Mapping properties of simple functions:
More on Conformal Mapping

Theorem 11.0 (Schwartz Lemma). Let D_1 be the unit disk. Let $f : D_1 \to D_1$ be analytic with $f(0) = 0$ and continuous in its closure $\bar{D}_1$. Then **i.** $|f(z)| \leq |z|$ for $z \in D_1$, **ii.** If $|f(z_0)| = |z_0|$ for some $z_0 \in D_1$, then $f(z) = e^{i\phi} z$ for some real ϕ. Further, **iii.** $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then $f(z) = e^{i\phi} z$.

Proof Note $g(z) = \frac{f(z)}{z}$. Note g is analytic in D. From maximum modulus Theorem, $|g(z)| \leq 1$. So, $|f(z)| \leq |z|$ as claimed in part **i**. If equality holds for $z = z_0$, then from maximum modulus theorem, $|g(z)| = 1$ everywhere, implying part **ii.**. Since

$$|g(h)| = |\frac{f(h) - f(0)}{h}| \leq 1$$

On taking the limit of $h \to 0$, $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then $f'(0) = e^{i\phi}$. Assume contrary to what needs to be proved, $f(z) \neq e^{i\phi} z$. Then, we may decompose

$$\frac{f(z)}{z} = e^{i\phi} \left(1 + z^m e^{i\psi} h(z)\right),$$

for some integer $m \geq 1$, with h analytic and $h(0) = 1$. Choose $z_0 = \epsilon e^{-i\psi/m}$ for ϵ small enough and we check from above that $|f(z_0)/z_0| > 1$, which contradicts **i**.

Corollary 11.0 If an analytic f maps unit disk D_1 into itself in a 1-1 manner with $f(0) = 0$, then $f(z) = e^{i\phi} z$ for some real constant ϕ.

Proof: We know from Schwartz Lemma that $|f(z)| \leq |z|$. But $f^{-1} : D_1 \to D_1$. and therefore $|f^{-1}(w)| \leq |w|$. Therefore, $|z| \leq |f(z)|$. Hence $|f(z)| = |z|$, implying from Schwartz Lemma that $f(z) = e^{i\phi} z$.

Lemma 11.0: The most general analytic 1-1 map $f : D_1 \to D_1$ (unit circle into itself) is given by

$$f(z) = S(z) = e^{i\phi} \left(\frac{\alpha - z}{1 - \bar{\alpha} z}\right) \quad \text{, where} \quad \alpha = f^{-1}(0)$$

Proof: First, we note on substituting $z = e^{i\theta}$, $\alpha = \rho e^{i\psi}$, $\rho < 1$, we obtain

$$|S(e^{i\theta})| = |e^{i\phi} \left(\frac{\rho e^{i\psi} - e^{i\theta}}{1 - \rho e^{-i\psi} e^{i\theta}}\right)| = 1$$

We then notice $h(\zeta) = f\left(S^{-1}(\zeta)\right)$ maps D_1 into itself with $h(0) = 0$, and hence $h(\zeta) = e^{i\phi'} \zeta$ implying $f^{-1}\left(e^{-i\phi'} \zeta\right) = S^{-1}(\zeta)$, *i.e.*, $e^{i\phi'} f(z) = S(z)$. From form of $S(z)$, we can choose $\phi' = 0$, without loss of generality.

Lemma 11.1 The most general analytic function f that maps the domain bounded by a circle/straight line to another domain bounded by a circle or a straight line is a Mobius (fractional linear) map.

Proof We already know Mobius transformation maps circles/straight lines into circles/straight lines. By using Lemma 11.0, we can complete the proof.

Example 4: $w = f(z) = z^2$ conformally maps the upper-half plane into the entire cut w plane, with the cut along the positive real w-axis. It also maps the region inside the semi-circle into the cut unit-circle; maps a quarter circle into a semi-circle. More generally $f(z) = z^n$ conformally maps the sector $0 < \arg < \frac{2\pi}{n}$ into the entire cut w plane, with cut along the positive real w axis.

Example 5: The exponential function $w = f(z) = e^z$ maps any strip parallel to the $Re\ z$ axis of width 2π into the entire w plane. It's inverse is the $\log\ w$ function with cut location dependent on the location of the strip in the z-plane. So, for a strip, lying between $\theta_0 < Im\ z_0 < 2\,\pi + \theta_0$, the branch cut in the w plane has to coincide with $\arg\ w = \theta_0$. with $\log\ w$ interpreted as $\log |w| + i \arg\ w$, $\arg\ w$ in $(\theta_0,\ 2\pi + \theta_0)$. Notice that e^z maps a strip of width π into a half-plane. Also, $f(z)$ maps the half-strip $x > 0, -\pi/2 < y < \pi/2$ onto the porton of the right half w plane that lies entirely outside the unit circle.

