Week 11 lectures, Math 804

1. More on Weirstrass Elliptic Function $\mathcal{P}(z)$: Integral formula

Lemma 1.1. *Introduce*

$$z = h(\zeta) = \int_{\zeta}^{\infty} (4t^3 - g_2 t - g_3)^{-1/2}\, dt \tag{1.1}$$

where the path of integration may be any curve which does not pass through a zero of $4t^3 - g_2 t - g_3$, *then* $h(\mathcal{P}(z)) = z$.

Proof. On differentiating (1.1), we obtain

$$\left(\frac{d\zeta}{dz}\right)^2 = 4\,\zeta^3 - g_2\,\zeta - g_3 \tag{1.2}$$

so that

$$\zeta = \mathcal{P}(\pm z + \alpha) = \mathcal{P}(z \pm \alpha) \tag{1.3}$$

Now, in (1.1), if we let $\zeta \to \infty$, we obtain $z \to 0$. This means that $\mathcal{P}(z + \pm\alpha)$ blows up at $z = 0$. This means that α is either 0 or a point congruent to it. Thus $\zeta = \mathcal{P}(z)$ and the Lemma is proved. ∎

Lemma 1.2. *The elliptic function $\mathcal{P}$ satisfies the following addition theorem:*

$$\begin{pmatrix} \mathcal{P}(z) & \mathcal{P}'(z) & 1 \\ \mathcal{P}(y) & \mathcal{P}'(y) & 1 \\ \mathcal{P}(z+y) & -\mathcal{P}'(z+y) & 1 \end{pmatrix} = 0 \tag{1.4}$$

Proof. Consider the following set of two linear equations for determining A and B:

$$\mathcal{P}'(z) = A\,\mathcal{P}(z) + B \tag{1.5}$$

$$\mathcal{P}'(y) = A\,\mathcal{P}(y) + B \tag{1.6}$$

These determine A and B uniquely in terms of z and y, unless $\mathcal{P}(z) = \mathcal{P}(y)$, i.e. unless $z = \pm y$ ($mod\ 2\omega_1, 2\omega_2$). For A and B as determined above, consider

$$\mathcal{P}'(\zeta) - A\,\mathcal{P}(\zeta) - B \tag{1.7}$$

It has a triple pole at $\zeta = 0$ and consequently three irreducible zeros (i.e. three zeros within a cell). Two of these zeros are clearly $\zeta = z$ and $\zeta = y$ and the third irreducible zero must be congruent to $-z-y$

(recall that the sum of pole location within a cell, which is zero in this case, is congruent to the sum of zero locations). Thus,

$$\mathcal{P}'(-z-y) \;=\; A\,\mathcal{P}(-z-y) \;+\; B$$

which implies

$$\mathcal{P}'(z+y) \;=\; A\,\mathcal{P}(z+y) \;+\; B \tag{1.8}$$

Since (1.5), (1.6) and (1.8) can be viewed as three equations for two unknowns A and B, their consistency conditions require that (1.4) is satisfied. ∎

Remark 1.3. *A more symmetrical form of (1.4) is*

$$det \begin{pmatrix} \mathcal{P}(u) & \mathcal{P}'(u) & 1 \\ \mathcal{P}(v) & \mathcal{P}'(v) & 1 \\ \mathcal{P}(w) & \mathcal{P}'(w & 1 \end{pmatrix} \;=\; 0 \tag{1.9}$$

for $u \;+\; v \;+\; w \;=\; 0$*.*

Lemma 1.4. *Another addition theorem for the Weirstrass elliptic function is that:*

$$\mathcal{P}(z \;+\; y) \;=\; \frac{1}{4}\left\{\frac{\mathcal{P}'(z) \;-\; \mathcal{P}'(y)}{\mathcal{P}(z) \;-\; \mathcal{P}(y)}\right\}^2 \;-\; \mathcal{P}(z) \;-\; \mathcal{P}(y) \tag{1.10}$$

