Week 3 notes

Discussion on multi-valued functions

Log **function** : Note that if z is written in its polar representation: $z = r\, e^{i\theta}$, where $r = \ |z|$ and $\theta \ = \ \arg\ z$, then

$$\log\ z \ \equiv\ log\, r \ +\ i\,\theta \ +\ 2in\pi \tag{1}$$

for integer n is consistent with *exp* (log z) $=$ z. This is easily shown through substitution. However, (1) is not unambiguous on the plane. To make (1) a single-valued function (for a particular n) on the complex plane, one needs to restrict θ to a specified interval of 2 π interval:

$$(\theta_0,\ 2\,\pi + \theta_0] \tag{2}$$

or

$$[\theta_0\ ,\ 2\,\pi\ +\ \theta_0)\ \text{ where } \theta_0\ \text{ is some constant} \tag{3}$$

Remark: One might wonder if (1) provides the most general representation of the inverse of *exp* function. The following lemma answers this in the positive.

Lemma: 1 If $\log_1$ and $\log_2$ are two functions satisfying definition (1), then then for a particular z,

$$\log_1\ z - \log_2\ z \ =\ 2\,\pi\ i\ n \tag{4}$$

where n is an integer.

Proof: Let $w_1 = \log_1\ z$ and $w_2\ =\ \log_2\ z$. Then, $exp(w_1 - w_2) = exp(w1)/exp(w2) = 1$. From previous lemma, $w_1 - w_2 = 2\,\pi\ i\ n$. **QED**

Remark: It is possible that $\log_1\ z\ =\ \log_2\ z$ for some set of z in the plane and not for others. For instance: if we define

$$\log_1\ z\ =\ \log\ r\ +\ i\,\theta, \quad \text{with}\ \ \theta\ \text{ in }\ (-\pi,\ \ \pi]\ \ , \tag{5}$$

$$\log_2\ z\ =\ \log\ r\ +\ i\,\theta, \quad \text{with}\ \ \theta\ \text{ in }\ [0,\ 2\,\pi)\ , \tag{6}$$

then clearly $\log_1\ z\ =\ \log_2\ z$ for z in the first and second quadrant but not equal in the third and fourth quadrant. In general, the integer n appearing in (1) varies with choice of z unless θ_0 is the same for $\log_1$ and $\log_2$ functions.

Definition: The particular choice $n = 0$ in (9), with $\theta_0 = 0$ in (7) defines a particular logarithmic function, called the *principal branch* of the logarithm, with the notation $\log_p\ z$ or $Log\ z$.

Definition The set of all z for which log is undefined are called it's branch points: (0 and ∞ in this case).

Definition The set of points across which log function undergoes a discontinuity for *a particular restriction of the function in a plane* is called a branch cut, i.e. $\{z \mid \arg\ z = \theta_0\}$ is the branch cut.

Definition For specificied branch cut, i.e. θ_0, the value of n is referred to as the branch of the logarithm.

Remark: Branch cuts (and therefore discontinuities) of log function cannot be avoided if $\log\ z$ is to be uniquely defined in the complex plane. However, a branch cut is quite arbitrary. It can be completely avoided if we equate the domain through the polar pair $(r,\ \theta)$, with no restriction on θ. In that case, one can define uniquely

$$\log\ z = \log\ r\ +\ i\theta \tag{7}$$

with

$$-\infty\ <\ \theta\ <\ \infty \tag{8}$$

In this representation, specification of z is not enough; we need to specify θ. This domain, based on choice of θ, is referred to a *Riemann surface.* Note that while $\log\ z$ is not continuous (across cuts), when restricted on the plane; it is indeed continuous on the Riemann surface.

Remark: Note that in general, $\log_p\ (z_1 z_2)\ \neq\ \log_p\ z_1\ +\ \log_p\ z_2$, though this is true on the Riemann surface, since on a Riemann surface, with appropriate choice of the argument,

$$\log\ (z_1\ z_2)\ =\ \log\ \left[r_1 r_2\ e^{i(\theta_1+\theta_2)}\right]\ =\ \log\ r_1 +\ \log\ r_2 + i\ \theta_1 + i\theta_2\ =\ \log\ z_1 +\ \log\ z_2 \tag{9}$$

Definition 2.1 A finite point z_0 is a branch point of a function $f(z)$ if for all sufficiently small $\epsilon\ >\ 0$,

$$f(z_0\ +\ \epsilon\ e^{i(\phi\ +\ 2\pi)})\ \neq f(z_0\ +\ \epsilon\ e^{i\phi})$$

when $f(z_0 + \epsilon e^{i\phi})$ changes continuously with ϕ. This means that if we circle around $z = z_0$ in a sufficiently small circle, we do not to the same value of f.

Eg. $\log\ (z+1)$ at $z = -1$. If $z + 1\ =\ \epsilon\ e^{i\phi}$, then $\log\ (z+1)\ =\ \log\ \epsilon\ +\ i\ \phi\ +\ 2\ i\ n\ \pi$. When θ is replaced by $\theta\ +\ 2\ \pi$, we do not return to the same value.

Definition 2.2 $z = \infty$ is a branch point of $f(z)$ if $f(1/w)$ has a branch point at $w = 0$.

Eg. $\log\ z$ has a branch point at ∞ since $\log\ 1/w\ =\ -\log\ w\ +\ 2in\pi$ has a branch point at $w = 0$. However, $\log\ \left(\frac{z+1}{z-1}\right)$ has no branch points at $z = \infty$.

