Week 4 notes

More examples on contour integration

Exercise 8.1: Compute

$$H(a) \ = \ \int_{-\infty}^{\infty} \frac{e^{-i \ a \ z}}{\sqrt{z+i} \ + \ \sqrt{z+3i}} \, dz \tag{1}$$

where $\sqrt{z+i} = |z+i|^{1/2} \ exp[i \ arg(z+i)/2]$, $\sqrt{z+3i} = |z+3i|^{1/2} \ exp[\ i \ arg(z+3i)/2]$, and $arg \ (z+i)$, $arg \ (z+3i)$ are each in $[-\pi/2, \ 3\pi/2)$.

Solution: Clearly, for $a \ < \ 0$, $H(a) \ = \ 0$ since on closing the contour from above, Jordan's Lemma applies. We now consider $a \ > \ 0$. Consider the closed contour C, as shown in Fig. 1.

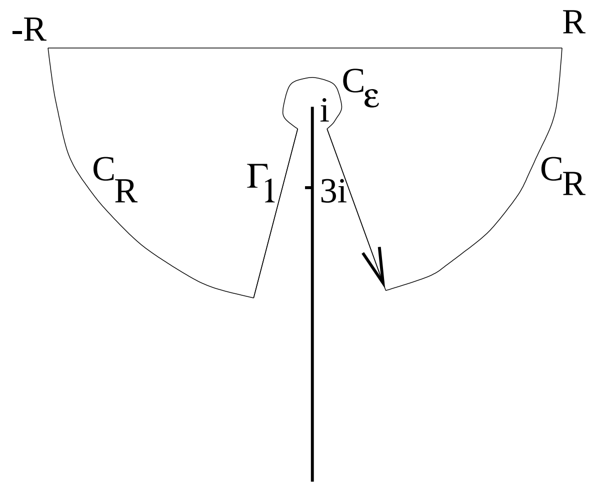

Figure 1: Closed contour C in eqn. 2

Then, since the integrand has no singularity within the contour,

$$\oint_C \frac{e^{-i \ a \ z}}{\sqrt{z+i} \ + \ \sqrt{z+3i}} \, dz \ = \ 0 \tag{2}$$

Using arguments as in Jordan's lemma, there is no contribution from the circular arc portion of the contours, as $R \ \to \ \infty$. We are only left with the contribution from the segment Γ_1 of the contour. For analyzing this contribution, it is prudent to get rid of the squareroot in the integrand in the denominator, by multiplying both the numerator and the denominator by $\sqrt{z+3i} - \sqrt{z \ + \ i}$. Then it is clear from (2) that as $R \ \to \ \infty$

$$H(a) = \ -\frac{i}{2} \int_{\Gamma_1} e^{-i \ a \ z} \sqrt{z+3i} \, dz \ + \ \frac{i}{2} \int_{\Gamma_1} e^{-i \ a \ z} \sqrt{z+3i} \, dz \tag{3}$$

Consider

$$H_1(a) \ = \ \int_{\Gamma_1} e^{-i \ a \ z} \sqrt{z+3i} \, dz \ = \ \int_{\Gamma_2} e^{-i \ a \ z} \sqrt{z+3i} \, dz \tag{4}$$

where Γ_2 is the contour shown in Fig. 2.

The latter equality in (4) follows from contour deformation of Γ_1 into Γ_2, as it crosses no singularities of the integrand. The local contribution from the circular arc C_ϵ around "weak" (*i.e.* with power > -1) branch point $z = -3\,i$ vanishes as $\epsilon \to 0$, following arguments from last lecture. On using $z + 3i = r\, e^{i3\pi/2}$ on the left leg of Γ_2 and $z + 3i = r\, e^{-3i\pi/2}$ on the right leg, we obtain in the limit $\epsilon \to 0$ and $R \to \infty$,

$$H_1(a) = 2\, e^{-3\, i\pi/4} \int_0^\infty dr\, r^{1/2}\, e^{-a\,(3\,+\,r)} = -2\, e^{-3a\,-\,3\,i\pi/4} \frac{\partial}{\partial a}\left[\int_0^\infty dr\, r^{-1/2}\, e^{-a\,r}\right]$$

$$= \frac{\sqrt{\pi}}{a^{3/2}}\, e^{-3a\,-\,3\,i\pi/4} \tag{5}$$

Going through a very similar process it is clear that

$$H_2(a) = \int_{\Gamma_1} e^{-i\,a\,z}\, \sqrt{z+3i}\, dz\, dz = = \frac{\sqrt{\pi}}{a^{3/2}}\, e^{-a\,-\,3\,i\pi/4} \tag{6}$$

From (3), it is clear that

$$H(a) = \frac{i}{2}\left[H_2(a) - H_1(a)\right] = \frac{\sqrt{\pi}}{2}\, a^{-3/2}\; e^{-i\pi/4}\left(e^{-a} - e^{-3a}\right) \tag{7}$$

Remark: Sometimes, in computing an integral, it is suitable to take derivative with respect to some parameter, as in (5) above, in an effort to simplify the integral. For instance, in calculating

$$I(t) = \int_{-i\,\infty}^{i\,\infty} \frac{e^{st}}{(s+a)\sqrt{s}}\, ds \tag{8}$$

we note that $I(t) = e^{-at}\, I_1(t)$, where

$$I_1(t) = \int_{-i\,\infty}^{i\,\infty} \frac{e^{(s+a)t}}{(s+a)\sqrt{s}}\, ds \tag{9}$$