Proof. Retaining the same notation as in Lemma 1.2, we see that the values of ζ for which $\mathcal{P}'(\zeta) \;-\; A\,\mathcal{P}(\zeta) \;-\; B$ vanish, are congruent to one of the points z, y, $-z-y$. Hence $\mathcal{P}'^2(z) \;-\; \{A\,\mathcal{P}(z) \;+\; B\}^2$ vanishes when ζ is congruent to the points z, y, $-z-y$. Recalling the differential equation satisfied by $\mathcal{P}$, this means that

$$4\,\mathcal{P}^3(\zeta) \;-\; A^2\,\mathcal{P}^2(\zeta) \;-\; (2AB \;+\; g_2)\mathcal{P}(\zeta) \;-\; (B^2 \;+\; g_3) \tag{1.11}$$

vanishes when $\mathcal{P}(\zeta)$ is equal to any one of $\mathcal{P}(z)$, $\mathcal{P}(y)$, $\mathcal{P}(z \;+\; y)$. For general values of z and y, $\mathcal{P}(z)$, $\mathcal{P}(y)$ and $\mathcal{P}(z \;+\; y)$ are unequal and so they are all the roots of

$$4\,Z^3 \;-\; A^2\,Z^2 \;-\; (2\,A\,B \;+\; g_2)\,Z \;-\; (B^2 \;+\; g_3) \;=\; 0 \tag{1.12}$$

Consequently, from the relation of the zeros of a cubic to its coefficients, it follows that

$$\mathcal{P}(z) \;+\; \mathcal{P}(y) \;+\; \mathcal{P}(z+y) \;=\; \frac{1}{4}\,A^2 \tag{1.13}$$

However, from solving for A and B in (1.5) and (1.6), and using the value of A, we obtain the relation (1.10) and the theorem is proved. ∎

Corollary 1.5. *The duplication formulae is given by*

$$\mathcal{P}(2z) \;=\; \frac{1}{4}\left\{\frac{\mathcal{P}''(z)}{\mathcal{P}'(z)}\right\}^2 \;-\; 2\,\mathcal{P}(z) \tag{1.14}$$

Proof. Consider the formula (1.10), with $y \;\to\; z$, then using L'Hopital's rule, (1.14) follows. ∎

Remark 1.6. *Differentiating*

$$\mathcal{P}'^2(z) \;=\; 4\,\mathcal{P}^3(z) \;-\; g_2\,\mathcal{P}(z) \;-\; g_3 \tag{1.15}$$

we obtain on cancellation of $\mathcal{P}'(z)$,

$$2\,\mathcal{P}''(z) \;=\; 12\,\mathcal{P}^2\,(z) \;-\; g_2 \tag{1.16}$$

Using (1.15) and (1.16) in the relation (1.14), it is possible to express $\mathcal{P}(2z)$ in terms of $\mathcal{P}(z)$ alone, though one ought to be careful in the proper interpretation of the square-root.

2. Relations satisfied by e_1, e_2 and e_3

Definition 2.1. *Define $e_1 \;=\; \mathcal{P}(\omega_1)$, $e_2 \;=\; \mathcal{P}(\omega_2)$ and $e_3 \;=\; \mathcal{P}(\omega_3)$, where $\omega_3 \;=\; -\omega_1 \;-\; \omega_2$.*

Lemma 2.2. *e_1, e_2 and e_3 are all unequal to each other and are distinct roots of*

$$4\,t^3 \;-\; g_2\,t \;-\; g_3 \tag{2.17}$$

Proof. First we note that

$$\mathcal{P}'(-\omega_1) \;=\; -\,\mathcal{P}'(\omega_1) \;=\; -\mathcal{P}'(2\,\omega_1 \;-\; \omega_1) \;=\; -\,\mathcal{P}'(\omega_1) \tag{2.18}$$

Hence $\mathcal{P}'(\omega_1) \;=\; 0$. The same is true for at ω_2 and ω_3. Since $\mathcal{P}'(z)$ is an elliptic function whose only singularities are triple poles at points congruent to the origin, $\mathcal{P}'(z)$ has three and only three irreducible zeros. Therefore the only zeros of $\mathcal{P}'(z)$ are at points congruent to ω_1, ω_2 and ω_3.