Recall also from last lecture that in general, for a specific branch, we may not have

$$\log(z_1\ z_2)\ =\ \log(z_1)\ +\ \log(z_2) \tag{1}$$

However, if we do not assign any explicit restriction on arg $(z_1\ z_2)$, but instead define it as:

$$\arg\ (z_1\ z_2)\ =\ \arg\ z_1\ +\ \arg\ z_2 \tag{2}$$

then (1) will hold. For instance, if we take $\arg\ (-1)\ =\ \pi$, and apply definition (2), $\arg\ (-1)^2\ =\ 2\ \arg(-1)\ =\ 2\ i\ \pi$ and hence $\log\ (-1)^2\ =\ 2\ i\ \pi$ and not 0.

In defining a composite function involving logarithm, *say.* $\log\ (z^2-1)$, it is convenient to place a $2\ \pi$ interval restriction on $\arg\ (z-1)$ and $\arg\ (z+1)$, but *not* on $\arg\ (z^2-1)$; instead we require

$$\arg\ (z^2-1)\ =\ \arg\ (z-1)\ +\ \arg(z+1)\ , \tag{3}$$

in accordance to (2). Thus, if $z-1=r_1 e^{i\theta_1}$ and $z_2\ =\ r_2\ e^{i\theta_2}$, then, using (3),

$$\log\ (z^2-1)\ =\ \log\ (r_1\, r_2)+i\arg(z^2-1)+2in\pi\ =\ \log\ r_1+\ \log\ r_2+i(\theta_1+\theta_2)+2in\pi \tag{4}$$

If θ_1, θ_2 chosen to be within $(-\pi,\ \pi]$, the branch cuts for $\log\ (z^2-1)$ will be those shown in Fig. 1. $z\ =\ \pm 1$ and $z\ =\ \infty$ are branch points as may be readily checked. There is of course a denumerably infinite set of branches of the function, corresponding to differing integral values of the integer n.

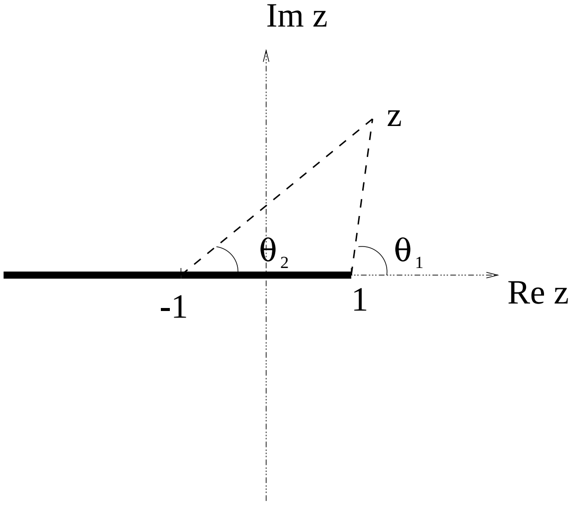

Figure 1: Branch cuts for $\log\ (z^2-1)$

Alternately, if we restricted θ_1 to $[0,\ 2\ \pi)$ and θ_2 to $(-\pi,\ \pi]$, then the corresponding branch cuts are as given below in Fig. 2. Once again we have infinitely many branches characterized by integer n in (4).

Consider another example of of a composite function involving logarithm:

$$\log\ \left(\frac{z-1}{z+1}\right) \tag{5}$$

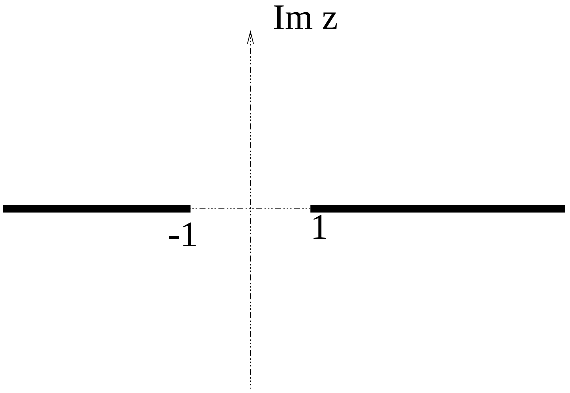

Figure 2: Another choice of cuts for log (z^2-1)

In this case, let $z-1 \ = \ r_1 \ e^{i \ \theta_1}$ and $z+1 \ = \ r_2 \ e^{i\theta_2}$. Then, if we put 2 π interval restriction on arg$(z \ \pm 1)$, but avoid putting direct restrictions on arg $\left(\frac{z-1}{z+1}\right)$ itself, then if is possible to define

$$\arg \ \left(\frac{z-1}{z+1}\right) \ = \ \arg \ (z-1) \ - \ \arg \ (z+1) \tag{6}$$

Then

$$\log \ \left(\frac{z-1}{z+1}\right) \ = \ \log \ r_1 \ - \ \log \ r_2 \ + \ i(\theta_1 \ - \ \theta_2) \ + \ 2 \ i \ n \ \pi \tag{7}$$

If both θ_1 and θ_2 are restricted to $(-\pi, \ \pi]$, then the corresponding branch cut is shown in Fig. 3.

