometric) functions.*

Definition 1.4. *The parallelogram obtained by joining 0, $2\,\omega_1$, $2\,\omega_2 + 2\,\omega_1$, $2\,\omega_2$ and 0 is called a fundamental period-parallelogram there is no point ω, except for the vertices, with the property that $f(z + \omega) = f(z)$.*

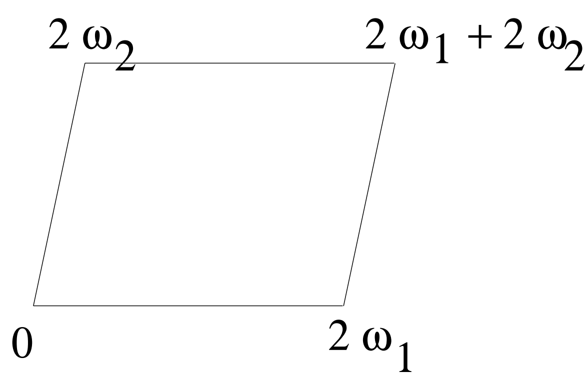

Figure 1. Fundamental Period Parallelogram

Remark 1.5. *It is clear that translations of the fundamental period-parallelogram (in Fig. 1) cover $\mathbb{C}$ (complex-plane) in a mesh, with vertices at $2\,m\,\omega_1 + 2\,n\,\omega_2$, $m, n \in \mathbb{Z}$ (integers). Doubly periodic property of an elliptic function implies that the same set of values is taken in each period parallelogram.*

Remark 1.6. *For integration purposes, it it is not convenient to deal with actual meshes if they have singularities of the function on the boundaries of the period- parallelogram. Because of periodicity, there is no loss of generality in choosing a contour, not necessarily coinciding* *with a period-parallelogram, but a mere-translation of it that guarantees no singularities on the boundary. Such a parallelogram will be referred to as a cell.*

Definition 1.7. *The set of poles (or zeros) of an elliptic function in any given cell is called an irreducible set; all other poles (or zeros) of the function are congruent to it, i.e. differ by an additive factor of* $2\ m\ \omega_1\ +\ 2\ n\ \omega_2$ *for* $m, n \in \mathbb{Z}$.

Lemma 1.8. *The number of poles of an elliptic function* $f(z)$ *in any cell must be finite.*

Proof. Suppose otherwise. Then, there must be a limit point of the set of poles within the finite cell. The limit point is a non-isolated singularity, which contradicts the definition of elliptic function. ∎

Lemma 1.9. *The number of zeros of an elliptic function* $f(z)$ *in any cell must be finite as well.*

Proof. Note $1/f$ is an elliptic function as well, with zeros of f becoming poles of its reciprocal. Applying 1.8, the proof follows. ∎

Lemma 1.10. *The sum of residues at all poles of an elliptic function in a cell is zero.*

Proof. Let f be an elliptic function. Let $\mathcal{C}$ be the the boundary of a cell with vertices t, $t\ +\ 2\omega_1$, $t\ +\ 2\ \omega_1\ +\ 2\ \omega_2$ and $t\ +\ 2\ \omega_2$. Then

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} f(z)dz\ =\ \frac{1}{2\pi i}\left\{\int_{t}^{t+2\omega_1}+\int_{t+2\omega_1}^{t+2\omega_2+2\omega_2}+\int_{t+2\omega_1+2\omega_2}^{t+2\omega_2}+\int_{t+2\omega_2}^{t}\right\}f(z)dz \tag{2}$$

The first and third integrals, as well as second and fourth integrals, can be combined as:

$$\frac{1}{2\pi i}\int_{t}^{t\ +\ 2\omega_1}[f(z)\ -\ f(z+2\omega_2)]dz-\frac{1}{2\pi i}\int_{t}^{t+2\omega_2}[f(z)-f(z+2\omega_1)]dz$$

and each of these integrands vanish due to periodicity. So, the lemma follows. ∎

Remark 1.11. *There cannot be an elliptic function with only one simple pole in a cell, as otherwise the sum of residues cannot be zero.*

Lemma 1.12. *An elliptic function with no poles in a cell is merely a constant.*

Proof. If elliptic f is analytic everywhere in a cell, it follows $|f(z)|\ \leq\ K$ (constant) for z on or within the cell, since the cell is a compact set.

