Week 2 Notes, Math 804

Phragmen-Lindelof Principle

Remark We seek to extend the maximum modulus theorem to unbounded geometry. This can be done with additional condition on growth rate of the function at ∞ as seen below.

Lemma Assume f is analytic everywhere in $\mathcal{H} = \{z : \Re z > 0\}$ and continuous in its closure $\overline{\mathcal{H}}$. Further, assume $|f(z)| \le M$ for $z \in \partial \mathcal{H}$ (*i.e.* on the imaginary axis). Further, assume that $|f(z)| \le e^{C_2 |z|^\alpha}$ for $z \in \mathcal{H}$ for constants C_2 and α, with $\alpha \in [0, 1)$. Then $|f(z)| \le M$ for $z \in \mathcal{H}$.

Proof: Fix any point $z_0 \in \mathcal{H}$. Choose $\beta \in (\alpha, 1)$. Consider $g_\epsilon(z) = e^{-\epsilon z^\beta}$ for $\epsilon > 0$. Consider the semi-circle

$$S \equiv \{z : \Re z \ge 0 \quad , \quad |z| \le R\}$$

For $z \in S$, we note

$$|g_\epsilon(z)| = \exp\left(-\epsilon r^\beta \cos \beta\theta\right) \le 1 \quad , \quad \text{where} \quad z = r e^{i\theta} \quad ,$$

and in $\mathcal{H}$, since $\cos \beta\theta \ge \cos \frac{\beta\pi}{2} > 0$. Further,

$$e^{C_2 |z|^\alpha} |g_\epsilon(z)| \to 0 \quad \text{as} |z| \to \infty$$

Consider function $h_\epsilon = g_\epsilon f$. For sufficiently large R, S contains z_0 and further on ∂S, $|h_\epsilon(z)| \le M$. Using maximum principle, $|h_\epsilon(z)| \le M$ for $z \in S$ that includes z_0. Therefore, for any ϵ,

$$|f(z_0)| \le M / |g_\epsilon(z_0)|$$

Since the left side is independent of ϵ, as is M, on taking the limit $\epsilon \to 0$, we obtain $|f(z_0)| \le M$. Since choice of $z_0 \in \mathcal{H}$ was arbitrary, the proof is complete.

Theorem (*Phragmen-Lindelof principle*) Let f be analytic everywhere in a sector S, with opening angle π/β at the origin for $\beta > \frac{1}{2}$ and continuous on its closure $\overline{S}$. Assume on the boundary ∂S, $|f(z)| \le M$. Also assume that everywhere in S, $|f(z)| \le e^{C_2 |z|^\alpha}$ for some constant C_2 and α, with $\alpha \in [0, \beta)$. Then $|f(z)| \le M$ everywhere in S.

Outline of Proof First, with appropriate rotation $z \to z e^{i\gamma}$, sector S becomes symmetric about the real axis. Further, $z \to z^\beta$ transforms a symmetric sector S into $\mathcal{H}$. Complete details of the proof as an exercise.

Dirichlet and Neumann problems for a harmonic function:

Remark: Suppose we want to determine a harmonic function u inside a circle given its value or the value of its normal derivate on the boundary. Cauchy integral formulae is helpful such an making an explicit determination.

Theorem 3.4 (Poisson integral formulae): If u is a harmonic function inside the unit circle and is continuous upto its boundary (the unit circle), then at a point described by polar coordinate (r, α),

$$u(r,\alpha) \;=\; \frac{1}{2\pi}\int_0^{2\pi} d\theta\; u(1,\theta)\;\frac{1-r^2}{1-2\;r\;cos\;(\alpha-\theta)\;+\;r^2} \tag{2}$$

Further, if v be the conjugate harmonic function to u, *i.e.* $u \;+\; iv \;=\; f$ is an analytic function of $z = r\;e^{i\alpha}$, then

$$v(r,\alpha) \;=\; v(0) \;+\; \frac{1}{\pi}\int_0^{2\pi} d\theta\; u(1,\theta)\;\frac{r\;sin\;(\alpha-\theta)}{1-2\;r\;cos\;(\alpha-\theta)\;+\;r^2} \tag{3}$$

Proof: From given conditions $f(z)$ (whose real part is u) is analytic everywhere inside the unit circle and is continuous upto the boundary. Take contour C to be the boundary of the unit circle; Cauchy integral formula implies

$$f(z) \;=\; \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta - z}\; f(\zeta) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\frac{f(\zeta)\zeta}{\zeta - z}\; d\theta \tag{4}$$

where $\zeta = e^{i\theta}$ on the boundary C. Since z is inside the unit circle, $\frac{1}{z^*}$ is clearly outside, and so

$$0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(\zeta)\zeta}{\zeta - 1/z^*}\; d\theta \;=\; -\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\zeta)z^*}{\zeta^* - z}\; d\theta \tag{5}$$

using $\zeta^* = 1/\zeta$. Subtracting or adding (5) from (4), we obtain

$$f(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta}{\zeta - z} \pm \frac{z^*}{\zeta^* - z^*}\right)\; d\theta \tag{6}$$

Taking the positive sign in (6), we get

$$f(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\;\frac{1-|z|^2}{|\zeta - z|^2}\; d\theta \tag{7}$$

On taking the real part of (7), we obtain

$$u(r,\;\alpha) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(1,\;\theta)\;\frac{1-r^2}{|e^{i\theta} - re^{i\alpha}|^2}\; d\theta$$

which leads to (2).

On taking the negative sign in (6), we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\;\frac{1-2\zeta z^* + |z|^2}{|\zeta - z|^2}\; d\theta \\ &= \;= \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left[1 \;+\; \frac{2iIm\;(z\zeta^*)}{|\zeta - z|^2}\right]\; d\theta \end{aligned} \tag{8}$$

On taking the imaginary part of (8), we obtain (3).

Remark: u harmonic does not mean it that $u(r,\ \alpha)$ satisfies $u_{rr} + u_{\alpha\alpha} = 0$, rather that $u(r(x,y),\ \alpha(x,y))$ satisfies $u_{xx} + u_{yy} = 0$ where $x = r\ cos\ \alpha$ and $y = r\ sin\ \alpha$. In the polar variable $(r,\ \alpha)$, this means that $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\alpha\alpha} = 0$

Remark: Note that if $u(1,\theta)$ is given, *i.e.* boundary value given, then (2) gives the unique expression for the harmonic function inside the domain (i.e. inside the unit circle in this case). This problem of finding a harmonic function given its boundary value is called the Dirichlet problem in a general context where the domain need not be a circle.