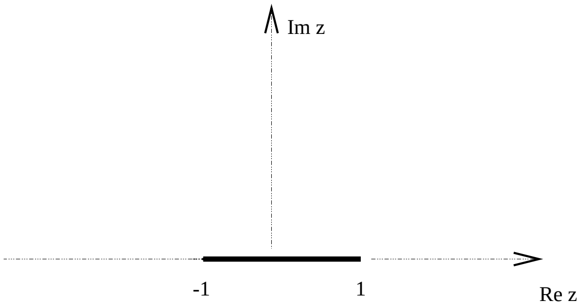

Figure 3: Branch cuts for log $\left(\frac{z-1}{z+1}\right)$

Note that there is no cut on the real axis, left of $z \ = \ -1$. The reason in this case is that the two cuts, for log $(z-1)$ and log $(z+1)$ have cancelled each other out. To see this, note that if we approach from the top a point on the real axis to the left of -1, $\theta_1, \ \theta_2 \ = \ \pi$; approaching from below, one gets $\theta_1, \ \theta_2 \ = \ -\pi$. In either case, $(\theta_1-\theta_2)$ appearing in (7) is equal to 0. So, for any fixed branch n, the function as defined in (7) has no discontinuity on the real axis to the left of -1 and hence no cut over there. Note if we restrict θ_1 and θ_2 to be in $[0,\ 2\ \pi)$, then the branch cut situation is still the same as in Fig. 3. In this case, the branch cut cancellation is on the positive real axis to the right of the branch point $z = 1$.

On the other hand if we restrict θ_1 to $[0,\ 2\pi)$ and θ_2 to $(-\ \pi,\ \pi]$, then the corresponding branch cut situation is described by Fig. 4.

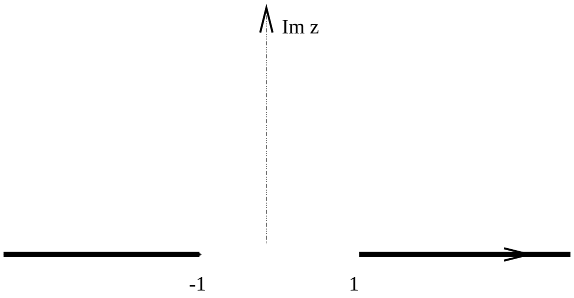

Figure 4: Another possible cut for log $\left(\frac{z-1}{z+1}\right)$

It is to be noted that the branch cut situation described in Fig. 3 and Fig. 4 are not the only ones possible. One can have other $2\ \pi$ interval restrictions on θ_1 and θ_2 that will give rise to other kinds of branch cuts for the function in (7). In practice, specific choices are made according to the requirement that the function be continuous in some part of the complex plane.

z^k for nonintegral k

Definition 2.3: For nonintegral k

$$z^k = exp[k\ \log\ z]\ =\ exp[k\ (\log\ r\ +\ i\theta\ +\ 2\ i\ n\ \pi)] \tag{8}$$

where for the purposes of uniqueness on a plance (for particular n), θ is restricted in the interval

$$[\theta_0,\ 2\ \pi + \theta_0) \quad \text{or} \quad (\theta_0,\ 2\ \pi + \theta_0\ ] \tag{9}$$

Remark: Since k is not an integer, $exp[2\ i\ n\ k\ \pi]\ \neq\ 1$; hence z^k can have more than one values for the same z, depending on n. A fixed choice of n in (8) and θ_0 in (9) specifies as unique function z^k on the plane. The value of n characterizes the branch, and the choice θ_0 defines the branch cut. Note however, that unlike the case of log , z^k need not have

infinite set of distinct branches since differing values of n can give the same value of $exp[2\ i\ n\ k\ \pi]$. In particular, if $k = p/q$, p and q being integers, then clearly $exp[2\ i\ n\ k\ \pi]$ can have q distinct values corresponding to the choice $n = 0, 1, 2, .., (q-1)$. Note that $n = q$ gives rise to the same value as $n = 0$, $n = q + 1$ the same as $n = 1$ and so on. For instance

$z^{1/2}$ has only two distinct branches corresponding to even and odd n in (8).

$$z^{1/2} = r^{1/2}\ e^{i\theta/2} \tag{10}$$

$$z^{1/2} \ = \ -\, r^{1/2}\ e^{i\theta/2} \tag{11}$$

Remark: We note that for a particular choice of n (i.e. branch) and a specific 2π restriction on the arg (...), it is possible that

$$(z_1\ z_2)^k \ \neq \ z_1^k\ z_2^k \tag{12}$$

On the other hand, if we only impose $2\ \pi$ interval restriction on arg z_1 and arg z_2, and not on arg $(z_1\ z_2)$, then as before in the context of log function, it is possible to define arg $(z_1\ z_2)$ so that

$$(z_1\ z_2)^k \ = \ z_1^k\ z_2^k \tag{13}$$

Exercise for reader: Determine appropriate branch cuts and branches for (i) $(z^2-1)^{1/2}$ and (ii) $\left(\frac{z+1}{z-1}\right)^{1/2}$. Suppose we have a branch for which $f(2) \ > \ 0$. Determine what is the value of $f(-2)$ consistent with a choice of cut such that $f(z)$ changes continously from $z = 2$ to $z = -2$ on a path connecting the two points.

Remark: Sometimes in determining branch cuts and branches of a composite function, involving logarithm and nonintegral powers in a complicated manner, it is useful to introduce suitable intermediate transformation $w = g(z)$ and determine the branch cut in the w variable. The pre-image of that cut in the z-variable gives the cut location in the z-plane. The example below illustrates this point.

Example: Describe the branch cuts, branch points and branches for the function

$$f(z) \ = \ \log\ (1+z^{1/2})$$

Solution: Let $w \ = \ g(z) \ = \ z^{1/2}$. Then $f(z) \ = \ \log\ [1 \ + \ g(z)]$. If we restrict arg $z \ = \ \theta$ to $(-\pi,\ \pi]$ (corresponding cut shown in Fig. 5), then there are two *distinct* possibilities for $g(z)$:

$$g_1(z) \ = \ r^{1/2}\ e^{i\theta/2}, \ \ g_2(z) \ = \ -\ r^{1/2}\ e^{i\theta/2} \tag{14}$$

where $r \ = \ |z|$. There is a branch point of each of $g_1(z)$ and $g_2(z)$ at $z = 0$ and $z \ = \ \infty$. These are clearly branch points of $f(z) \ = \ \log(1+g(z))$ as well.