Then, if it is permitted to take derivative with respect to t inside the integral, then we note that

$$I_1'(t) = \int_{-i\,\infty}^{i\,\infty} \frac{e^{(s+a)t}}{\sqrt{s}}\, ds = e^{at} \int_{-i\,\infty}^{i\,\infty} \frac{e^{st}}{\sqrt{s}}\, ds \tag{10}$$

which is a lot easier integral to calculate. In order to find $I_1(t)$ and hence $I(t)$, we integrate the answer in (10). To determine the constant of integration, we note that the integral for $I_1(0)$ in (9) can be calculated easily, through a change in variable ($s_1 = \sqrt{s}$). However, care must be taken to differentiate under under the integral sign and be certain that this operation is valid.

Exercise 8.2: Evaluate

$$f(z) = \int_{-1}^{1} \frac{dt}{(t-z)\,\sqrt{1-t^2}} \tag{11}$$

for z not in $[-1, 1]$, and squareroot interpreted in the usual sense of a real positive number.
Solution: Consider

$$\oint_C \frac{1}{(\zeta - z)\ (\zeta^2 - 1)^{1/2}}\ d\zeta \tag{11}$$

for a contour C enclosing the cut between -1 and +1, but not enclosing $\zeta = z$ (Fig. 3). Here we choose $arg\ (\zeta\ \pm 1)$ in $(-\pi,\ \pi]$.

Note that since the contribution from the small circular arc contours C_{ϵ_1} and C_{ϵ_2} cannot contribute in the limit as radius $\epsilon\ \rightarrow\ 0$, since each is around a weak branch point, with power $>\ -1$. Note that on the straight segment of Fig. 3 above the branch cut, $arg\ (\zeta - 1)\ =\ \pi$, where as $arg\ (\zeta + 1)\ =\ 0$, where as below the cut, $arg\ (\zeta - 1)\ =\ -\pi$, $arg\ (\zeta + 1)\ =\ 0$. Therefore, as $\epsilon\ \rightarrow\ 0$,

$$\oint_C \frac{1}{(\zeta - z)\ (\zeta^2 - 1)^{1/2}}\ d\zeta\ =\ -2\ i\ \int_{+1}^{-1} \frac{1}{(t - z)\ (1 - t^2)^{1/2}}\ dt\ =\ 2\ i\ f(z) \tag{12}$$

Now it is clear from Cauchy's integral formulae for a multiply connected (see Fig. 4) region that

$$\left(\oint_{C_R} - \oint_C \right) \frac{1}{(\zeta - z)\ (\zeta^2 - 1)^{1/2}}\ d\zeta\ =\ 2\pi i\ (\text{residue at}\ \zeta\ =\ z)$$

$$=\ \frac{2\ \pi\ i}{(z^2 - 1)^{1/2}} \tag{13}$$

However, on C_R, as $R\ \rightarrow\ \infty$

$$|\int \frac{1}{(\zeta - z)\ (\zeta^2 - 1)^{1/2}}\ d\zeta|\ \leq\ \int_0^{2\pi} \frac{R}{(R - |z|)\ (R^2 - 1)^{1/2}} d\theta\ \rightarrow\ 0 \tag{14}$$

Therefore, from (12) and (13), it follows that

$$f(z)\ =\ -\ \frac{\pi}{(z^2 - 1)^{1/2}} \tag{15}$$

Remark: In other exercises of this type, we may find that the contribution from $\oint_{C_R}$ tends to some known constant as $R\ \rightarrow\ \infty$. Even in that case, if there is no branch point at ∞ and no cuts going there, it is useful to expand out an initial closed contour C around a cut to a large contour C_R.
Remark: For integrals involving periodic function over a period (or something that can be extended to a period), it is useful to relate to a closed complex contour through a change in variable. Here is an example below.
Exercise 8.3: Compute

$$J = \int_0^{\pi} \frac{d\theta}{1\ +\ \epsilon\ cos\ \theta} \tag{16}$$

for $0 \ < \ \epsilon \ < \ 1$

Solution: Note that since the integrand is even,

$$J = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{1 \ + \ \epsilon \ cos \ \theta} \tag{17}$$

Substituting $z = e^{i\theta}$, $cos \ \theta \ = \ \frac{1}{2} \ (z + 1/z)$ and $d\theta \ = \ dz/(iz)$. So, the integral (17) becomes

$$J = \frac{1}{2}\oint_{|z|=1} \frac{dz}{iz\left(1 \ + \ \frac{\epsilon}{2}(z+1/z)\right)} \ = \ \frac{1}{i\epsilon}\oint_{|z|=1} \frac{dz}{z^2 \ + \ \frac{2}{\epsilon}z \ + \ 1} \tag{18}$$

Denoting the two roots of the quadratic in the denominator of the integrand in (18) by z_1 and z_2, it is clear

$$z_{1,2} \ = \ -\frac{1}{\epsilon} \ \pm \ \sqrt{\frac{1}{\epsilon^2} - 1} \tag{19}$$

The product of the roots $z_1 \ z_2 \ = \ 1$ and only z_1 is inside a contour of radius 1. Note that the denominator in the integrand in (18) can also be written as $(z - z_1)(z - z_2)$. So,

$$J \ = \ 2\pi i \ \frac{1}{i\epsilon} \lim_{z \ \to \ z_1} \frac{(z - z_1)}{(z - z_1) \ (z - z_2)} \ = \ \frac{\pi}{\sqrt{1 - \epsilon^2}}$$

Remark: Actually the answer above holds for $-1 \ < \ \epsilon \ \leq \ 0$ as well.