Remark: If instead for some harmonic function v, only $\frac{\partial v}{\partial n}$, i.e. the normal derivative is specified on the boundary (see Fig. 1), that is usually referred to as the Neumann problem. Note that if we are to associate v to be the imaginary part of an analytic function, then from the Cauchy Riemann conditions,

$$\frac{\partial v}{\partial n} = -\frac{\partial u}{\partial s}, \tag{9}$$

where s, denotes the arclength variable in the sense shown in Fig. 1, and u is another harmonic function so that $f = u + iv$ is an analytic function of z.

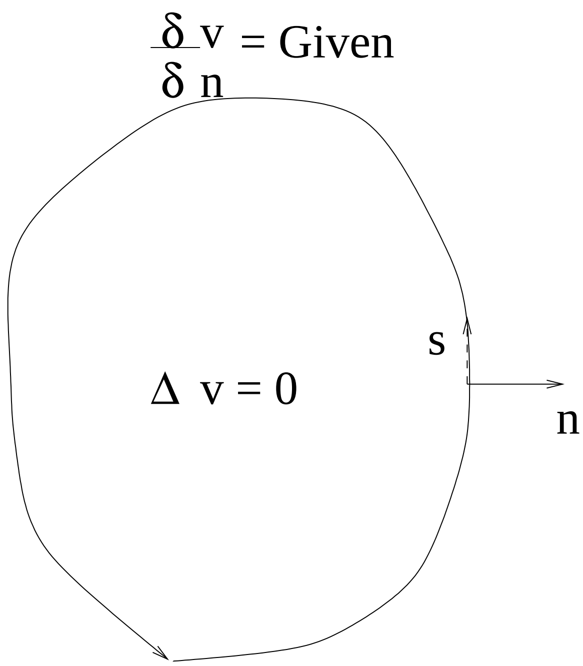

Figure 1: Neumann Problem; n denotes the normal

Equation (9) implies that a Neumann solution would not have a solution for arbitrarily specified $\frac{\partial v}{\partial n}$. Indeed it is necessary that

$$\oint ds\, \frac{\partial v}{\partial n} = 0 \tag{10}$$

in order that u returns to the same value after traversing the closed contour. Once (10) is satisfied, then it is clear from (9) that specifying $\frac{\partial v}{\partial n}$ on the boundary is equivalent to specifying u on the boundary (upto some constant). In the case of a unit circular domain, equation (3) determines the harmonic function $v(r,\ \alpha)$ at a point $z\ =\ re^{i\alpha}$ inside the unit circle. The constant is nailed by knowledge of v at the origin, as in (3), or at some other point within the domain.

Remark: It is to be noted that (2) and (3) can be combined to give

$$f(z)\ =\ i\ v(0)\ +\ \frac{1}{2\pi}\int_0^{2\pi}\ u(1,\ \theta)\ \frac{\zeta+z}{\zeta-z}\ d\ \theta \tag{11}$$

by simply noting that the Kernel

$$\frac{1-r^2}{1-2\ r\ cos\ (\alpha-\theta)+r^2}\ =\ Re\ \left(\frac{\zeta+z}{\zeta-z}\right)\ =\ Re\ \left(\frac{e^{i\theta}\ +\ re^{i\alpha}}{e^{i\theta}\ -\ r\ e^{i\alpha}}\right)$$

Exercise 3.5 Determine function $\phi(x,y)$ that is harmonic and single valued at every point in $x^2+y^2\ \leq\ 1$ except at $(x_0,0)$, where

$$\phi(x,y)\ =\ \ln\ \left[(x-x_0)^2+y^2\right]\ +\ O(1)$$

Further we require the Dirichlet boundary condition $\phi(x,y)=0$ on $x^2+y^2=1$.

Theorem 3.6: Let $f(z)$ be analytic everywhere for $|z-z_0|\ \leq\ R$. Then for $|z-z_0|\ <\ R$,

$$f(z)\ =\ \sum_{n=0}^{\infty}\ a_n\ (z-z_0)^n\quad,\quad a_n\ =\ \frac{f^{(n)}(z_0)}{n!} \tag{12}$$

Proof: Choose a counter-clockwise circular path of radius R around z_0. Let ζ be a point on the circle, while z be a point inside. Then, from Cauchy integral formulae:

$$f(z)\ =\ \frac{1}{2\pi i}\oint_C\ \frac{f(\zeta)\ d\zeta}{\zeta-z}\ =\ \frac{1}{2\pi i}\oint_C\frac{f\zeta)}{\zeta-z_0}\ \left(1-\frac{z-z_0}{\zeta-z_0}\right)^{-1}\ d\zeta \tag{13}$$

Since

$$\left(1-\frac{z-z_0}{\zeta-z_0}\right)^{-1}\ =\ 1+\frac{z-z_0}{\zeta-z_0}+\left(\frac{z-z_0}{\zeta-z_0}\right)^2\ +\ ... \tag{14}$$

converges uniformly for any compact subset of $|z-z_0|\ <\ R$, term by term integration of the series (14) is possible. Using the relation

$$\frac{f^{(n)}(z_0)}{n!}\ =\ \frac{1}{2\pi i}\oint_C\ \frac{f(\zeta)\ d\zeta}{(\zeta-z_0)^{n+1}}$$

we obtain (12).

Remark: The radius of convergence of the Taylor series about z_0 of a function is the distance d of z_0 to the nearest singularity in the complex plane. This follows from the fact that R in Thm. 3.6 can be chosen to be anything $< d$. However, R cannot be greater than d since otherwise $f'(z)$ will exist (from term by term differentiation of power series). even for $|z - z_0| = d$. This contradicts the information that there is a singularity at this distance. As an example,

$$ln\ (1 + z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \tag{15}$$

is the Taylor series about $z = 0$ and has radius of convergence $= 1$, which is the distance of the nearest singularity of $ln\ (1 + z)$ from $z = 0$, the singularity being at -1.