We also note that $w \ = \ -1$ is not in the range of g_1, but it is of g_2 (since $g_2\ (1) \ = \ -1$). Now, consider $\log\ (1 \ + \ w)$. We choose the restriction arg $(1+w)$ in $(-\pi, \pi]$, with the cut in Fig. 6. we define

$$\log(1 \ + \ w) \ = \ \log|1+w| \ + \ i\ \arg(1 \ + \ w) \ + \ i\ 2\ m\ \pi \tag{15}$$

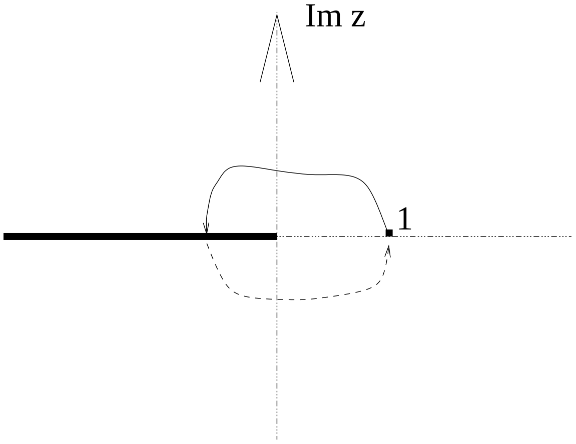

Figure 5: Branch cuts for $z^{1/2}$ and $f_{1,m}(z)$. A cut-crossing path also shown

Since no point on the chosen cut in the w plane is in the range of g_1, it is clear that

$$f_{1,m}(z) \;=\; \log_m\;(1\;+\;g_1(z)) \tag{16}$$

will have no branch points and cuts in the z plane corresponding to the branch point and cut in the w-plane. So, the only branch point and branch cut for the function $f_{1,m}(z)$ are those shown in Fig. 5– only the ones corresponding to $g_1(z)$.

However, the pre-image in the z-plane of the chosen branch cut in the w-plane under g_2 is the positive real z axis, to the right of the branch point $z = 1$ (corresponds to the cut in the w-plane in Fig. 6). Thus, the function

$$f_{2,m}(z) \;=\; \log_m\;(1\;+\;g_2(z)) \tag{17}$$

will have branch cuts shown in Fig. 7.

The function $f(z)$ has two set of infinite number of branches given by (16) and (17), corresponding to integer m. The cuts are different for (16) and (17). The reader might be interested to note that if we start from $z = 1$ where $f(1) \;=\; \log\;2$, i.e. choose $f_{1,0}(z)$ for this point evaluation, and follow the change of $f(z)$ on the Riemann surface along the path shown in Fig. 5, then on crossing the cut in Fig. 5, $f(z)$ ceases to be $f_{1,0}(z)$. Instead, it becomes $f_{2,0}(z)$ for which Fig. 7 describes the branch cuts and points. We have used dotted lines on the path in Fig. 5 to denote our journey to the other sheet for which Fig. 7 is applicable.

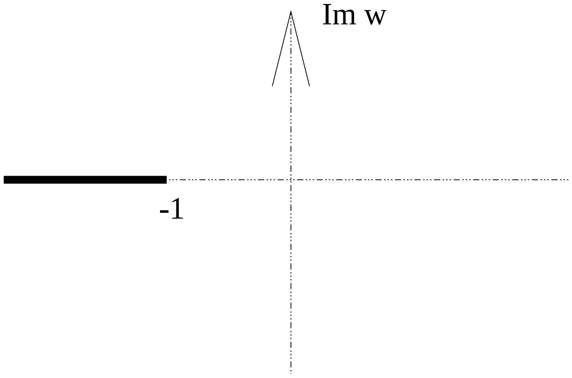

Figure 6: Branch cut in the w plane for $\log\ (1+w)$

Lecture 7: Contour Integration

Remark: Cauchy's theorem and its extension are of course directly related to determination of closed contour integrals of analytic functions that contain only isolated singularities within the contour. We simply use the Residue theorem and compute the sum of the residues.

Remark: It can, however, also be used to compute:

(i) Certain definite integerals

(ii) Principal value integrals

The main objective in such exercises is to relate the desired integral to some closed contour in the complex plane, either by appending additional open contours to the original open contour so as to make the union closed. This is helpful if the answer on the additional open contours are known independently or can be related to the original open contour contribution. Sometimes, change of variables relates the contour integration to a complex closed path integration.