From periodicity, f is bounded everywhere. From Liouville's theorem, f must be a constant. ∎

Definition 1.13. *The order of an elliptic function is the number of roots in a cell of the equation*

$$f(z) \ = \ c \tag{1.2}$$

Remark 1.14. *As will be shown, this is independent of the value of* c *chosen, but only depends on the elliptic function under consideration.*

Lemma 1.15. *The order of an elliptic function* f *is the same as the number of poles (including multiplicities) of* f *in a cell.*

Proof. Consider

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f'(z)}{f(z) - c} \, dz \tag{1.3}$$

Without loss of generality, the cell boundary $\mathcal{C}$ is chosen so that it not only avoids poles of $f(z)$, but also zeros of $f(z) \ - \ c$. Breaking up the integral (1.3) into four parts, as in the proof of Lemma 1.10 and using periodicity of the integrand as before, we conclude that the expression in (1.3) is zero. However, (1.3) is the merely the difference between the number of zeros of $f(z) - c$ and the number of poles of $f(z) - c$ (the same as $f(z)$), within the cell. Thus, the Lemma follows. ∎

Remark 1.16. *Note that the number of poles of* f *is not dependent on* c *and hence the order of an elliptic function, as defined in definition 1.13, is independent of* c*, as claimed above.*

Lemma 1.17. *The sum of the irreducible zeros (including multiplicity) is congruent to the sum of poles (including multiplicity).*

Proof. Let $\mathcal{C}$ denote the boundary of a cell, that will be chosen, without loss of generality, to avoid any zeros or poles of f. Then,

(1.4)

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{zf'(z)}{f(z)}dz &= \frac{1}{2\pi i}\left\{\int_t^{t+2\omega_1} + \int_{t+2\omega_1}^{t+2\omega_2+2\omega_2} + \int_{t+2\omega_1+2\omega_2}^{t+2\omega_2} + \int_{t+2\omega_2}^{t}\right\}\frac{z\ f'(z)}{f(z)}\,dz \\
&= \frac{1}{2\pi i}\int_t^{t+2\omega_1}\left\{\frac{zf'(z)}{f(z)} - \frac{(z+2\omega_2)\ f'(z+2\omega_2)}{f(z+2\omega_2)}\right\}dz \\
&\quad - \frac{1}{2\pi i}\int_t^{t+2\omega_2}\left\{\frac{zf'(z)}{f(z)} - \frac{(z+2\omega_1)f'(z+2\omega_1)}{f(z+2\omega_1)}\right\}dz \\
&= \frac{1}{2\pi i}\left\{-2\omega_2\int_t^{t+2\omega_1}\frac{f'(z)}{f(z)}dz + 2\omega_1\int_t^{t+2\omega_2}\frac{f'(z)}{f(z)}\ dz\right\} \\
&= \frac{1}{2\pi i}\left\{-2\omega_2\left[\ln f(z)\right]_t^{t+2\omega_1} + 2\omega_1\left[\ln\ f(z)\right]_t^{t+2\omega_2}\right\} = 2m\omega_1 + 2n\omega_2\ ,
\end{aligned}$$

for $m, n \in \mathbb{Z}$. However, we note that $zf'(z)/f(z)$ has a simple pole with residue z_0 at a simple zero (z_0 being the location of the zero) and has a simple pole with residue $-z_p$ at a simple pole z_p. For an higher order zero or poles, the residues are multiplied by the order of zero or pole. Thus, from residue theory, and relation (1.4), it follows that

$$\sum_j z_{0_j} - \sum_k z_{p_k} = 2\ m\omega_1 + 2\ n\ \omega_2$$

where the multiplicity of zeros or poles are included in the summation (i.e. if there is a second order zero at a point, then two of the indices j in the summation correspond to that point). Hence the Lemma follows. ∎

2. Construction of Weirstrass elliptic function $\mathcal{P}(z)$

Definition 2.1. *We define the Weirstrass elliptic function as:*

(2.5)
$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}\right\}$$

where

(2.6)
$$\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$$

and the summation in (2.5) extends to all $m, n \in \mathbb{Z}$, except for $(m, n) = (0, 0)$.