Remark: Taylor series provides an alternate representation of a harmonic function within a unit circle. Suppose $z = r\ e^{i\theta}$. Then, if $u(r, \theta) = Re\ f(z)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n\ z^n \tag{16}$$

implies

$$u(r\ , \theta) = \sum_{n=0}^{\infty} Re\ \left(|a_n|\ e^{i\phi_n}\ r^n\ e^{i\ n\ \theta}\right) = \sum_{n=0}^{\infty} |a_n|\ r^n\ cos\ (n\theta + \phi_n) \tag{17}$$

where $\phi_n = arg\ a_n$. So, if $u(1,\ \theta) = g(\theta)$, then

$$g(\theta) = = \sum_{n=0}^{\infty} |a_n|\ cos\ (n\theta + \phi_n) \tag{18}$$

This means that $|a_n|\ cos\ \phi_n$ and $-|a_n|\ sin\ \phi_n$ are the Fourier cosine and sine coefficients of $g(\theta)$.

Theorem 3.7: Let $f(z)$ be single-valued and analytic everywhere in a region R described by: $r_1 \leq |z - z_0| \leq r_2$. Then for any z satisfying $r_1 < |z - z_0| < r_2$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n\ (z - z_0)^n \tag{19}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}} \tag{20}$$

where C is any contour around z_0 that is contained within R.

Proof: We know that

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta) d\zeta}{\zeta - z} + \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta) d\zeta}{\zeta - z} \tag{21}$$

where C_2 is the circular contour of radius r_2 traversed positively, while C_1 is the circular contour of radius r_1, traversed clockwise. Now (21) can be rewritten as

$$f(z) \;=\; \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)d\zeta}{\zeta - z_0}\left(1 - \frac{z-z_0}{\zeta - z_0}\right)^{-1} \;-\; \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)d\zeta}{z - z_0}\left(1 - \frac{\zeta-z_0}{z - z_0}\right)^{-1} \tag{22}$$

Expanding each of the integrands, using binomial theorem, and integrating term by term (possible because of uniform convergence in any compact subset of $r_1 \;<\; |z-z_0| \;<\; r_2$, we obtain

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n\,(z-z_0)^n \;+\; \sum_{n=1}^{\infty} a_{-n}\,(z-z_0)^{-n} \tag{23}$$

where for $n \;\geq\; 0$,

$$a_n \;=\; \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}} \tag{24}$$

and for $n \;>\; 1$

$$a_{-n} \;=\; \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}} \tag{25}$$

Using the fact that for evaluation of a_n, as well as a_{-n} in (24) and (25), one can deform contours C_1 and C_2 to contour C, the result (19) and (20) follow immediately from (23)-(25).
Remark: The choice of r_1 and r_2 is restricted by singularities of $f(z)$.
Remark: Laurent/Taylor expansion of a function is unique because if

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n \tag{26}$$

is known to converge uniformly for z on some circle C around $z = z_0$, then multiplying (26) by $(z-z_0)^{-m-1}$ and integrating term by term, we obtain

$$\oint_C \frac{f(z)}{(z-z_0)^{m+1}}\,dz \;=\; \sum_{n=-\infty}^{\infty} a_n \oint_C \frac{dz}{(z-z_0)^{m+1-n}} \tag{27}$$

Now, on the circle C, $z - z_0 \;=\; \rho\,e^{i\theta}$ and $dz \;=\; i\,\rho\,e^{i\theta}\,d\theta$ and so we get for integer j that

$$\oint_C \frac{dz}{(z-z_0)^j} \;=\; \int_0^{2\pi} i\,R^{-j+1}\,e^{-i(j-1)\theta}\,d\theta \;=\; 0 \text{for } j \;\neq\; 1 \text{ and } 2\pi i \text{ for } j = 1 \tag{28}$$

Thus, from (27), we do indeed get the unique expression (29) for a_n.
Remark: Sometimes, it is convenient to determine Laurent and/or Taylor expansion of a function directly, via use of binomial theorem or information on other functions.
Example: $e^{\frac{1}{z}}$ about $z = 0$:

$$e^{\frac{1}{z}} \;=\; \sum_{n=0}^{\infty} \frac{1}{z^n\;n!} \tag{29}$$

Example: $\frac{1}{z(z-2}$ for $0 \ < \ z \ < \ 2$:

$$\frac{1}{z(z-2)} \ = \ \frac{1}{-2z} \frac{1}{1-z/2} \ = \ -\frac{1}{2z} \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j \tag{30}$$

Example: $\frac{1}{z(z-2}$ for $2 \ < \ z$:

$$\frac{1}{z(z-2)} \ = \ \frac{1}{z^2} \frac{1}{1-2/z} \ = \ \frac{1}{z^2} \sum_{j=0}^{\infty} \left(\frac{2}{z}\right)^j \tag{31}$$

Remark about Taylor and Laurent series:

Taylor and Laurent series of functions involving product of more elementary functions are sometimes more easily calculated by using the Taylor expansion of each term and then collecting all terms containing the same powers. In this connection, it is to be noted that if $f(z)$ and $g(z)$ have both convergent laurent expansion in some common domain $r_1 \ < \ |z - z_0| \ < \ r_2$. then the Laurent expansion of $g(z)\ f(z)$ is found simply by considering the series for the product. This result can be proved in general. As an example:

$$f(z) \ = \ sin\ z\ e^z \ = \ (1+z+z^2/2! \ + \ ...)(z \ - z^3/3! \ + \ ..) \ = \ z+z^2 \ + \ z^3(1/2! \ - \ 1/3!) \ + \ ..$$

Also, if $f(z)$ has a Taylor/Laurent expansion (TE) or Laurent expansion about $z = z_0$ and $g(w)$ has a Taylor/Laurent expansion about $w_0 = f(z_0)$, then the Taylor expansion of $g(f(z))$ about $z = z_0$ can be found simply by subsituting one Taylor/Laurent expansion into another and collecting terms of like power. This can be proved to converge for $|z - z_0|$ small enough so that it is within the radius of convergence of the TE of $f(z)$ and that $|f(z) - f(z_0)|$ is less than the radius of convergence of the TE of $g(w)$ about $w = w_0$.

Eg.: $\frac{1}{cos\ z}$ about $z = 0$. Take $f(z) = cos\ z$, $g(w) = 1/w$. Then $z_0 = 0$, $w_0 = 1$.

$$f(z) = 1 - z^2/2! + z^4/4! \ + \ ... \quad , \quad g(w) \ = \ [1 \ + \ (1-w) \ + \ (1-w)^2 \ + \ ...]$$

Then

$$\frac{1}{cos\ z} \ = \ 1 \ + \ [1-(1-z^2/2! \ + \ ..)] \ + \ [1-(1-z^2/2! \ + \ ..)]^2 \ + \ [1-(1-z^2/2! \ + \ ..)]^3 \ + \ ..$$

$$= \ 1 \ + \ z^2/2! \ + \ z^4\ [-1/4! \ + \ (1/2!)^2] \ + \ ...$$

will converge for z close enough to 0 so that $|cos\ z\ | \ < \ 1$

Lecture 4: Isolated singularity

Definition 4.1: Suppose $f(z)$ is single valued and analytic everywhere for $0 \ < \ |z-z_0| \ < \ R$, but not at z_0 itself. Then $z = z_0$ is an isolated singularity at z_0.