Eg. 1: Compute $I \ = \ \int_0^\infty \frac{dx}{x^4\ +\ 1}$

Solution: Consider the closed path contribution from the contour shown in Fig. 8.

$$\oint_C \frac{dz}{1\ +\ z^4} \ = \ \int_0^R \frac{dx}{1\ +\ x^4} \ + \ \int_0^{\pi/2} \frac{i\ R\ e^{i\theta}\ d\theta}{1\ +\ R^4\ e^{4i\theta}} \ + \ \int_R^0 \frac{i\ dr}{1\ +\ r^4} \tag{1}$$

It is to the noted that the first and third integral on the right side of (1) combine to give us, in the limit $R\ \rightarrow\ \infty$, $(1\ -\ i)\ I$. Now, as far as the second integral, we note that

$$|\ \int_0^{\pi/2} \frac{i\ R\ e^{i\theta}\ d\theta}{1\ +\ R^4\ e^{4i\theta}}\ | \ < \ \int_0^{\pi/2} \frac{R\ d\theta}{R^4\ -\ 1} \ \rightarrow\ 0\ \text{ as }\ R\ \rightarrow\ \infty$$

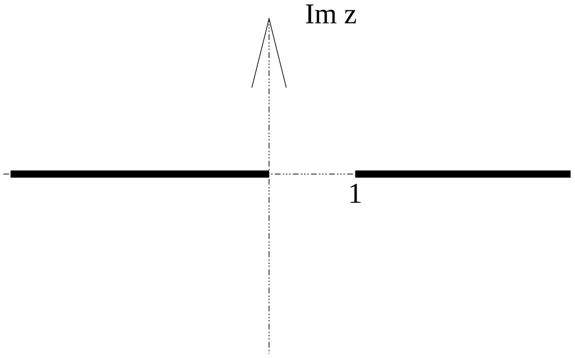

Figure 7: Branch cut for $f_{2,m}\ (z)$, as given by (17)

Thus, from (1), as $R \to \infty$

$$\oint_C \frac{dz}{1 + z^4} = (1 - i)\, I = 2\pi \text{ Residue at } z = e^{i\ \pi/4} \tag{2}$$

since $z = e^{i\pi/4}$ is the only contour within C. Residue at that point found to be $\frac{1}{4\ e^{i\ 3\ \pi/4}}$. So, from (2),

$$I = \frac{1}{4(1-i)}\ 2\ \pi\ i\ e^{-i\ 3\ \pi/4} = \frac{\pi}{2\sqrt{2}}$$

Remark: For integrands of the type $\frac{1}{1\ +\ x^p}$, it is prudent, to choose closed path with a leg on the real axis, another one a straight line, at an angle $2\pi/p$ with respect to the real axis while the third is a circular arc of radius R, which is found not to contibute as $R \to \infty$.

Lemma 7.3 Jordan's Lemma: If $|g(R\ e^{i\theta})| \to 0$ as $R \to \infty$ for θ in $[0,\ \pi]$, then

$$\int_{C_R} dz\ g(z)\ e^{i\alpha\ z} \ \to\ 0 \ \text{ as } R \to \infty$$

where $\alpha > 0$ and C_R is a semi-circle in the upper-half plane of radius R about the origin.

Proof: We note that on C_R, $z = R\ e^{i\theta} = R\ cos\ \theta + i\ R\ sin\ \theta$, $dz = i\ R\ e^{i\theta}\ d\theta$. Further for any ϵ, there exists R (independent of θ) so that $|g(R\ e^{i\theta}| < \epsilon$. Further, we note from the graph of $sin\ \theta$ that on the interval $[0,\ \pi/2]$, $sin\ \theta \geq \frac{2}{\pi}\theta$ Using these information, we get for sufficiently large R,

$$|\int_{C_R} dz\ g(z)\ e^{i\alpha\ z}\ | < \int_0^{\pi} R\ d\theta\ \epsilon\ e^{-\alpha\ R\ sin\ \theta} = 2\ \epsilon\ R \int_0^{\pi/2} d\theta\ e^{-\alpha\ R\ sin\ \theta}$$

$$< 2\ \epsilon\ R \int_0^{\pi/2} d\theta\ exp[-\alpha\ R\ \frac{2}{\pi}\theta] < \frac{\epsilon\ \pi}{\alpha}\ [1 - exp[-\alpha\ R]] \tag{3}$$

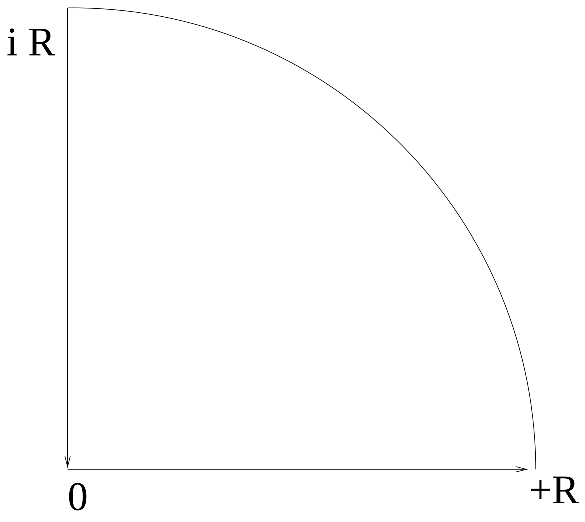

Figure 8: Closed contour C in (2)

Hence the lemma follows.

Corollary 7.2: If $|g(R\ e^{i\theta})|\ \rightarrow\ 0$ as $R\ \rightarrow\ \infty$ for θ in $[-\pi,\ 0]$, then

$$\int_{C_R} dz\ g(z)\ e^{-i\ \alpha\ z}\quad \rightarrow\ 0\ \text{ as } R\ \rightarrow\ \infty$$

where $\alpha\ >\ 0$ and C_R is a semi-circle in the lower-half plane of radius R about the origin.

Proof is very similar to Corollary 7.1 and is left to the reader.

Corollary 7.3: If $|g(R\ e^{i\theta})|\ \rightarrow\ 0$ as $R\ \rightarrow\ \infty$ for θ in $[-\pi/2,\ \pi/2]$, then

$$\int_{C_R} dz\ g(z)\ e^{-\alpha\ z}\quad \rightarrow\ 0\ \text{ as } R\ \rightarrow\ \infty$$

where $\alpha\ >\ 0$ and C_R is a semi-circle in the righthalf plane of radius R about the origin.