Remark: Contour integation can also be used to compute infinite sum and product representation of meromorphic functions (analytic functions, whose only singularities are poles).

Example 8.4: Consider

$$P_N(z) \ = \ \frac{1}{2\pi i}\oint_{C_N} \frac{tan \ (\pi \ \zeta) \ d\zeta}{\zeta(\zeta - z)} \tag{20}$$

where C_N is a square contour with corners at $N(\pm 1 \ \pm i)$, where N is a positive integer that will eventually be made to tend to ∞ (See Fig. 5).

Now, using the exponential representation of *sin* and *cos*, it can be easily be shown that $tan \ [\pi \ (\pm N \ + i \ y)]$ and $tan \ [\pi \ (x \ + \pm i \ N)]$ are bounded by a constant independent of N. Thus, it is clear from (20) that as $N \ \to \ \infty$, $P_N \ \to \ 0$ (leave the details for the reader to fill up). But P_N encloses singularities at $\zeta = z$ and at $\pm\zeta_n$, where $\zeta_n \ = \ (n + 1/2)$ for $N - 1 \ \geq \ n \ \geq \ 0$ At $\zeta = z$ the residue is clearly $\frac{tan \ \pi z}{z}$. At $\zeta = \pm\zeta_n$, the residue is clearly $-\frac{1}{\pi(\pm\zeta_n) \ (\pm\zeta_n \ - \ z)}$. Collecting all the residues

$$P_N \ = \ \frac{tan \ \pi z}{z} - \frac{1}{\pi}\sum_{n=0}^{N-1}\left\{ \frac{1}{z - \zeta_n} \ \frac{1}{z + \zeta_n}\right\}$$

Taking the limit $N \ \to \ \infty$, we get

$$tan \ \pi z \ = \ \frac{2 \ z}{\pi}\sum_{n=0}^{\infty} \frac{1}{\zeta_n^2 - z^2} \tag{21}$$

where $\zeta_n = (n + 1/2) \ \pi$.

Lecture 9: Conformal Mapping

Remark: Please check O. Costin's Complex Variable notes, pages 34-67. The treatment here is similar, though not as extensive.

Remark: Laplace equation is common in the physical sciences, electro-statics, potential flow in fluid mechanics, steady-state heat diffusion problems to name a few:

$$\Delta \phi \ \equiv \ \phi_{xx} \ + \ \phi_{yy} \ = \ 0 \tag{1}$$

for $x + iy \in D \subset \mathbf{C}$, with either *Dirichlet* or *Neumann* boundary conditions on $\Gamma \equiv \partial D$:

$$\phi = g(t) \quad \text{(Dirichlet)}, \quad \text{or} \quad \frac{\partial \phi}{\partial n} \ = \ g(t) \ \ \text{(Neumann)} \ \ , \tag{2}$$

where $a \le t \le b$ parametrizes Γ and $\frac{\partial}{\partial n} \equiv \mathbf{n} \cdot \nabla$ denote normal derivative on Γ, and g is a known function For now, we will assume D to be simply connected.

When D is not finite, additional conditions on ϕ have to be appended at ∞. In electro-statics for instance, electric field $\mathbf{E} = -\nabla \phi$ In a region without charges, $\nabla \cdot \mathbf{E} = 0$; so $\Delta \phi = 0$.

Uniqueness of solution to (1) and (2): For Dirichlet problem, we already know from application of maximum modulus theorem, that ϕ attains a maximum or minium on Γ. Therefore, uniqueness of solution to (1) satisfying (2) for *Dirichlet* problem in a finite domain if solution is assumed to be continuous upto the boundary. Similar arguments using conjugate harmonic function show that the solution to the *Neumann* problem is unique upto an additive constant iff $\int_a^b g(t) dt = 0$, when t is the arc-length parametrization.

Existence of solution to (1) and (2):

When $D = D_1$, the unit circle, then Poisson-integral formula explicitly provides solution to the boundary value problem (1)-(2). In an effort to find solution to (1)-(2), we seek to find an analytic 1-1 mapping (analytic homeomorphism) $f : D_1 \to D$ since Laplace's equation is invariant under such mapping as proved in the following:

Lemma: If ϕ is harmonic in (x, y) for $z = x + iy \in D$, a simply connected domain, and $h : D \to D_1$ is an analytic homeomorphism, then for $z = f^{-1}(\zeta) = u(\xi, \eta) + iv(\xi, \eta)$, where $\zeta = \xi + i\eta \in D_1$,

$$\Phi(\xi, \eta) = \phi(u(\xi, \eta), v(\xi, \eta))$$

is harmonic in (ξ, η).