Example 1: $\frac{1}{z}$, $e^{\frac{1}{z}}$ and $sin\ 1/z$ all have isolated singularity at the origin.

Example 2: One branch of $[(z-1)^{1/2} - 1]^{-1}$ has an isolated singularity at $z = 2$, while the other branch has no singularity. Notice that each branch is single-valued in a neighborhood of $z = 2$, even as the function is not single valued in a neighborhood of $z = 1$.

Example 3:

$$f(z) \ = \ e^z \quad \text{for} \ \ z \ \neq \ 0, \quad f(0) \ = \ 0$$

In this case $f(z)$ is not continuous at $z = 0$ and therefore not analytic; yet it is analytic in a neighborhood of $z = 0$.

Example 4: $f(z) \ = \ \frac{1}{sin\ \frac{1}{z}}$ has a singularity at $z \ = \ 0$, but it is not isolated since any neighborhood of $z = 0$ contains singular points $z = \frac{1}{n\pi}$ for sufficiently large integer n.

Remark: From conditions for Laurent expansion, we know that $f(z)$ has a convergent Laurent expansion in a neighborhood of $z = z_0$ of the form:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \ (z - z_0)^n \tag{1}$$

Definition 4.2: An isolated singularity z_0 is called a *removable* singularity of $f(z)$ if a_n in (1) are all zero for $n \ < \ 0$. In that case, one obtains an analytic function $g(z)$ at $z = z_0$ as well by defining

$$g(z) \ = \ f(z) \ \ \text{for} \ z \ \neq \ 0 \ \ ; \ \ g(z_0) \ = \ a_0 \ = \ \lim_{z \ \to \ z_0} \ f(z)$$

Example: Example 3 above constitutes a removable singularity at $z \ = \ 0$, which can be removed by merely changing the value of $f(z)$ at $z = 0$ to:

$$f(0) \ = \ \lim_{z \ \to \ 0} \ e^z \ = \ 1$$

Example: $sin\ z/z$ has a removable singularity at $z = 0$, that point being undefined. Yet, if we define this function to be 1, in accordance to $\lim_{z \ \to \ 0}$, then $sin\ z/z$ becomes analytic at $z = 0$.

Definition 4.3: The part $\sum_{n=-\infty}^{-1} a_n \ (z - z_0)^n$ of the Laurent series of $f(z)$ in (1) is called the *principal part* of $f(z)$ at $z = z_0$.

Definition 4.4: If the principal part of $f(z)$ at $z = z_0$ has only a finite number of nonzero terms, then $z = z_0$ is called *a pole.* This pole is said to be of order m if $a_{-m} \ \neq \ 0$; but $a_n \ = \ 0$ for $n \ < \ -m$. A pole of order one is also called a *simple pole.*

Example: $f(z) = \frac{1}{z(z+1)^2}$ has a simple pole at $z = 0$ and a double pole at $z = -1$. This is because about $z = 0$, the Laurent series is:

$$\frac{1}{z}\,[1 - 2/z + 3z^2 \ + \ ..] = \frac{1}{z} - 2 \ + \ \text{positive powers of } z$$

Thus, for the Laurent series about $z = 0$, $a_{-1} \ \neq \ 0$, but $a_n \ = \ 0$ for $n \ < \ -1$. On the other hand, about $z = -1$,

$$f(z) = -\frac{1}{(z+1)^2}\left[\frac{1}{1-(z+1)}\right] \ = \ -\frac{1}{(z+1)^2} \ - \ \frac{1}{z+1} \ - \ 1 \ + \ ..$$

Thus, about $z = -1$, $a_n \ = \ 0$ for $n \ < \ -2$, while $a_{-2} \ \neq \ 0$.

Definition 4.5: If the principal part of $f(z)$ about $z = z_0$ contains infinitely many terms, then $z = z_0$ is an *essential* singularity of $f(z)$.

Example

$$e^{\frac{1}{z}} \ = \ 1 \ + \ \frac{1}{z} \ + \ \frac{1}{2! \ z^2} \ + \ ..$$

about $z = 0$. Hence $z = 0$ is an essential singularity of $e^{1/z}$.

Remark: An function $f(z)$ can be shown to attain all possible values, with the exception of at most one, in any neighborhood of an essential singularity (Titchmarsh, Theory of Functions, See also Rudin, section on essential singularity)

Remark: An isolated singularity, in our definition, must a *removable* singularity, a *pole* of some finite order or an *essential* singularity.

Definition 4.6: The term a_{-1} in the Laurent series expansion (1) about $z = z_0$ is called the *residue* of $f(z)$ at $z = z_0$.

Lemma 4.1: Let $f(z)$ be analytic on and inside some contour C, except for a finite number of isolated singularities within this contour: z_1, z_2,... z_N. Then

$$\oint_C f(z)\ dz \ = \ 2\pi i \sum_{j=1}^{N} (\text{Residue at } \ z = z_j) \tag{2}$$

Proof: Recall from Cauchy integral formulae for a multiply connected region that

$$\oint_C f(z)\ dz \ = \ \sum_{j=1}^{n} \oint_{C_j} f(z)\ dz \tag{3}$$

where C_j is a small circular contour of radius ϵ around $z = z_j$ (see Fig. 1), chosen so small that it contains no other singularities of $f(z)$. Now, term by term of the Laurent series (1) (about $z = z_j$) gives

$$\oint_{C_j} f(z)\ dz \ = \ \sum_{n=-\infty}^{\infty} a_n^{(j)} \oint_{C_j} (z - z_j)^n\ dz \tag{4}$$

where superscript j denotes the Laurent series coefficients about $z = z_j$. But on C_j, $z - z_j = \epsilon\, e^{i\theta}$, $dz = i\,\epsilon\, e^{i\theta}\, d\theta$. Therefore:

$$\oint_{C_j} (z - z_j)^n \, dz = i\,\epsilon^{1+n} \int_0^{2\pi} e^{i(n+1)\theta}\, d\theta = 2\pi i \ \text{ for } n = -1 \text{ and } \ 0 \text{ otherwise} \tag{5}$$

Using (5) in (4) and then in (3), we obtain

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{N} a_{-1}^{(j)}$$

and hence (2) follows.