Proof is left to the reader.

Corollary 7.4: If $|g(R\ e^{i\theta})|\ \rightarrow\ 0$ as $R\ \rightarrow\ \infty$ for θ in $[\pi/2,\ 3\pi/2]$, then

$$\int_{C_R} dz\ g(z)\ e^{\alpha\ z}\quad \rightarrow\ 0\ \text{ as } R\ \rightarrow\ \infty$$

where $\alpha\ >\ 0$ and C_R is a semi-circle in the left-half plane of radius R about the origin.

Proof is left to the reader.

Corollary 7.5: If $|\ g(R\ e^{i\theta})\ |/R^{p-1}\ \rightarrow\ 0$, as $R\ \rightarrow\ \infty$ for θ in $[0,\ \pi/p]$, then

$$\int_{C_R} dz\ g(z)\ e^{i\ \alpha\ z^p}\quad \rightarrow\ 0\ \text{ as } R\ \rightarrow\ \infty$$

where $\alpha\ >\ 0$ and C_R is an arc of a circle of radius R about the origin, starting on the positive real axis and subtending an angle π/p at the origin.

Proof follows simply from Jordan's Lemma, once we carry out the substitution $z_1 = z^p$.

Exercise 2: Compute

$$I(a) \; = \; \int_0^\infty \frac{cos\ ax}{1 \; + \; x^2} \, dx$$

Solution: Note that, without loss of generality, $a \; > \; 0$. Further,

$$I(a) \; = \; \frac{1}{2} \int_{-\infty}^\infty \frac{cos\ ax}{1 \; + \; x^2} \, dx \; = \; \frac{1}{2} \, Re \, I_1(a) \tag{4}$$

where

$$I_1(a) \; = \; \int_{-\infty}^\infty \frac{e^{iax} \, dx}{1 \; + \; x^2} \tag{5}$$

Now consider the integral on the closed path C, shown in Fig. 9.

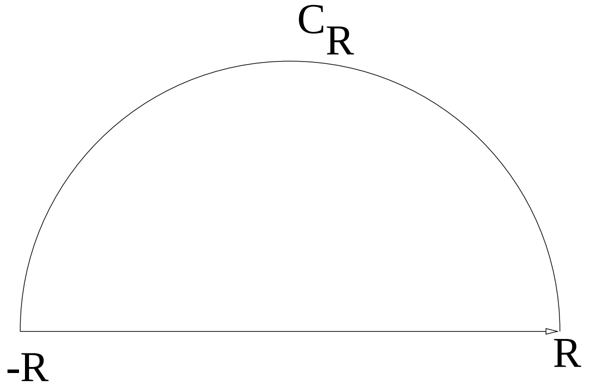

Figure 9: Closed contour C in (6)

The round trip contour integral can be broken up into parts:

$$\oint_C \frac{e^{i \ a \ z} dz}{1 \; + \; z^2} \, dz \; = \; \int_{-R}^R \frac{e^{i \ a \ x}}{1 \; + \; x^2} \, dx \; + \; \int_{C_R} \frac{e^{i \ a \ z}}{1 \; + \; z^2} \, dz \tag{6}$$

In the limit $R \; \to \; \infty$, from Jordan's Lemma, there is no contribution from $\int_{C_R}$. On the other hand from (6), in this limit, we get $I_1(a)$. So

$$I_1(a) \; = \; 2\pi i \text{ (Residue at } z = i) \; = \; 2 \, \pi \, i \, \frac{e^{-a}}{2i} \; = \; \pi \, e^{-a}$$

Hence $I(a) \; = \; \frac{\pi}{2} \, e^{-a}$.

Remark: Note if we were to consider the complex integral $\oint_C \frac{cos\ az}{1 \; + \; z^2} \, dz$ instead of the chosen integrand in (6), we could not chose a convenient contour, since Jordan type lemma is invalid for *cos*; it does not go to zero in the upper-half or lower-half plane at large distances.

Exercise 7.2 Evaluate

$$\int_0^\infty \frac{x^{-p}}{1 + x} \, dx \tag{7}$$

where $1 > p > 0$. In this case it is prudent to consider

$$\oint_C \frac{z^{-p}\ dz}{1\ +\ z} \quad \text{where} \quad \arg\ z \quad \text{is in } [0,\ 2\pi) \tag{8}$$

and C is the contour shown in Fig. 10.

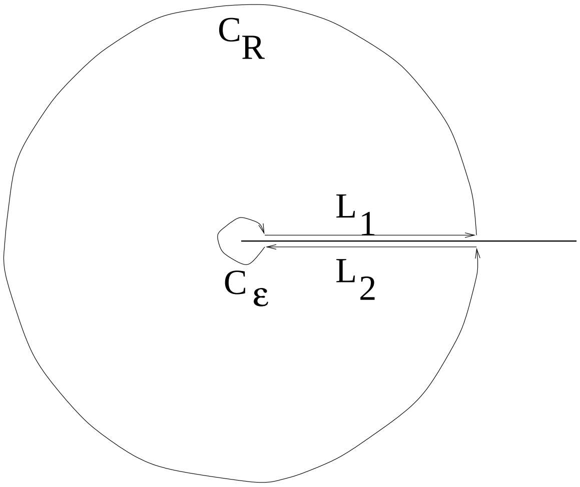

Figure 10: Contour C for Exercise 7.2

The contour C consists of straight segments L_1, L_2 and near circles of radius R and ϵ. Note that the cut from 0 to ∞ along the positive real axis is being avoided. Now, as $\epsilon \rightarrow 0$,

$$|\int_{C_\epsilon} \frac{z^{-p}\ dz}{1\ +\ z}| \ < \ |\int_{2\pi}^{0} i\ \epsilon^{1-p}\ \frac{d\theta\ e^{i(1-p)\theta}}{1\ +\ \epsilon\ e^{i\theta}}\ | \ \leq \ 2\ \pi\ \frac{\epsilon^{1-p}}{|1-\epsilon|} \ \rightarrow \ 0$$