Proof: Define $g(z) = g(x + iy) = \phi(x, y) + i\psi(x, y)$, where g is analytic in D. Since $x + iy = z = f^{-1}(\zeta) = f^{-1}(\xi + i\eta)$, it follows that

$$\Phi(\xi, \eta) = \Re \left\{ g \left(f^{-1}(\zeta) \right) \right\}$$

Since composition of analytic functions is analytic, $g(f^{-1}(\zeta))$ is an analytic function of ζ, implying Φ to be harmonic in (ξ, η).

Remark We now turn to the question of existence of such analytic homeomorphism. This is guaranteed by the Riemann mapping theorem, as stated below for simply connected domains.

Riemann Mapping Theorem: Assume D is a simply connected open domain with more than one boundary point. Then there exists an analytic homeomorphism $f : D \to D_1$. This map is unique if some $z = z_0 \in D$ is required to to correspond to $0 \in D_1$ with $f'(z_0)$ real and positive.

Remark: There are similar theorems for mappings between multiply connected regions, though with additional restrictions between the domains. In general, any two domain are *conformally equivalent* if there is an analytic homeomorphism between the two.

Exercise 9.1: Let Γ_1 and Γ_2 be the half-lines arg $z = 0$ and arg $z = \frac{3\pi}{2}$, respectively, and let D be the domain $0 < arg\ z < \frac{3\pi}{2}$ (See Fig. 6). Find a bounded solution $\phi(x, y)$ to Laplace's equation $\Delta\, \phi = 0$, subject to the conditions $\phi = a$ on Γ_1 and $\phi = a + k$ on Γ_2, where a and k are real constants.

Solution: It is to be noted that if we apply the transformation

$$\zeta = h(z) = \frac{2}{3\pi}\, ln\ z \ \text{ with } \ 0 < arg\ z < \frac{3\pi}{2} \tag{7}$$

to the domain D, then the corresponding image in the ζ plane corresponds to the region D' between two parallel straight lines $Im\ \zeta = 0$ and $Im\ \zeta = 1$ (see Fig. 7). It is to be noted that the inverse of the transformation in (7) is given by

$$z = f(\zeta) = exp\left[\frac{3\pi}{2}\,\zeta\right] \tag{8}$$

The given boundary conditions in D translate to

$$\Phi = a \ \text{ on } \eta = 0 \ , \ \text{ and } \Phi = a + k \text{ on } \eta = 1 \tag{9}$$

In the domain D', a solution $\Phi(\xi, \eta)$ is sought in the form $\Phi(\eta)$. In that case, Laplace's equation reduces to

$$\Phi_{\eta\eta} = 0 \tag{10}$$

The general solution to (10) is in the form $c_1 + c_2\, \eta$. Since it must satisfy the boundary conditions (9), one finds

$$\Phi = a + k\, \eta = Re\, (-i\, k\, (\xi + i\eta) + a) \tag{11}$$

This means that at least one solution for the complex potential is

$$\chi(\zeta) = a - i\, k\, \zeta \tag{12}$$

Therefore

$$\Omega(z) \;=\; \chi(h(z)) \;=\; a \;-\; \frac{2ik}{3\pi}\; ln\; z \;\;, \tag{13}$$

implying that

$$\phi \;=\; \frac{2k}{3\pi}\; arg\; z \;+\; a \;\; \text{with } 0 \;<\; arg\; z \;<\; \frac{3\pi}{2} \tag{14}$$

We now argue that the solution (14) is the unique solution. Suppose we had two different χ, χ_1 and χ_2, whose real part Φ_1 and Φ_2, each satisfy Laplace's equation and (9). Then $\chi_3(\zeta) \;= \chi_1(\zeta) \;-\; \chi_2(\zeta)$ is a bounded analytic function within the strip D', with vanishing real part on $\partial D'$. From repeated application of Schwarz reflection principle, χ_3 is an *entire* function of ζ, as is e^{χ_3}. Boundedness assumption on solution ϕ, implies e^{χ_3} is bounded, *i.e* it must be a constant. However, since the real part of χ_3 is zero on the Γ', which extends to ∞, χ_3 is generally some imaginary constant. Thus, (14) is indeed unique solution to the problem posed.

Remark: If we did not include the boundedness condition on ϕ, then the solution is not unique, since one can add to (13) $\Lambda_j(z) \;=\; i\; z^{2j/3}$ for any nonzero integer j and yet given satisfy boundary conditions on $\phi \;=\; Re\; \Omega$. However, using Phragmen-Lindeloff principle, we can weaken the assumption on boundedness of ϕ to some degree and yet obtain a unique solution.

Remark: A problem of considerable interest in potential theory to detemine *Green's function* $g(x, y; x_0, y_0)$ for a simply connected domain D. In 2-D, it is defined to be a single-valued harmonic function of (x, y), except only at (x_0, y_0), where

$$g(x, y; x_0, y_0) - \frac{1}{2\pi} \ln\; |z - z_0| \;=\; O(1) \tag{15}$$

where $z = x \;+\; iy$ and $z_0 \;=\; x_0 \;+\; i\; y_0$. For *Dirichlet problem*, we require $g \;=\; 0$ on the boundary $\Gamma = \partial D$. From Riemann mapping theorem, there exists $f : D_1 \to D$ (D_1 unit disk around the origin) with $f(0) = z_0$; this mapping is unique upto a rotational degree inherent in choosing $\arg f'(0)$. We denote the inverse mapping $h(z; z_0) \;=\; f^{-1}(z)$, the additional parameter z_0 in h indicates dependence of mapping f on $z_0 = f(0)$. Since $h(z) = h'(z_0)(z - z_0) + \text{regular terms}$, it follows $\Re \log h = \log |z - z_0| + \text{harmonic}$. Further, since h is 1-1 and has no other zeros in D, it follows that $\log h$ is analytic elsewhere and so $\Re \log h(x + iy; z_0)$ is harmnonic in D and takes 0 value on ∂D, $|h(z)| = 1$. Hence