Example 6: $z = f(\zeta) = \sin \frac{\pi}{2}\zeta$ conformally maps the half-strip $-1 < Re\ \zeta < 1$, $Im\ \zeta > 0$ to the upper-half z plane. This is because for ζ on the real axis, in (-1,1), the image is also in (-1, 1). For $\zeta = \pi/2 + i\ \eta$, the image is the real z axis in $(1,\ \infty)$ since

$$\frac{1}{2i}\ \left[e^{i\pi/2-\eta} - e^{-i\pi/2+\eta}\right] = \cosh\ \eta$$

Similarly, the image of $\zeta = -\pi/2 + i\ \eta$ is the real z axis in $(-\infty,\ 1)$. From the orientation of the points mapped on the boundary it is clear, that the mapping is to the upper half z plane rather than the lower-half plane.

Example 7: Joukowski's transformation

$$z = f(\zeta) = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)$$

conformally maps the region exterior to $|\zeta| > 1$ into the region exterior of the straightline slit connecting $z = -1$ to $z = 1$. Note that $f'(\zeta) = 0$ iff $\zeta = \pm 1$. Hence the mapping is not conformal at those boundary points, though it is in the interior. Note, however, it maps circles $|\zeta| = a > 1$ into ellipses since

$$z = x + iy = \frac{1}{2}\left((a + \frac{1}{a}) \cos\ \theta + (a - \frac{1}{a}) \sin\ \theta\right)$$

Therefore, $f(\zeta)$ maps the exterior of circle $|\zeta| = a > 1$ into the exterior of an appropriate ellipse in the z-plane. The inverse of the mapping from the exterior of a circle to the exterior of an ellipse (or a slit) is given by

$$\zeta = h(z) = z + (z^2 - 1)^{1/2}$$

with cuts chosen to coincide with the slit and branch chosen so that $(z^2 - 1)^{1/2} \rightarrow z$, as $z \rightarrow \infty$. Note that Joukowsky's map also maps the interior of a unit semi-circle in the upper-half ζ plane into the lower-half z plane.

Remark: It is to be noted that the Joukowski's transformation also maps the interior of the unit circle in the same way because the mapping has the property $f(1/\zeta) = f(\zeta)$. The inverse in this case is the other branch

$$\zeta = h(z) = z - (z^2 - 1)^{1/2}$$

In this case, it is to be noted that $\zeta \rightarrow 0$ as $z \rightarrow \infty$, as it should be.

Remark: The examples above can be combined to solve quite nontrivial problems of potential theory.

Exercise: Find a solution to $\Delta\phi =$ inside a unit quarter-circle in the first quadrant so that $\phi = 0$ on straight segments and and $\phi = 1$ on the circular arc.

Solution: Let $\phi(x, y) = Re\ \Omega(z)$, $z = x + iy$. Introduce

$$z_1 = f_1(z) = z^2 \tag{11.1}$$

This maps the quarter circle into the interior of the upper-half unit-semi-circle. $\phi = 0$ on the real diameter, while $\phi = 1$ on the circular arc. Now introduce

$$z_2 = f_2(z_1) = -\frac{1}{2}(z_1 + 1/z_1) \tag{11.2}$$

This maps the inside of the semi-circle into the upper half-plane. Note in the z_2 plane, $\phi = 1$ in $(-1, 1)$ and equals 0 on the real axis outside this interval. Now, introduce the Mobius map:

$$z_3 = f_3(z_2) = \frac{z_2 - 1}{z_2 + 1} \tag{11.3}$$

This is easily seen to map the upper-half plane to the upper-half plane, with $z_2 = 1, -1, \infty$ corresponding to $z_3 = 0, \infty, 1$, respectively. Thus, now on the real z_3 axis, $\phi = 1$ for $z_3 < 0$ and $\phi = 0$ for $z_3 > 0$. Now, introduce

$$\zeta = f_4(\zeta_3) = \frac{1}{\pi} \log z_3 \quad \text{with} \quad \arg z_3 \ \text{ in } \ [0, \pi] \tag{11.4}$$

It is clear that f_4 maps the upper-half plane into a strip of width 1 in the $\zeta = \xi + i\eta$ plane, where $\phi = 1$ on $\eta = 0$ and $\phi = 0$ on $\eta = 1$. A solution to Laplace's equation with the given boundary condition in the ζ plane is:

$$\Phi(\xi, \eta) = \eta = Re\ i(\xi + i\eta) \tag{11.5}$$

This means that, without loss of generality,

$$\chi(\zeta) = i\zeta \tag{11.6}$$

Therefore

$$\Omega(z) \; = \; \chi(f_4(f_3(f_2(f_1(z))))) \; = \; \frac{2i}{\pi} \; \log \; \left[\frac{1+z^2}{1-z^2} \right] \tag{11.7}$$

and $\phi(x, y) = Re\ \Omega(x \ + \ iy)$ It can be checked that the boundary conditions are indeed satisfied, with suitable choice of cuts and branches. Also, this is unique solution for ϕ if it can be assumed to be bounded. This is proved in a manner similar to what was done in class.