Next, consider $\mathcal{P}(z) - e_1$. This vanishes at ω_1 and since $\mathcal{P}'(\omega_1) \;=\; 0$, it has a double zero at ω_1. Since $\mathcal{P}(z)$ is a second order elliptic function, the only zeros of $\mathcal{P}(z) \;-\; e_1$ is at ω_1. In the same manner, the only irreducible zero of $\mathcal{P}(z) \;-\; e_2$ is at ω_2 and the only irreducible zero of $\mathcal{P}(z) \;-\; e_3$ at ω_3. It follows therefore that $e_1 \;\neq\; e_2 \;\neq\; e_3$, as otherwise one of $\mathcal{P}(z) \;-\; e_j$ would have a zero at more than one point. Also, from the differential equation for $\mathcal{P}(z)$ and the vanishing of $\mathcal{P}'$ at ω_j, it follows that

$$4\,\mathcal{P}^3(\omega_j) \;-\; g_2\,\mathcal{P}(\omega_j) \;-\; g_3 \;=\; 0 \tag{2.19}$$

Thus, each e_j is a distinct root of (2.17) and the lemma is proved. ∎

Remark 2.3. *From the formulae connecting roots of (2.17) with their coefficients, it follows that*

$$e_1 + e_2 + e_3 = 0 \tag{2.20}$$

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{1}{4} g_2 \tag{2.21}$$

$$e_1 e_2 e_3 = \frac{1}{4} g_3 \tag{2.22}$$

Once e_1*,* e_2 *and* e_3 *are found in terms of* g_2 *and* g_3*, expressions for* ω_1*,* ω_2 *and* ω_3 *follow from the inverse relation given in Lemma 1.1.*

Exercise 1: Suppose g_2 and g_3 are real and the discriminant $g_2^3 - 27 g_3^2 > 0$. Show that e_j are all real; choosing them so that $e_1 > e_2 > e_3$, show that

$$\omega_1 = \int_{e_1}^{\infty} (4t^3 - g_2 t - g_3)^{-1/2} dt \tag{2.23}$$

$$\omega_3 = -i \int_{-\infty}^{e_3} (4t^3 - g_2 t - g_3)^{-1/2} dt \tag{2.24}$$

so that ω_1 is real and ω_3 is purely imaginary.

Exercise 2: Show that, in the circumstances of Exercise 1, $\mathcal{P}(z)$ is real on the perimeter of the rectangle whose corners are 0, ω_3, $\omega_1 + \omega_3$ and ω_1. Determine that $\mathcal{P}(z)$ conformally maps this rectangle to the upper-half plane. We know that the upper-half plane can be mapped to the interior of the rectangle given through Schwartz Christoeffel transformation. Relate this formulae to that given as the inverse of $\mathcal{P}(z)$ in Lemma 1.1.

Lemma 2.4. *Expression for the addition of a half-period to the argument of* $\mathcal{P}(z)$ *is given as:*

$$\mathcal{P}(z + \omega_1) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\mathcal{P}(z) - e_1} \tag{2.25}$$

Proof. Using (1.10), we get

$$\mathcal{P}(z + \omega_1) + \mathcal{P}(z) + \mathcal{P}(\omega_1) = \frac{1}{4} \left\{ \frac{\mathcal{P}'(z) - \mathcal{P}'(\omega_1)}{\mathcal{P}(z) - \mathcal{P}(\omega_1)} \right\}^2 \tag{2.26}$$

Since

$$\mathcal{P}'^2(z) = 4 \, \Pi_{r=1}^3 \, [\, P(z) - e_r] \tag{2.27}$$

we have

$$(2.28) \qquad \mathcal{P}(z + \omega_1) = \frac{[\mathcal{P}(z) - e_2]\,[\mathcal{P}(z) - e_3]}{\mathcal{P}(z) - e_1} - \mathcal{P}(z) - e_1$$

This results in (2.25) and the Lemma is proved. ▮

3. Representation of general elliptic function in terms of Whittaker

Lemma 3.1. *An arbitrary even elliptic function $\phi(z)$, with irreducible zeros and poles at $\pm a_j$ and $\pm b_j$ respectively, with $j = 1,2,\ldots n$ can be written as*

$$(3.29) \qquad \phi(z) = A_1\, \Pi_{r=1,n} \left\{ \frac{\mathcal{P}(z) - \mathcal{P}(a_r)}{\mathcal{P}(z) - \mathcal{P}(b_r)} \right\}$$

for some constant A_1. If any of the constants a_r or b_r is congruent to the origin, the corresponding factor $\mathcal{P}(z) - \mathcal{P}(a_r)$ or $\mathcal{P}(z) - \mathcal{P}(b_r)$ is omitted in the product in (3.29).