$$g(x, y; x_0, y_0) = \frac{1}{2\pi} \log |h(x + iy; x_0 + iy_0)| \tag{16}$$

Lecture 10: Conformal Transformations

Remark: The examples from last time illustrate the need to find an analytic function $f(\zeta)$ that maps some standard domain D' into the region D in a one-one manner. We now discuss such mapping functions and their properties without reference to Laplace's equation.

Definition: A transformation $f : D' \to D$ is said to be *conformal* at a point $\zeta_0 \in D'$ if the angle of intersection between any two smooth curves at that point as well as their relative orientation is preserved. The transformation is conformal in D', then this property is valid at every point in D' (See Fig. 8).

Lemma 10.1: An function f analytic at ζ_0 is *conformal* at ζ_0 if $f'(\zeta_0) \neq 0$.

Proof: Consider two differentiable curves $\Gamma(t)$ and $\gamma(s)$ intersecting at $s = 0, t = 0$, such that $\zeta_0 = \Gamma(0) = \gamma(0)$. Consider the effect of the mapping $z = f(\zeta)$. The angle between the mapped curves $f(\Gamma)$, $f(\gamma)$ at $z_0 = f(\zeta_0)$ is clearly

$$\arg\left[\frac{d}{dt} f(\Gamma(t))\right]_{t=0} - \arg\left[\frac{d}{ds} f(\gamma(s))\right]_{s=0} = \arg\left[f'(\zeta_0)\Gamma'(0)\right] - \arg\left[f'(\zeta_0)\gamma'(0)\right] = \arg\left[\Gamma'(0)\right] - \arg\left[\gamma'(0)\right]$$

Therefore the angle and orientation are both preserved.

Remark: The converse is also true. If a map f on a domain $D \in \mathbf{C}$ is conformal and orientation preserving, then it can be proved that f is analytic with $f' \neq 0$.

Transformation of arc-length and area under conformal map f:

Consider curve $\Gamma(t)$ in the last Lemma for $a \leq t \leq b$. The infinitesimal arc-length change on the transformed curve $f(\Gamma)$ is clearly $|df(\Gamma(t))| = |f'(\Gamma)||\Gamma'|dt$. Therefore arclength is of transformed curve $f(\Gamma)$ is

$$\int_a^b |f'\left((\Gamma(t)\right)||\Gamma'(t)|dt$$

So, small length elements at ζ_0 are changed by factor of $|f'(\zeta_0)$. Using Cauchy Riemann conditions, it is clear that if

$$z = x + iy = f(\zeta) = f(\xi + i\eta) = x(\xi, \eta) + iy(\xi, \eta),$$

then area element $d\xi d\eta$ in the ζ plane is transformed to

$$|f'(\zeta_0)|^2 d\xi d\eta = \left|\frac{\partial\ (x, y)}{\partial\ (\xi, \eta)}\right| d\xi d\eta \tag{4}$$

Remark: Note that from Taylor expansion, that is if $f'(\zeta_0) = 0$, then

$$\delta z \ \to \ \frac{f^{(m)}(\zeta_0)}{m!}\,(\delta\zeta)^m$$

where $f^{(m)}(\zeta_0) \neq 0$ is the first nonvanishing derivative. Further, as $\delta\zeta \ \to \ 0$,

$$arg\ \delta z \ \to \ m\ arg\ \delta\zeta \ + \ arg\ f^{(m)}(\zeta_0) \tag{6}$$

The angle between any two infinitesimal line elements at the point ζ_0 is increased by the factor m, and therefore the transformation is not conformal at ζ_0.

Remark: Thus, from Lemma 10.1 and the above remark, a transformation $z = f(\zeta)$ for an analytic function f is conformal at ζ_0 if and only if $f'(\zeta_0) \neq 0$.

Remark: In an earlier lecture, we noted that for an analytic function f, $f'(\zeta_0) \neq 0$ implies that f is locally invertible and that the inverse function is analytic at $z_0 = f(\zeta_0)$. However, the condition $f' \neq 0$ at each point of D' does not necessarily mean that f is a one-one map between D' and D. For instance, $f(\zeta) = e^{\zeta}$ is locally invertible (since $f' \neq 0$); yet if we take D' to be the region $\{\zeta \mid 0 \leq Im\, \zeta < 4\pi\}$, then there exists more than one point in D' (actually, exactly two) with the same image in D.

Lemma 10.2: Assume $f : D' \to D$ is an analytic homeomorphism. Then f and f^{-1} is *conformal* at each interior point.