More examples on use of conformal mapping

Example 1: Let D be the domain which is in the intersection of the two circles: one around $z = 1$ and the other around $z = i$, each of radius 2 (See Fig. 1). Solve for $\phi(x, y)$ satisfying the Laplace's equation in D, with boundary conditions $\phi \ = \ 0$ on the arc of the circle around $z = 1$, and $\phi \ = \ 1$ on the arc of the other circle.

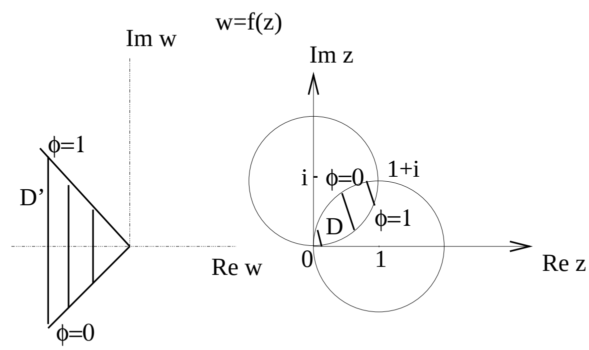

Figure 1: Domain D in the z and w-planes

Solution: Because the domain is bounded by circular arcs, both of which pass through $z = 0$ and $z = 1 + i$, it is natural to use a Mobius map

$$w \; = \; f(z) \; = \; \frac{z}{z - 1 - i} \tag{11.8}$$

The resulting domain in the w plane is bounded by the rays arg $w \ = \ 3\ \pi/4$ and arg $w \ = \ 5\ \pi/4$. $\phi = 1$ is being satisfied on arg $w = 3\pi/4$ and $\phi = 0$ o the other boundary. Now, further use transformation

$$\zeta \; = \; \frac{2}{\pi} \; \left(\; \log \; w \; - \; i \; \frac{3\pi}{4} \right) \; \text{ where arg } w \text{ is in } (0, \; 2\pi) \tag{11.9}$$

Note the image of D in the $\zeta \ = \ \xi \ + \ i\eta$ plane is simply the region between $\eta \ = \ 0$ and $\eta = 1$. (See Fig. 2)

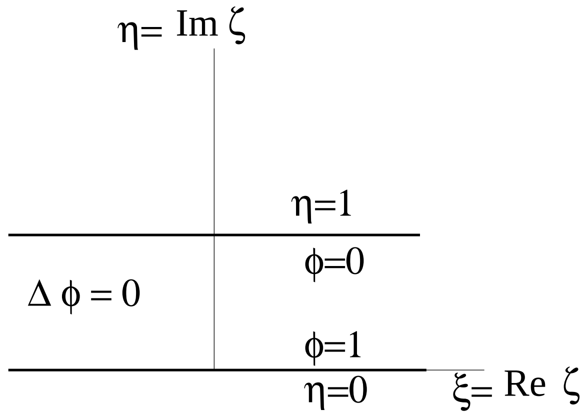

Figure 2: Boundary value problem in the ζ-plane

The appropriate solution is given by

$$\Phi \;=\; (1-\eta) \tag{11.10}$$

Therefore,

$$\chi \;=\; 1 \;+\; i\zeta, \tag{11.11}$$

where $\Phi = \Re\chi(\xi + i\eta)$ Therefore, $\phi(x,y) = \Re\Omega(x+iy)$, with

$$\Omega(z) \;=\; 1 \;+\; \frac{2\,i}{\pi}\left(\;\log\;\frac{z}{z-1-i}\;-\;i\;\frac{3\pi}{4}\right) \tag{11.12}$$

with appropriate choice of branch of log. On taking the real part, we get

$$\phi\;(x,y) \;=\; \frac{2}{\pi}\left(\frac{\pi}{4}\;-\;\arctan\;\frac{x-y}{x(x-1)+y(y-1)}\right)$$

where $-\pi/2 \;<\; \arctan\;(.)\;<\;\pi/2$.

Lecture 12:

Remark on Uniqueness in Riemann mapping Theorem The mapping is not unique. Recall that the most general 1-1 analytic map from the unit circle to the unit circle is $S(\zeta) = e^{i\phi}\left(\frac{\alpha-\zeta}{1-\bar{\alpha}\zeta}\right)$, with three real parameters ϕ, $\Re\alpha$, $\Im\alpha$. If $f : D \to D_1$ is a particular mapping function, the most general map will be $S(f(z))$, which has three real degrees of freedom. For instance, if we specify a particular interior point $z = z_0$ to correspond to $\zeta = 0$, this uses up two real degrees of freedom. The third degree of freedom, in the form of rotation, can be used to fix ϕ. Alternately, we may specify points on ∂D to correspond to three points on ∂D_1 in the right order, in terms of orientation.