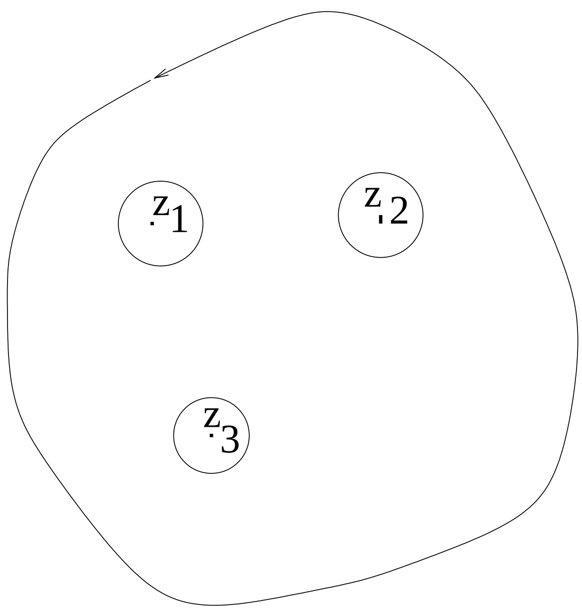

Figure 2: Contours C_j around each singularity z_j

Lemma 4.2: If $f(z)$ has a pole of order m at $z = z_0$ then the residue

$$a_{-1} = \frac{1}{(m-1)!} \left(\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m \, f(z) \right] \right)_{z \to z_0} \tag{6}$$

Proof: On multiplying (1) by $(z - z_0)^m$, we get

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}\,(z - z_0) + \ldots. + a_{-1}\,(z - z_0)^{m-1} + a_0\,(z - z_0)^m + \ldots$$

On taking the $m-1$ derivative term by term and evaluating the result at $z = z_0$, we obtain (6). For $m = 1$, no derivative is necessary, $\frac{d^0}{dz^0}$ in (6) is understood simply as the functional evaluation.

Example: Calculate

$$\oint_{|\zeta|=2} \frac{d\zeta}{\zeta^2 \; sin \; \zeta} \tag{7}$$

Answer: Note that the integrand is singular at $\zeta = 0$ and at $\zeta \;=\; n\;\pi$ for integer n. However, within the contour $|\zeta| = 2$, we have only $\zeta = 0$ included. So, we have to calculate residue at that point. In a small neighborhood of $\zeta = 0$,

$$\frac{1}{\zeta^2 \; \sin \; \zeta} = \frac{1}{\zeta^3 \; (1 - \zeta^2/3! + \zeta^4/5!....)} = \frac{1}{\zeta^3}\left[1 \; + \; (\zeta^2/3! + ...) \; + \; (....)^2 \; + \; ..\right] = \frac{1}{\zeta^3} + \frac{1}{3!\zeta} + .. \tag{8}$$

Therefore, we get a pole of order 3 at $\zeta = 0$ with residue equals 1/6, and the answer to (7) is $2\pi i/6 = \pi \; i/3$.

Example: Calculate

$$\oint_{|\zeta|=4} \frac{d\zeta}{\zeta^2 \; sin \; \zeta} \tag{9}$$

Answer: In this case, additional residues at singularities $\zeta = \pm\pi$ have to be collected. At $\zeta = \pi$, residue is

$$\lim_{\zeta \to \pi} \frac{(\zeta - \pi)}{\zeta^2 \; sin \; \zeta} = \frac{1}{\pi^2 \; cos \; \pi} = -\frac{1}{\pi^2}$$

Replacing π by $-\pi$ in the above argument, gives an additional residue $-1/\pi^2$ at $\zeta = -\pi$. Thus, the answer in this case is

$$2\pi i \; \left[\frac{1}{3!} - \frac{1}{\pi^2} - \frac{1}{\pi^2}\right]$$

Note Assume $f(z) - f(0) = \sum_{n=1}^{\infty} a_n z^n$ and $g(w) - g(w_0) = \sum_{m=1}^{\infty} b_m (w - w_0)^m$, where $w_0 = f(0)$. Then $g(f(z)) - g(f(z_0)) = \sum_{k=1}^{\infty} c_k z^k$, where

$$c_k = \sum_{m=1}^{k} \sum_{k_j \geq 1} b_m \prod_{j=1}^{m} a_{k_j} \quad \text{with} \quad \sum_{j=1}^{m} k_j = k$$

This follows from plugging from noticing that the coefficient of z^k in

$$(w - w_0)^m = (a_1 z + a_2 z^2 + ..)(a_1 z + a_2 z^2 \; + \; ..)....(a_1 z + a_2 z^2 \; + \; ...)$$

is given by $\sum_{k_j \geq 1} \prod_{j=1}^{m} a_{k_j}$ with the constraint that $\sum_j k_j = k$. We know that $g(f(z))$ has a convergent series expansion if the $f(z)$ is analytic at 0 and $g(w)$ is analytic at $w_0 = f(0)$ since the composite function is analytic. So, the series $\sum c_k z^k$ will converge under these conditions.

Lemma 4.3: Let $f(z)$ be an analytic function on and inside a contour C, except that, within C,

(1) $f(z)$ has a finite number (P) of poles, counting their multiplicity (*i.e.* if $z = z_0$ is an m-th order pole, it is counted as m poles).
(2) $f(z)$ has a finite number (N) of zeros (counting multiplicity). Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, dz \;=\; N - P \tag{10}$$

Proof: We will carry out the proof only for the case when the zeros and poles are all simple (The proof is very similar in the more general case). In the neighborhood of a simple zero, z_0 (See Fig. 3), from the Taylor series expansion,

$$f(z) \;=\; a_1(z-z_0) + a_2(z-z_0)^2 + ... = (z-z_0)\,[a_1 + (z-z_0)\,a_2 + ...] \;=\; (z-z_0)\,h(z) \tag{11}$$

where $h(z)$ is analytic and $h(z_0) \;=\; a_1 \;\neq\; 0$ (because f has a simple zero). From continuity of $h(z)$, there exists a sufficiently small neighborhood of $z = z_0$ where $h(z) \;\neq\; 0$ as well. Therefore, in that neighborhood, it is clear from the expression

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_0} \;+\; \frac{h'(z)}{h(z)} \tag{12}$$

that the integrand in (10) has a simple pole at $z = z_0$ with residue 1.