Proof: Assume, other wise; *i.e.* that there exists point $\zeta_0 \in D'$ such that $f'(\zeta_0) = 0$. Denote $z_0 = f(\zeta_0)$. Let m be the smallest integer with $f^{(m)}(z_0) \neq 0$. Since the zeros of a nontrivial f are isolated, there exists ϵ such that or $0 < |\zeta - \zeta_0| < \epsilon$, $f(\zeta) - z_0 \neq 0$. Choose a contour C around ζ_0 and define

$$\delta = \min_C |f(\zeta) - z_0|$$

Now for $|z - z_0| < \delta$, it is clear from Rouche's theorem that the functions $f(\zeta) - z_0$ and $f(\zeta) - z_0 + (z_0 - z)$ have exactly the same number of zeros, which is m (counting multiplicity) > 1. This is a contradiction since f is given to be an analytic homemorphism. So the mapping is conformal at all interior points of D'. The same arguments can be made for $f^{-1} : D \to D'$ since it is also an analytic homeomorphism.

Remark: Note in the above proof that if $f'(\zeta_0) = 0$, there are many ($m > 1$) branches of an inverse in a neighborhood of $z = z_0$; i.e. $z = z_0$ is a branch point of an inverse function of f.

Lemma 10.3: Assume C is a piecewise smooth oriented simple closed curve enclosing a simply connected domain D and f analytic in D and continuous in $\bar{D}$. If $C' = f(C)$ is also a simple closed curve, then f maps D into the interior of C' in a conformal, one to one manner and C' has the same orientation as C.

Proof: We parametrize the closed path C by a real parameter t in [0,1], and assume it is traversed positively (counter-clockwise) as t increases. Since C is a simple closed curve, this implies that for $t_1 < t_2$, $\zeta(t_1) \neq \zeta(t_2)$ except when $(t_1, t_2) = (0, 1)$. From given condition, $C' = f(C)$ is is a simple closed curve as well; or otherwise at least two distinct points on C would map to the same point on C'. Let C' contain the domain D'. Take a point $w_0 \in D'$.

We will show that $f(\zeta)$ attains the value w_0 exactly once in D. We note

$$\frac{1}{2\pi i}\oint_C \frac{f'(\zeta)d\zeta}{f(\zeta)-w_0} = \frac{1}{2\pi i}\int_0^1 \frac{f'(\zeta(t))\zeta'(t)\ dt}{f(\zeta(t))-w_0} \tag{7}$$

Now, since $w(t) = f(\zeta(t))$ is the parametrization of the simple closed C', it is clear that (7) equals

$$\frac{1}{2\pi i}\oint_{C'} \frac{dw}{w-w_0} = \pm 1 \tag{8}$$

depending on the orientation of C'. However since expression (7) is the number of times f attains w_0, which must be non-negative, and is equal to the expression (8), it must equal $+1$. This implies (i) f is an analytic homeomorphism in D and (ii) it preserves the orientation of C. From Lemma 10.2, it is also is also *conformal*.

Remark: Because of Lemma 10.3, in order to determine if an analytic f is conformal and defines a 1-1 map between D and $f(D)$, where D is simply connected with piecewise ∂D, it is enough to that f is one to one between ∂D and $f(\partial D)$.

Remark: If the region of interest in the D plane is the exterior of some closed simple curve C on which the analytic map f is one to one, then it is helpful to introduce an intermediate transformation g: $g(\zeta) = \frac{1}{\zeta-\zeta_0}$, where ζ_0 is inside C. g is seen to map the region exterior of C into a domain interior of $g(C)$ in a 1-1 manner. Lemma 10.3 can then be applied to $f \cdot g^{-1}$ which is a mapping between finite domains $g(D)$ and $f(D)$.

Remark: If the region of interest is one side of an infinite non-self-intersection curve C in the ζ plane, which is piecwise smooth, we can once again use the transformation g, defined by $g(\zeta) = \frac{1}{\zeta-\zeta_0}$ for ζ_0 on the other side of C, before applying Lemma 10.3. On transformation, the infinite curve C can be applied to the resulting domain in the ζ_1 domain. This has the effect of transforming an infinite contour into a finite one and mapping one side of it into the interior/exterior of that curve.

Remark: In the case, f has a simple pole in a finite domain D, we consider first the mapping properties of g defined by $g(\zeta) = \frac{1}{f(\zeta)-a}$, where $a \neq f(\zeta)$ for $\zeta \in \bar{D}$. Since g is analytic and free of singularities, we can use Lemma 10.3 when applicable. Once mapping properties of g is determined, the explicit relation between f and g allows determination of the mapping properties of f as well.

Mapping properties of simple functions:

Example 1: Consider linear mapping $f(\zeta) = a\ \zeta + c$; this corresponds to a dilation of $|a|$ and counter-clockwise rotation by $\arg a$, followed by translation by c.

Example 2: Consider $w = f(\zeta) = \frac{1}{\zeta}$. It maps the exterior of unit circle centered at the the origin into its interior, with ∞ mapped to the origin. More generally, it maps any domain exterior to a curve C that contains the origin into a finite closed domain. It also maps a finite region containing the origin into an infinite region, exterior to the curve $f(C)$. Generally, it maps circles (straightlines) into circles (or straightlines). To see this, suppose $\zeta = \xi + i\eta$; then

$$u + iv = f(\zeta) = \frac{1}{\xi + i\eta}$$

Thus,

$$\xi = \frac{u}{u^2 + v^2} \quad , \quad \eta = -\frac{v}{u^2 + v^2} \tag{9}$$

Then, if

$$\xi^2 + \eta^2 + A\,\xi + B\,\eta = C$$

then from (2)

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{Au}{u^2 + v^2} - \frac{Bv}{u^2 + v^2} = C$$

which simplifies to

$$1 + Au - B\,v = C(u^2 + v^2)$$

If $C = 0$, this is a straight line in the w plane; otherwise it is a circle. Similarly, you can easily show that the map of a straight line in the ζ plane is a circle (or straight line) in the w plane.