Further, as $R \rightarrow \infty$,

$$|\int_{C_R} \frac{z^{-p}\ dz}{1\ +\ z}| \ < \ |\int_{0}^{2\pi} i\ R^{1-p}\ \frac{d\theta\ e^{i(1-p)\theta}}{1\ +\ R\ e^{i\theta}}\ | \ \leq \ 2\ \pi\ \frac{R^{1-p}}{R-1} \ \rightarrow \ 0$$

Further, we note that on the straightline segment L_1,

$$\int_{L_1} \frac{z^{-p}\ dz}{1\ +\ z} \ = \ \int_{\epsilon}^{R} \frac{x^{-p}\ dx}{1\ +\ x}$$

while on L_2,

$$\int_{L_2} \frac{z^{-p}\ dz}{1\ +\ z} \ = \ \int_{R}^{\epsilon} \frac{r^{-p}\ e^{-i2\pi p}\ dr}{1\ +\ r} \ = \ -\ e^{-2\pi i p} \int_{L_1}$$

Therefore, as $\epsilon \rightarrow 0$ and $R \rightarrow \infty$,

$$\oint_C \frac{z^{-p}\ dz}{1\ +\ z} \ = \ (1 - e^{-2\pi i p})\ I \ = \ 2\pi\ i\ \left(\text{residue at } z\ =\ e^{i\pi}\right) \ = \ 2\ \pi\ i\ e^{-i\pi p}$$

Therefore,

$$I = 2\ \pi\ i\ \frac{e^{-i\pi p}}{1\ -\ e^{-2\ \pi\ i\ p}}\ =\ \frac{\pi}{sin\ \pi\ p}$$

Remark: The method of calculating above also tells us that I exists in a limiting process involving $\int_{\epsilon}^{R}\ \frac{x^{-p}\ dx}{1+x}$, with $\epsilon\ \rightarrow\ 0$ and $R\ \rightarrow\ \infty$.

Remark: It is to be noted from this exercise that if we have contour integration around a small circular arc of radius ϵ around a branch point point z_0, where the integrand is like $(z - z_0)^{-p}$ for $p\ <\ 1$, then there is no contribution from such an integral as $\epsilon\ \rightarrow\ 0$. This fact will simplify future calculations of such integral.

Remark: The contour C in Fig. 10 can be used to calculate

$$\int_0^{\infty}\ \frac{dx}{\text{Polynomial in x of degree}\ \geq\ 2}$$

by considering

$$\oint_C\ \frac{\log\ z\ dz}{\text{Polynomial in}\ z}$$

Exercise 7.3 Compute

$$I(a)\ =\ \int_0^{\infty}\ e^{i\ a\ x^2}\ dx \tag{9}$$

Solution: Consider

$$\oint_C\ dz\ e^{i\ a\ z^2} \tag{10}$$

where C is the contour shown in Fig. 11.

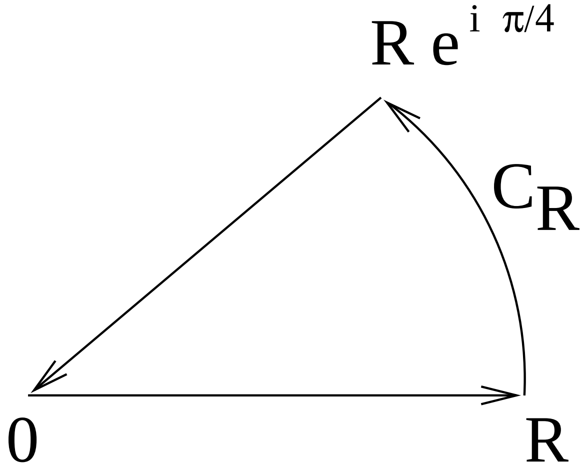

Figure 11: Contour for exercise 10.3

Note from a corollary of Jordan's lemma, the contribution for $\int_{C_R}$ must go to zero as $R\ \rightarrow\ \infty$. Also, the contribution from the straightline segment along the real axis as $R\ \rightarrow\ \infty$

is clearly I. Also, on the straightline, where arg $z = \pi/4$, since $i\, a\, z^2 = i\, a\, (r\, e^{i\pi/4})^2 = -a\, r^2$,, the contribution, as $R \to \infty$ is

$$\int_R^0 dr\ e^{i\pi/4}\ e^{-a\ r^2} \quad \to \quad -\, e^{i\pi/4}\ \frac{\sqrt{\pi}}{2\sqrt{a}} \tag{11}$$

from well known result about area under a Gaussian curve. Since (10) involves an integral of an analytic function without any singularity on and within C, the $\oint_C = 0$. Combining with (11), we therefore get $I(a)$ to be negative of the answer in (11). So,

$$I(a) \;=\; e^{i\pi/4}\ \frac{\sqrt{\pi}}{2\sqrt{a}} \tag{12}$$

Remark: If a in exercise 7.3 were negative, we would have chosen a contour to return to the origin at an angle $-\pi/4$, so as to allow use of Jordan's lemma for $a < 0$ to conclude no contribution from the circular arc.