At a simple pole $z = z_p$ (See Fig. 3), using the Laurent series at $z = z_p$, it is clear that $f(z) = \frac{g(z)}{z-z_p}$, where $g(z)$ is analytic and nonzero in a sufficiently small neighborhood of $z = z_p$. Using

$$\frac{f'(z)}{f(z)} = -\frac{1}{z - z_0} \;+\; \frac{g'(z)}{g(z)}$$

it is clear that the integrand in (10) has a simple pole at $z = z_p$ with residue -1.

Adding up all the residues of $f'(z)/f(z)$ (+1 for a zero and -1 for a pole of $f(z)$) equation (10) follows.

Remark: Since the indefinite integral of $f'(z)/f(z)$ is $ln\ f(z) \;=\; ln\ |f(z)| \;+\; i\ arg\ f(z)$, the relation (10) implies that $2\pi\ (N - P)$ will be the total change of $arg\ f(z)$ resulting from a complete traversal of $f(z)$. Here, we are assuming a continuous change in arg.

Lemma 4.4 (Rouche's Theorem): Let $f(z)$ and $g(z)$ be analytic inside and on C, with $|g(z)| \;<\; |f(z)|$ on C. Then $f(z)$ and $f(z) \;+\; g(z)$ have the same number of zeros within C.
Proof: Note that

$$\oint_C dz\, \frac{f'(z) + g'(z)}{f(z) + g(z)} \;=\; i \text{ Change in } \; arg\ (f(z) \;+\; g(z)) \tag{13}$$

But $arg\ (f(z) \;+\; g(z)) \;=\; arg\ f(z) \;+\; arg\ (1 \;+\; g(z)/f(z))$ with suitable choice of definition. Further since on C, $|g(z)/f(z)| \;<\; 1$, the image of C under the map $w(z) = (1 \;+\; g(z)/f(z))$

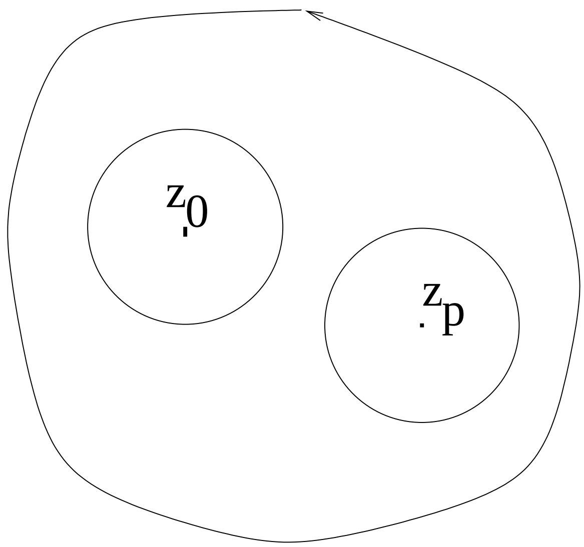

Figure 3: Contour C containing one zero z_0 and a pole z_p

cannot circle the origin (See Fig. 4). Hence there is no change of $arg\ [1\ +\ g(z)/f(z)]$ as the closed path C is traversed. Therefore, the only net-change of $arg\ (f(z)\ +\ g(z))$ over the closed path C is due to change of $arg\ f(z)$. From remark above, and the fact that there are no poles, this is $2\pi i\ N$, where N is the number of zeros of $f(z)$ within C. Thus, within C, $f(z)\ +\ g(z)$ has the same number of zeros as $f(z)$

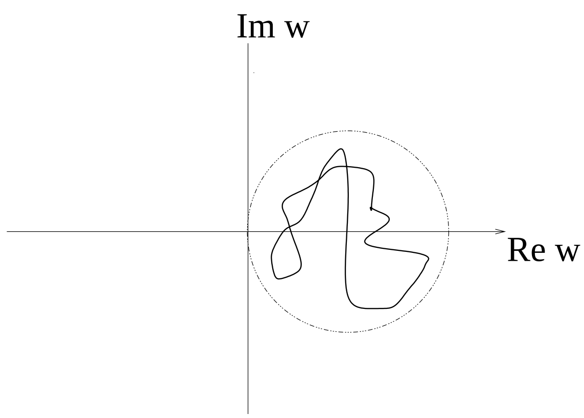

Figure 4: Image of C in the w plane, where $w(z) = 1\ +\ g(z)/f(z)$

Remark: Rouche's theorem is useful in proving that locally the inverse of an analytic function exists at every point where $f'(z)\ \neq\ 0$.

Lemma 4.5: Suppose $f(z)$ is analytic at $z\ =\ z_0$, with $f'(z_0)\ \neq\ 0$. Then there exists an analytic function $g(w)$ at $w\ =\ w_0\ =\ f(z_0)$ and a local neighborhood of w_0 where

$g(f(z)) \; = \; z.$

Proof: Assume $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$. From the Taylor expansion about z_0,

$$f(z) - w_0 = (z - z_0) \left[f'(z_0) \; + \; \frac{1}{2}(z - z_0) \; f''(z_0)(z) \; + \; ..\right] \; = \; (z - z_0) \; h(z) \qquad (14)$$

where $h(z)$ is analytic and nonzero at z_0. Therefore, it is nonzero in a small neighborhood of z_0. It follows that in that neighborhood, $|f(z) - w_0| \; = \; 0$ only once, at $z \neq 0$. Choose contour C encircling z_0, but within this neighborhood. Then

$$\min_{\mathcal{C}} \; |f(z) - w_0| \; = \; \delta \; > \; 0 \qquad (15)$$

Then for any w satisfying

$$|w - w_0| \; < \; \delta \qquad (16)$$

the functions $|f(z) - w_0|$ and $f(z) - w_0 + w_0 - w \; = \; f(z) - w$ must have the same number of zeros, since on $\mathcal{C}$, (16) implies $|f(z) \; - \; w_0| \; > \; |w_0 - w|$. Hence for given w within $|w - w_0| \; < \; \delta$, there exists unique z for which $f(z) = w$. Define $g(w)$ to be that value of z. Hence, by construction $g(f(z)) \; = \; z$ Use of a limiting process (as in proving chain rule) shows that $g'(w)$ exists and that $g'(f(z)) \; f'(z) \; = \; 1$. Therefore, $g'(w) \; = \; 1/f'(g(w))$. Therefore $g(w)$ is analytic for $|w - w_0| \; < \; \delta$.