Example 3: A fractional linear map (or Mobius map) is defined by

$$w = f(z) = \frac{az + b}{cz + d} \tag{10}$$

where $a/c \neq b/d$ (otherwise w is a constant). This can be generally viewed as a composition of a linear mapping, an inversion, followed by a linear mapping:

$$w_1 = cz + d \quad ; \quad w_2 = \frac{1}{w_1} \quad ; \quad w = \left(b - \frac{ad}{c}\right) w_2 + \frac{a}{c}$$

If $c = 0$, then the fractional linear mapping simply reduces to a linear mapping. It is to be noted that the inverse of (10) is once again a fractional linear transformation (as can be checked):

$$z = h(w) = \frac{dw - b}{-cw + a}$$

Fractional linear transformation maps a circle (and straightlines) into a circle or straightlines. It also has the property that if w_1, w_2, w_3 and w_4 are four points corresponding to z_1, z_2, z_3 and z_4, then the cross ratios are equal, *i.e.*

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

It is to be noted that there are only three independent parameters in (3). For instance, for nonzero c, these independent parameters can be take to be a/c, b/c and d/c. These three parameters are uniquely set when the images of any three distinct points z_1, z_2 and z_3 are specified to be the distinct values w_1, w_2 and w_3. The image w_4 of a fourth point z_4 is immediately determined by the cross-ratio relation above.

Composition of Mobius map is also a Mobius map as is readily checked. Further, since inversion of a Mobius map is again a Mobius map and the set of mappings include the identity mapping, the set of Mobius maps form a group. Indeed, it is easily proved that any analytic homeomorphism between circles is necessarily a Mobius map. This map be used to prove the uniqueness claim of the Riemann-mapping theorem in the following manner. Consider a particular $f : D \to D_1$ (D_1 is the unit disk), whose existence is given by Riemann mapping theorem. Now, the most general map from D_1 to D_1 is a Mobius map g. Therefore, the most general map $F : D \to D_1$ must have the form $F = g \cdot f$. By demanding $F(z_0) = w_0$ and $F'(z_0) > 0$, the Mobius map g is uniquely determined.

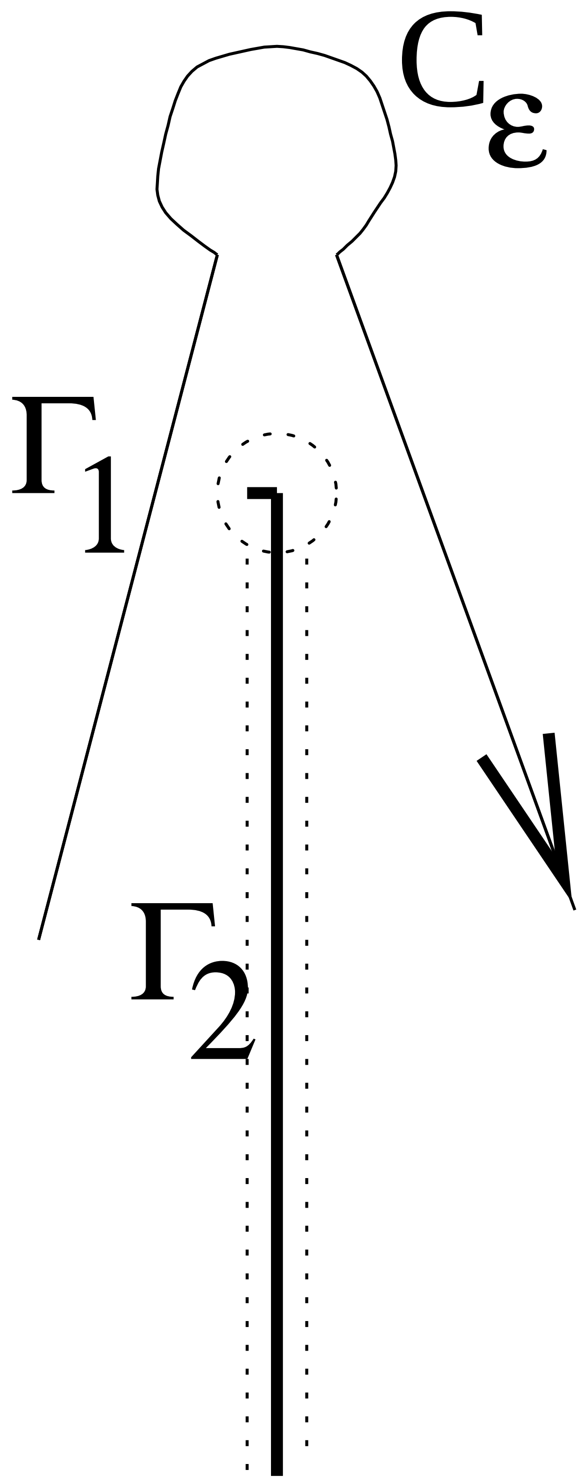

Figure 2: Contour Γ_2, deformed from Γ_1

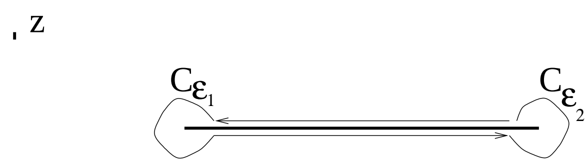

Figure 3: Closed contour C in eqn. (11)