Proof. Consider the function

$$(3.30) \qquad \frac{1}{\phi(z)}\, \Pi_{r=1,n} \left\{ \frac{\mathcal{P}(z) - \mathcal{P}(a_r)}{\mathcal{P}(z) - \mathcal{P}(b_r)} \right\}$$

It is an elliptic function of z. It has no poles since the zeros of $\phi(z)$ concide with the zeros of the numerator; while the poles of $\phi(z)$ coincide with the zeros of the denominator. Thus, from Liouville's theorem, (3.30) must be some constant A_1. Hence (3.29) follows and the lemma is proved. ▮

Lemma 3.2. *Any elliptic function can be expressed in terms of the the Weirstrass elliptic function $\mathcal{P}(z)$ and its derivatives.*

Proof. We note that any elliptic function $f(z)$ can decomposed as:

$$(3.31) \qquad f(z) = [f(z) + f(-z)] + [f(z) - f(-z)]\, \frac{\mathcal{P}'(z)}{\mathcal{P}'(z)}$$

Each of the terms $f(z) + f(-z)$ and $[f(z) - f(-z)]/\mathcal{P}'(z)$ are clearly elliptic functions. Applying Lemma 4.1, each of these terms can be written in the form (3.29). Thus the Lemma is proved. ▮

Remark 3.3. *There is an expression connecting two elliptic functions with the same period. This can be obtained as follows: As discussed before, each elliptic function can be expressed in terms of $\mathcal{P}(z)$ and $\mathcal{P}'(z)$.*

Eliminating $\mathcal{P}(z)$ and $\mathcal{P}'(z)$ algebraically between these two equations and

$$\mathcal{P}'^2(z) = 4\,\mathcal{P}^3(z) - g_2\,\mathcal{P}(z) - g_3 \tag{3.32}$$

we can obtain an algebraic relation connecting the two elliptic functions directly.

Remark 3.4. *One cannot perform the following integration in terms of elementary functions:*

$$\int^x (a_0x^4 + 4\,a_1\,x^3 + 6a_2\,x^2 + 4\,a_3x + a_4)^{-1/2}\,dx \tag{3.33}$$

However, ellptic functions provide an alternate expression for the answer.

Lemma 3.5. *Define*

$$f(x) = a_0x^4 + 4\,a_1\,x^3 + 6a_2\,x^2 + 4\,a_3x + a_4 \tag{3.34}$$

and let

$$z = \int_{x_0}^{x} [f(t)]^{-1/2}\,dt \tag{3.35}$$

where x_0 is any root of $f(x) = 0$; then it is possible to express x as a rational function of $\mathcal{P}(z)$, with invariants g_2 and g_3, where

$$g_2 = a_0\,a_4 - 4\,a_1\,a_3 + 3\,a_2^2 \tag{3.36}$$

$$g_3 = a_0\,a_2\,a_4 + 2\,a_1\,a_2\,a_3 - a_2^3 - a_0\,a_3^2 - a_1^2\,a_4 \tag{3.37}$$

Proof. By Taylor's theorem:

$$f(t) = 4\,A_3\,(t-x_0) + 6\,A_2\,(t-x_0)^2 + 4\,A_1\,(t-x_0)^3 + A_0(t-x_0)^4 \tag{3.38}$$

where

$$A_0 = a_0\,,\ A_1 = a_0x_0 + a_1\,,\ A_2 = a_0\,x_0^2 + 2\,a_1\,x_0 + a_2 \tag{3.39}$$

$$A_3 = a_0\,x_0^3 + 3a_1\,x_0^2 + 3\,a_2\,x_0 + a_3 \tag{3.40}$$

On writing $\tau = (t-x_0)^{-1}$ and $(x-x_0)^{-1} = \xi$, we have

$$z = \int_{\xi}^{\infty} [4\,A_3\,\tau^3 + 6A_2\,\tau^2 + 4A_1\,\tau + A_0]^{-1/2}\,d\tau \tag{3.41}$$