Remark 2.2. *When $|\Omega_{m,n}|$, is large, it is clear that the general term of the series in (2.5) is $O(|\Omega_{m,n}|^{-3})$, and so the double sum over m and*

n converges absolutely and uniformly in any compact set that excludes the set of points $\Omega_{m,n}$. Thus $\mathcal{P}(z)$ as defined above is indeed an analytic function, except for double poles at $\Omega_{m,n}$.

To show periodicity and other properties of $\mathcal{P}(z)$, it is easier to first consider the properties of $\mathcal{P}'(z)$. Because of uniform convergence, it follows that

$$\mathcal{P}'(z) \; = \; -2\sum_{m,n} \frac{1}{(z-\Omega_{m,n})^3} \tag{2.7}$$

On rearranging the shifting the index m in the above summation, it is clear that

$$\mathcal{P}'(z \; + \; 2\omega_1) \; = \; \mathcal{P}'(z) \tag{2.8}$$

Similarly, $2\omega_1$ in the above, can be replaced by $2\omega_2$. Thus, it is clear that $\mathcal{P}'(z)$ is a doubly periodic analytic function (elliptic function) with periods $2\omega_1$ and $2\omega_2$, and order three. Further,

$$\mathcal{P}'(-z) \; == \; 2\sum_{m,n} \frac{1}{(z+\Omega_{m,n})^3} \tag{2.9}$$

However, we can replace $\Omega_{m,n}$ in the above by $-\Omega_{m,n}$, since the set of values attained by $-\Omega_{m,n}$ is the same. Therefore

$$\mathcal{P}'(-z) \; == \; 2\sum_{m,n} \frac{1}{(z-\Omega_{m,n})^3} \; = \; -\mathcal{P}(z) \tag{2.10}$$

In the same manner, it follows that

$$\mathcal{P}(-z) \; = \; \frac{1}{z^2} \; + \; \sum_{m,n}{}' \left\{ \frac{1}{(z+\Omega_{m,n})^2} \; - \; \frac{1}{\Omega_{m,n}^2} \right\} \; = \; \mathcal{P}(z) \tag{2.11}$$

On integrating (2.8), it follows that

$$\mathcal{P}(z \; + \; 2\;\omega_1) \; = \; \mathcal{P}(z) \; + \; A \tag{2.12}$$

for some constant A. However, if we now substitute $z \; = \; -\omega_1$ into (2.11), we obtain

$$\mathcal{P}(-\omega_1) \; = \; \mathcal{P}(\omega_1) \; + \; A$$

Using (2.12), it follows that $A \; = \; 0$. Hence, from (2.12), we obtain

$$\mathcal{P}(z+2\omega_1) \; = \; \mathcal{P}(z) \tag{2.13}$$

Similarly

$$\mathcal{P}(z+2\omega_2) \; = \; \mathcal{P}(z) \tag{2.14}$$

Thus $\mathcal{P}(z)$ is a second order elliptic function, with fundamental periods $2\omega_1$ and $2\omega_2$. Notice it cannot possibly have a smaller period, because

$\mathcal{P}(z)$ has a pole at $z = 0$ and there are no other poles in the period parallelogram, with vertices 0, $2\omega_1$, $2\omega_1 + 2\omega_2$ and $2\omega_2$.