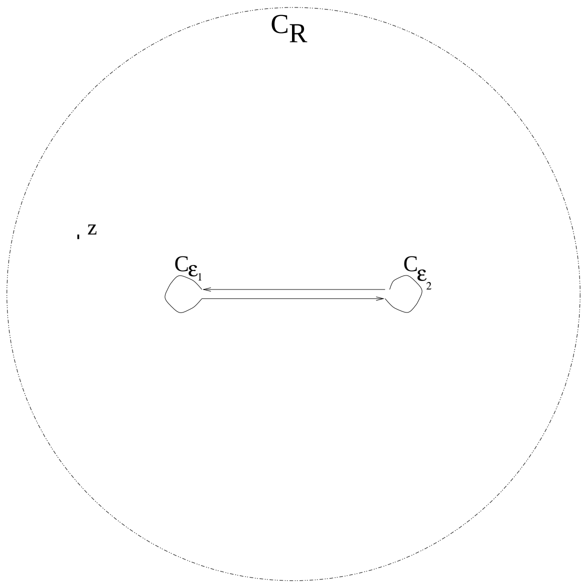

Figure 4: Cauchy integral formulae applied the region between C and C_R

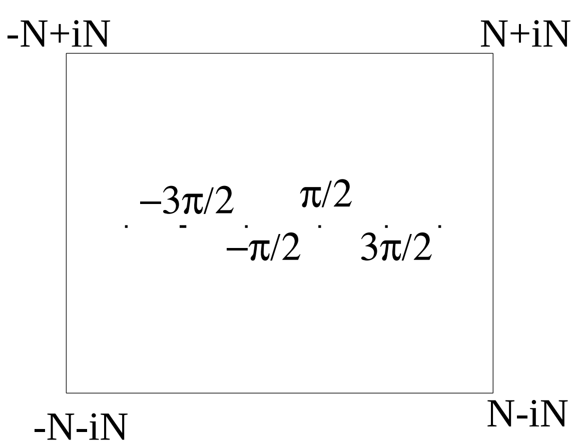

Figure 5: Square contour C_N in eqn. 20

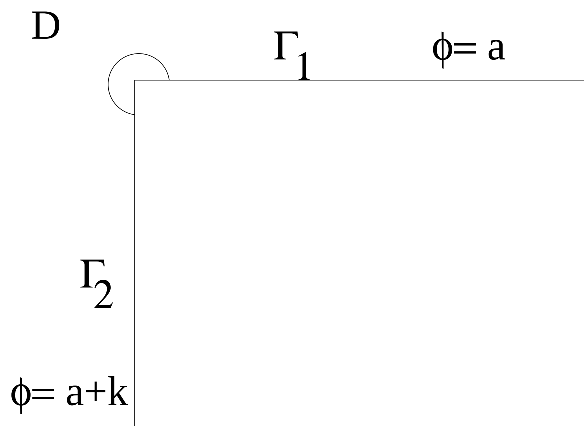

Figure 6: Boundary value problem in D, bounded by Γ_1 and Γ_2

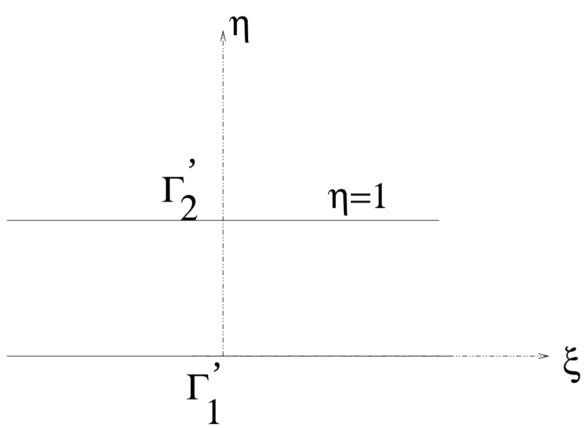

Figure 7: Domain D', bounded by Γ_1' and Γ_2', in the $\zeta = \xi + i\eta$ plane

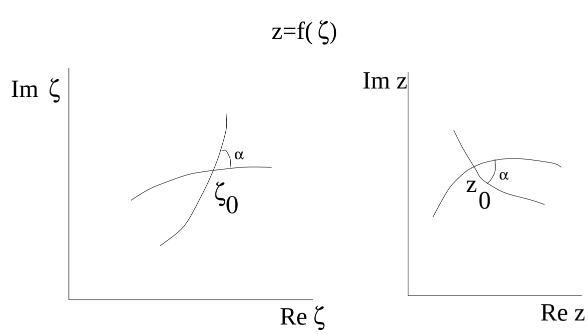

Figure 8: Preservation of angle α and orientation, under transformation $z = f(\zeta)$