Remark: There is an alternate argument for showing analyticity of $g(w)$. This hinges on the identity

$$g(w) \; = \; \frac{1}{2\pi i} \oint_C \frac{z \; f'(z)}{f(z) - w} \; dz \qquad (17)$$

where C is a contour, as in the proof of Lemma 4.5. Then from (17), it follows that g' exists and that

$$g'(w) \; = \; \frac{1}{2\pi i} \oint_C \frac{z \; f'(z)}{[f(z) - w]^2} \, dz \; = \; \frac{1}{2\pi i} \oint_C \frac{dz}{f(z) - w} \; = \; \frac{1}{2\pi i} \oint_C \frac{dz}{f(z) - w_0} \left[1 - \frac{w - w_0}{f(z) - w_0} \right]^{-1} \qquad (18)$$

Thus, if

$$g(w) \; = \; \sum_{n=0}^{\infty} \; b_n \; (w - w_0)^n \qquad (19)$$

then from expanding out (18),

$$n \; b_n \; = \; \frac{1}{2\pi i} \oint_C \frac{d\zeta}{(f(\zeta) - w_0)^n} \qquad (20)$$

Remark: Frequently, it is more convenient to figure out the first few terms of the power series (20) directly through substitution into the relation $g(f(z)) = z$ where

$$f(z) = \sum_{n=0}^{\infty} \; a_n \; (z - z_0)^n \qquad (21)$$

i.e.

$$z_0 + (z - z_0) \;=\; b_0 + b_1 \left[a_1(z-z_0) + a_2(z-z_0)^2 + ...\right] + b_2 \left[a_1(z-z_0) + a_2(z-z_0)^2 + ...\right]^2 + ... \tag{22}$$

So, equating the constant, $(z - z_0)$ and $(z - z_0)^2$ terms,

$$z_0 \;=\; b_0 \quad , \quad 1 \;=\; b_1\, a_1 \quad , \quad 0 \;=\; b_1\, a_2 \;+\; b_2\, a_1^2$$

Thus b_0, b_1, b_2 are determined. In principle, this process can continue *ad infinitum*, though it becomes tedious after the first few terms.

Remark: Using some of the arguments in the proof of Lemma 4.5, we note that we have also shown that the zeros of an analytic function are isolated.

Remark: The poles of an analytic function $f(z)$ must also be isolated or otherwise the zeros of $\frac{1}{f(z)}$ cannot be.

Lecture 5: Analytic continuation:

Definition: Suppose an analytic function has some representation $f_1(z)$ (say in the form of a series or an integral) that is valid only over some region (open connected set) R_1. If a different representation of an analytic function $f_2(z)$ is found in some region R_2 so that in the interesection of R_1 and R_2 (assumed to be a nontrivial region),

$$f_1(z) \;=\; f_2(z) \tag{1}$$

Then $f_2(z)$ is said to be the analytic extension of $f_1(z)$ to the entire region R_2.

Remark: From Moerara's theorem, the definition

$$f(z) = f_1(z) \quad \text{for } z \text{ in } R_1 \text{ and } f(z) = f_2(z) \quad \text{for } z \text{ in } R_2 \tag{2}$$

gives rise to an analytic function in $R_1 \; U \; R_2$.

Example:

$$f_1(z) \;=\; 1 \;+\; z \;+\; z^2 \;+\; z^3 \;+\; ... \tag{3}$$

defines an analytic function for $|z| \; < \; 1$.

$$f_2(z) \;=\; \frac{1}{1-z} \tag{4}$$

defines an analytic function everywhere, except for $z = 1$, and this agrees with (3) inside the unit circle . $f_2(z)$ is therefore an analytic continuation of $f_1(z)$ outside of $|z| \; < \; 1$. Note in this case, region R_1 is contained within R_2. This need not be the case, in general.

Example: $f_1(z) \;=\; ln_p \, z$ the principal branch of the logarithm is a representation of an analytic function for domain, with a cut along the negative real axis. Let $f_2(z) \;=\; ln|z| + i \, arg \, z$, where $0 \;\leq\; arg \; z \;\leq\; 2\pi$. Claim $f_2(z)$ is analytic continuation of $f_1(z)$ across the branch cut from above. This is because above the cut (in the second quadrant), $f_1(z) \;=\; f_2(z)$. Note that the representation $f_2(z)$ is analytic on the negative real axis, unlike $f_1(z)$.

Remark: The fact that $f_1(z)$ and $f_2(z)$ do not agree on the third and fourth quadrant in the plane is not surprising. The analytic continuation across the cut to the lower half-plane need not agree with the original $f_1(z)$ in the third quadrant, since the two evaluations involve circling around the singular point $z = 0$.

Lemma 5.3 (Identity theorem) If $f_1(z)$ and $f_2(z)$ are known to be analytic in some common region R and it is also known that $f_1(z) \;=\; f_2(z)$ for z in a subset R', where R' is a nontrivial subregion, a line or even a set of infinite points, with limit point in R. Then $f_1(z) \;=\; f_2(z)$ everwhere in R.

Proof: Let $g(z) \;=\; f_2(z) - f_1(z)$. Now, suppose $g(z)$ is not identically 0. Since $g(z)$ is analytic, its zeros must be isolated. However, if R' is a subregion, line or a set of infinite points with its limit in R, then clearly there exists an infinite set of zeros $\{z_n\}$ of $g(z)$, with its limit point z_L inside R. At the limit point, from continuity, $g(z_L) = 0$. z_L is therefore also a zero. But, this is impossible since z_L is not isolated. Thus $g(z)$ is trivially zero everywhere in R.

Remark: Analytic continuation of a function $f(z)$ outside some original region R can be attempted, in principle, through the following procedure based on Taylor series expansion. At a point z_0 inside R (see Fig. 5), from knowledge of f and all its derivatives at that point, we can compute the Taylor expansion. This provides a representation of $f(z)$ inside some circle. This circle can intersect with points outside R. If it does, we take a point z_1 in this intersection From knowledge of f and all its derivatives at z_1, we compute the Taylor series at $z = z_1$. This series is once again convergent inside some circle that may stray outside R, as well as the previous circle. We can keep continuing this process. From identity theorem, it follows that the precise choice of the points z_0, z_1 are not important, and give rise to the same analytic function, as long there are no singular points involved, between the different regions. Note, that if this process makes us go around some point outside R, then the value of the analytically continued function need not agree with its original representation (see shaded region in Fig. 5). This is the case for $f(z) = ln_p\, z$, when analytically across the branch cut from above.