Lemma 2.3. *$\mathcal{P}(z)$ satisfies the differential equation:*

$$\mathcal{P}'^2(z) = 4\,\mathcal{P}^2(z) - g_2\,\mathcal{P}(z) - g_3 \tag{2.15}$$

where

$$g_2 = 60 \sum_{m,n}{}' \Omega_{m,n}^{-4}\ , \ g_3 = 140 \sum_{m,n}{}' \Omega_{m,n}^{-6} \tag{2.16}$$

Proof. We first note that $\mathcal{P}(z) - z^{-2}$ is a regular function in a neighborhood of $z = 0$ and has the Taylor series expansion:

$$\mathcal{P}(z) - z^{-2} = \frac{1}{20} g_2\, z^2 + \frac{1}{28} g_3\, z^4 + O(z^6) \tag{2.17}$$

where g_2 and g_2 are as defined by (2.16). Further on differentiating, it follows that

$$\mathcal{P}'(z) = -2\, z^{-3} + \frac{1}{10}\, g_2\, z + \frac{1}{7}\, g_3\, z^3 + O(z^5) \tag{2.18}$$

Cubing and and squaring relations (2.17) and (2.18), respectively, we get

$$\mathcal{P}^3(z) = z^{-6} + \frac{3}{20}\, g_2\, z^{-2} + \frac{3}{28}\, g_3 + O(z^2) \tag{2.19}$$

$$\mathcal{P}'^2(z) = 4\, z^{-6} - \frac{2}{5}\, g_2\, z^{-2} - \frac{4}{7}\, g_3 + O(z^2) \tag{2.20}$$

Hence

$$\mathcal{P}'^2(z) - 4\,\mathcal{P}^3(z) = -g_2\, z^{-2} - g_3 + O(z^2) \tag{2.21}$$

Therefore, in a neighborhood of $z = 0$,

$$\mathcal{P}'^2(z) - 4\,\mathcal{P}^3(z) + g_2\,\mathcal{P}(z) + g_3 = O(z^2) \tag{2.22}$$

Since, the expression on the left hand side of (2.22) is clearly an elliptic function with possible singularities only at singularities of $\mathcal{P}$ and $\mathcal{P}'$, *i.e* at $z = 0$ and all points congruent to it, it follows from (2.22) that it has no singularities anywhere (since it cannot have one at 0). Thus, the left hand side of (2.22) must be a constant as it is entire and bounded. Equation (2.22) implies that the constant must be zero, since the right hand side is 0 at $z = 0$. Thus, (2.15) follows. ∎

Remark 2.4. *Conversely, given a differential equation in the form:*

$$\left(\frac{dy}{dz}\right)^2 \;=\; 4\, y^3 \;-\; g_2\, y \;-\; g_3 \tag{2.23}$$

with given g_2 and g_3, if numbers ω_1 and ω_2 can be determined so that (2.16) is valid, then the general solution of the differential equation is given by

$$y(z) \;=\; \mathcal{P}(\pm z \;+\; \alpha), \tag{2.24}$$

where α is some constant of integration. This can be proved by simply transforming the dependent variable y through the transformation $y \;=\; \mathcal{P}(u)$. It is then seen that the differential equation (2.23) becomes

$$\left(\frac{du}{dz}\right)^2 \;=\; 1$$

Remark 2.5. *The differential equation (2.23) arises naturally in studying the travelling wave solution to the Korteweg Devries equation for $u(x,t)$ of the form:*

$$u_t \;+\; 6u\; u_x \;-\; u_{xxx} \;=\; 0 \tag{2.25}$$

For a travelling wave solution $u(x,t) \;=\; F(x \;-\; c\; t)$. Then

$$-\,c\; F' \;+\; 6\; F\; F' \;-\; F''' \;=\; 0 \tag{2.26}$$

Integrating once,

$$-\,c\; F \;+\; 3\; F^2 \;-\; F'' \;=\; d \tag{2.27}$$

for some constant d. Multiplying (2.27) by $2\; F'$ and integrating again, we get

$$F'^2 \;=\; -c\; F^2 \;+\; 2\; F^3 \;-\; 2\; d\; F \;+\; b \tag{2.28}$$

for some constant b. By shifting F by a constant, i.e. introducing $F \;=\; G +$ constant and choosing constant suitably, we get an equation for G of the form

$$G'^2 \;=\; 2\; G^3 \;-\; g_2\; G \;-\; g_3 \tag{2.29}$$

This can be further transformed into (2.24) by merely scaling both the dependent and independent variables by constants. Thus, the elliptic function is useful in studying this integrable partial differential equation.