Remark: On taking the real and imaginary parts of the (9), it is possible to compute $\int_0^\infty\ cos\ a\ x^2\ dx$ and $\int_0^\infty\ sin\ a\ x^2\ dx$.

Exercise 7.4 Compute

$$I \;=\; ⨍_{-\infty}^{\infty} \frac{e^{ix}}{x}\ dx \tag{13}$$

Solution: It is prudent to choose a contour as shown in Fig. 12.

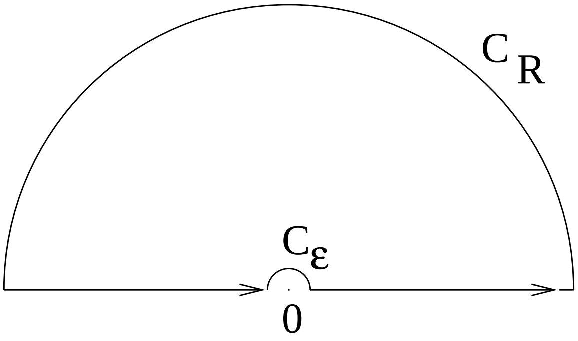

Figure 12: Contour C for Exercise 10.4

It is clear there are no singularities within the contour, So,

$$0 \;=\; \oint_C \frac{e^{iz}}{z}\ dz \;=\; = \;\left(\int_{-R}^{-\epsilon} \;+\; \int_{\epsilon}^{R}\right) \frac{e^{ix}\ dx}{x} \;+\; \int_{C_R} \frac{e^{iz}}{z}\ dz \;+\; \int_{C_\epsilon} \frac{e^{iz}}{z}\ dz \tag{14}$$

From Jordan's lemma, there is no contribution from C_R as $R \to \infty$. Further, as $\epsilon \to 0$, since on C_ϵ, $e^{i\ z} \to 1$,

$$\int_{C_\epsilon} \;\to\; \int_\pi^0 \frac{i\ \epsilon\ e^{i\theta} d\theta}{\epsilon\ e^{i\theta}} = -\pi\ i \tag{15}$$

Therefore, from (14), in the limit $R \to \infty$, $\epsilon \to 0$,

$$\left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \right) \frac{e^{ix}\, dx}{x} \to \pi\, i \tag{16}$$

But the integral in (16) in this limit is precisely I.

Remark: By taking the imaginary part of (16), and using the even nature of the integrand, it is clear that

$$\int_0^\infty \frac{sin\ x}{x} = \frac{\pi}{2}$$

Note in this case, the principal value integral reduces to a regular integral, since the integrand has a finite limit as $x \to 0$.

Remark: On taking the real part of (16), it follows that

$$⨍_{-\infty}^{\infty} \frac{cos\ x}{x} = 0$$

Exercise 7.5 For $-\pi < \alpha < \pi$,

$$I(\alpha) = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{cosh\ \pi\ x}\, dx \tag{17}$$

Solution: It is to be noted that if the integrand in (17) is denoted by $f(x)$, then it has the property (using representation of cosh interms of exp) that

$$f(x + i) = -\, e^{i\alpha}\, f(x) \tag{18}$$

From property (18), it is prudent to consider

$$\oint_C \frac{e^{\alpha z}}{cosh\ \pi\ z}\, dz \tag{19}$$

where C is a rectangular contour, shown in Fig. 13.

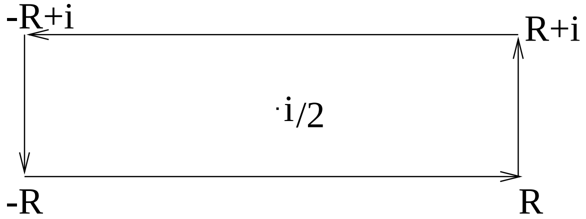

Figure 13: Contour C for Exercise 10.5

Since the integrand

$$f(\pm R + iy) = \frac{e^{\pm \alpha R}\, e^{i\alpha y}}{e^{\pi R}\, e^{i\pi y} + e^{-\pi R}\, e^{-i\pi y}}$$

is bounded in absolute value by $\frac{e^{|\alpha|R}}{e^{\pi R}-1}$, which tends to zero as $R \to \infty$. In this limit, there is no contribution from the vertical segments of C (since the integration is over a fixed range in y, independent of R). Thus, in this limit, using (18), we get

$$\oint_C = (1 + e^{i\ \alpha})\ I \tag{20}$$

Now consider the residues within the contour C. This is where $cosh\ \pi\ z = 0$. From the exponential representation, it is clear that this happens where $e^{2\pi z} = -1$, i.e. $z = i\ (1/2 + n)$ for integer n. The only one inside the contour is at $i/2$, where the residue is

$$\lim_{z \to i/2} (z - i/2)\ f(z) = \frac{e^{i\alpha/2}}{\pi\ sinh\ (i\ \pi/2)} = \frac{e^{i\alpha/2}}{i\ \pi} \tag{21}$$

So, from residue theorem and (20),

$$I = (1 + e^{i\alpha})^{-1}\ 2\pi i\ \frac{e^{i\alpha/2}}{i\ \pi} = \frac{1}{cos\ (\pi\alpha/2)} \tag{22}$$

Remark: The result in (22) is independent of the sign of α. Hence it follows that

$$\int_{-\infty}^{\infty} \frac{cosh\ \alpha x}{cosh\ \pi\ x}\ dx = \frac{1}{cos\ (\alpha/2)}$$

and

$$\int_{0}^{\infty} \frac{cosh\ \alpha x}{cosh\ \pi\ x}\ dx = \frac{1}{2\ cos\ (\alpha/2)}$$