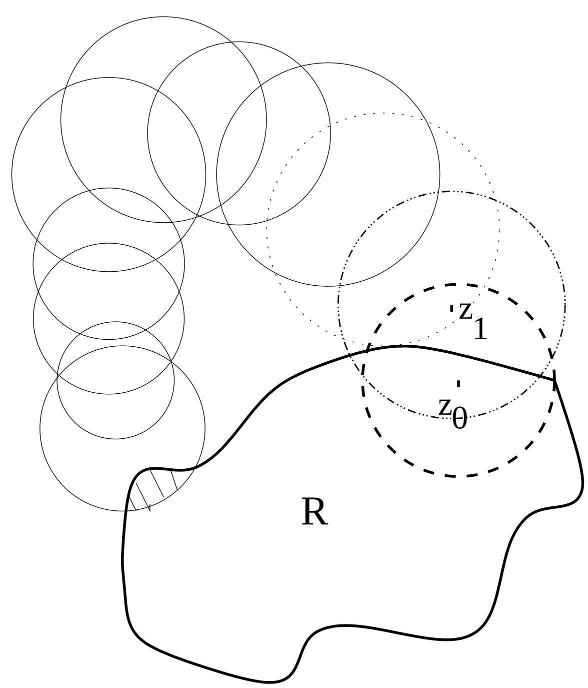

Figure 5: Illustration of analytic continuation via Taylor series

Remark: Analytic continuation is not always possible, due to *natural boundaries*.

Remark: If R_1 and R_2 be two adjacent regions, with a common boundary Γ. Let $f_1\ (z)$ be analytic within R_1 and $f_2\ (z)$ analytic within R_2 with each continuous upto Γ, where $f_1\ (z)$ and $f_2\ (z)$ agree. Then from Moerara's theorem, if we define

$$f(z) = f_1(z) \quad \text{for } z \text{ in } R_1 \text{ and } f(z) = f_2(z) \quad \text{for } z \text{ in } R_2$$

gives rise to an analytic function in $R_1\ U\ \Gamma\ U\ \ R_2$.

Lemma 5.4 (Schwarz's Reflection Principle) Let $f(z)$ be analytic in a region R of the upper-half plane adjacent to a part Γ of the real axis, with $f(z)$ continuous upto Γ and purely real on Γ. Then $f(z)$ can be analytically continued onto an image region in the lower half plane, by setting $f(z) = f^*(z^*)$, for points z in the lower-half plane.

Proof: Let R^* denote the image of R in the lower half-plane, on reflection about Γ. (See Fig. 6). Then consider $g(z) = f^*(z^*)$. Since $f(z)$ is analytic in the upper-half plane, $f(x + iy) = u(x, y) + i\ v(x, y)$ satisfies the Cauchy Riemann conditions; i.e.

$$u_x\ (x, y) = v_y\ (x, y) \quad ; \quad u_y\ (x, y) = -\ v_x\ (x, y) \tag{5}$$

Now, in the lower-half plane, from definition, $g(x + iy) = u\ (x, -y) - i\ v(x, -y)$. Note that the corresponding Cauchy Riemann conditions for $g(z)$ as given below are satisfied because of (5):

$$\frac{\partial}{\partial x}\ u(x, -y) = \frac{\partial}{\partial y}\ [-v(x, -y)] \ ; \ \frac{\partial}{\partial y}\ u(x, -y) = -\ \frac{\partial}{\partial x}\ [-v(x, -y)] \tag{6}$$

The continuity of the partials follow from the analyticity of $f(z)$ itself. Thus $g(z)$ is analytic in R^*. Further, on the real axis; $g(x) = f^*(x) = f(x)$ for given conditions. In order to complete the proof of $g(z)$ being an analytic continuation of $f(z)$, we need to ensure that the continuation is analytic on the Γ as well. For that purpose, define

$$h(z) = f(z) \ \text{ for } \ z \ \text{ in } \ R\ ; \quad = g(z) \ \text{ otherwise} \tag{7}$$

Using Moerara's theorem, it is clear that $h(z)$ is analytic everwhere in $R\ U\ \Gamma\ U\ R^*$. Thus, the analytically continued $f(z)$ is analytic on Γ as well.

Remark: Theorems similar to the above hold if the region R above is adjacent to some analytic curve Γ (i.e. there exists an analytic function $\Gamma(t)$ that maps part of the real axis to Γ). where f is real and continuous. In the general case, however, there is no explicit representation of the analytic continuation.

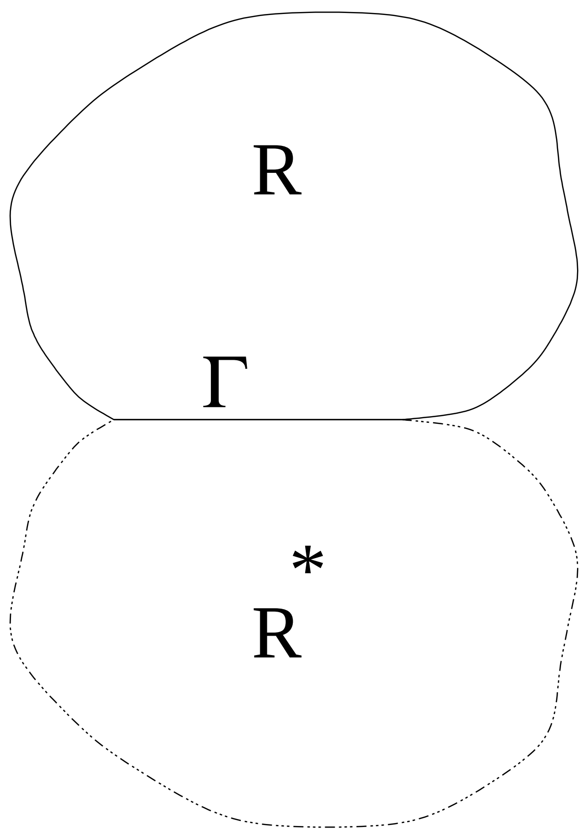

Figure 6: Reflected region R^*.