We remove the quadratic term in the cubic involved through the transformation

$$\tau = A_3^{-1}\left(\sigma - \frac{1}{2}\,A_2\right) \tag{3.42}$$

(3.43) $$\xi = A_3^{-1} (s - \frac{1}{2} A_2)$$

and we get

(3.44) $$z = \int_s^\infty \{4 \sigma^3 - g_2 \sigma - g_3\}^{-1/2} d\sigma$$

where

(3.45) $$g_2 = 3 A_2^2 - 4 A_1 A_2 , \; g_3 = 2 A_1 A_2 A_3 - A_2^3 - A_0 A_3^2$$

On substituting (3.39) and (3.40) into (3.45), (3.36) and (3.37) follow. From (3.44), it follows that

(3.46) $$s = \mathcal{P}(z)$$

with invariants g_2 and g_3. From the transformations given above,

(3.47) $$x = x_0 + \frac{1}{4} f'(x_0) \left\{ \mathcal{P}(z; g_2, g_3) - \frac{1}{24} f''(x_0) \right\}^{-1}$$

So the lemma is now proved. ∎

4. Quasi-Periodic function $\zeta(z)$

Definition 4.1. *We introduce $\zeta(z)$ through the equation*

(4.48) $$\frac{d\zeta}{dz} = -\mathcal{P}(z)$$

coupled with the condition that

(4.49) $$\lim_{z \to 0} [\zeta(z) - z^{-1}] = 0$$

Since the series for $\mathcal{P}(z) - z^{-2}$ converges uniformly throughout any domain from which we exclude the neighborhoods of the points $\Omega'_{m,n}$ (i.e. not including $m = n = 0$ term), we get

(4.50)
$$\zeta(z) - z^{-1} = - \int_0^z \left\{ \mathcal{P}(z) - z^{-2} \right\} dz = - \sum_{m,n}{}' \int_0^z \left\{ (z - \Omega_{m,n})^{-2} - \Omega_{m,n}^{-2} \right\} dz$$

Hence

(4.51) $$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left\{ \frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2} \right\}$$

It is easily seen that the general term of this series is $O(|\Omega_{m,n}^{-3}|)$ as $|\Omega_{m,n}| \to \infty$ and hence the series in (4.51) defines an analytic function over the whose $z-$plane except at simple poles, with residues 1, at all the points of the set $\Omega_{m,n}$. It is easily shown that

(4.52) $$\zeta(-z) \ = \ -\zeta(z)$$

and hence it is an odd-function. We now show below the quasi-periodicity of $\zeta(z)$: On integrating the relation

(4.53) $$\mathcal{P}(z \ + \ 2\omega_1) \ - \ \mathcal{P}(z) \ = \ 0$$

it follows that

(4.54) $$\zeta(z \ + \ 2\omega_1) \ = \ \zeta(z) \ + \ 2\eta_1$$

Evaluating this relation at $\zeta \ = \ -\omega_1$ and using the oddness property of ζ, it follows that

(4.55) $$\eta_1 \ = \ \zeta(\omega_1)$$

In like manner,

(4.56) $$\zeta(z \ + \ 2\omega_2) \ = \ \zeta(z) \ + \ 2\eta_2$$

where

(4.57) $$\eta_2 \ = \ \zeta(\omega_2)$$

Lemma 4.2. *For η_1 and η_2 defined as above,*

(4.58) $$\eta_1\omega_2 \ - \ \eta_2\omega_1 \ = \ \frac{1}{2}\ \pi\ i$$

Proof. To obtain this relation, consider $\int_C \ \zeta(z)\ dz$, taken around the boundary of a cell. There is one pole of $\zeta(z)$ inside the cell with residue $+1$; hence $\int_C \ \zeta(z)\ dz \ = \ 2\ \pi\ i$. However, the contour integral can be alternately expressed as:

(4.59)
$$2\pi i \ = \ \int_t^{t+2\omega_1} [\zeta(z){-}\zeta(z{+}2\omega_2)]\, dz - \int_t^{t+2\omega_2} [\zeta(z){-}\zeta(z{+}2\omega_1)]\, dz \ = \ -4\eta_2\omega_1 + 4\eta_1\omega_2$$

Thus, the relation (4.58) follows. ∎