1 Intuitive Construction of the Schwarz-Christoeffel map between upper-half plane and the interior of an n-sided polygon

We now turn to mappings between the upper-half $\zeta = \xi + i\eta$-plane to the interior/exterior of a polygon in the $z = x + iy$-domain. First, we will consider the interior problem. We denote by $z = f(\zeta)$ a mapping from the upper-half plane to the interior of the polygon. Let the n vertices of the polygon, enumerated in some counter clockwise order, be the points z_1, z_2, $...z_n$ and let their image points on the ξ axis be ξ_1, ξ_2 through ξ_n. Without any loss of generality, we take $\xi_1 < \xi_2 < \xi_3 < ...\xi_n$.

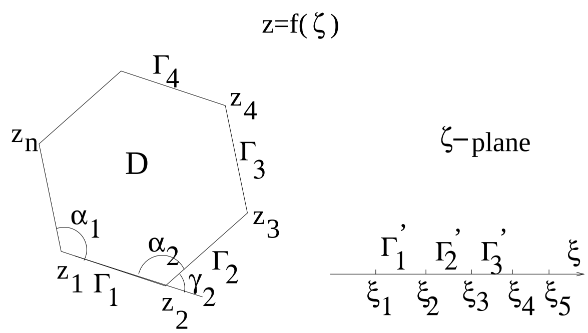

Figure 3: Domain D in the z-plane and the upper-half ζ plane

The n sides of the polygon are denoted by Γ_1, Γ_2, $..\Gamma_n$ (See Fig. 3), and their respective image in the ξ-axis given by Γ_1', Γ_2', $..\Gamma_n'$ (If $\xi_1 \neq -\infty$ and $\xi_n \neq +\infty$, Γ_n' consists of two segments $\xi < \xi_1$ and $\xi > \xi_n$). Let the interior angle of the polygon be denoted by α_j, $j = 1, ..n$. Define $\gamma_j = \pi - \alpha_j$. Notice that a counter-clockwise rotation of Γ_{j-1} through an angle γ_j alligns it with Γ_j. From properties of polygons,

$$\sum_{j=1}^{n} \alpha_j = (n-2)\,\pi \quad \text{or} \quad \sum_{j=1}^{n} \left(1 - \frac{\alpha_j}{\pi}\right) = \sum_{j=1}^{n} \frac{\gamma_j}{\pi} = 2 \tag{12.1}$$

Let $z = f(\zeta)$ denote a conformal map from the upper-half plane It is to be noted that $\arg\ f'(\zeta)$ is a piecewise constant on each segment Γ_j'. Further, it is to be noted that

$$\arg\ f' + \sum_{j=1}^{n} \frac{\gamma_j}{\pi} \arg(\zeta - \xi_j) = \text{constant} \tag{12.2}$$

on each segment Γ_j', with $\arg\ (\zeta - \xi_j) \in [0,\ \pi]$. It is clear that within each Γ_j', there is no variation of the function on the left of (12.2). As ζ moves from Γ_{k-1}' to Γ_k', $\arg\ f'$ clearly increases by γ_k, while the summation decreases by $-\gamma_k$, since the term $\arg\ (\zeta - \zeta_k)$ in the summation changes from π to 0, while other terms within the summation undergoes no change. This suggests that for some constant $\tilde{a}$,

$$f'(\zeta) \; = \; \tilde{a} \; \Pi_{j=1}^{n} \; (\zeta - \zeta_j)^{-\gamma_j/\pi}$$

i.e.

$$f(\zeta) \; = \; z_1 \; + \; \tilde{a} \; \int_{\xi_1}^{\zeta} d\zeta' \;\; \Pi_{j=1}^{n} \; (\zeta' - \zeta_j)^{-\gamma_j/\pi} \tag{12}$$

This can be directly verified by noting that it maps the real axis into the polygonal boundary. Clearly, it maps ξ_1 into ζ_1. Further, if $\zeta \in \Gamma'_1$, it is clear on using (12.1) that for some complex a, related to $\hat{a}$,

$$z \; = \; f(\zeta) \; = \; z_1 \; + \; a \; e^{i \; \gamma_1} \; \int_{\xi_1}^{\zeta} d\zeta' \; (\zeta' - \xi_1)^{-\gamma_1/\pi} \; \Pi_{j=2}^{n} \;\; (\xi_j - \zeta')^{-\gamma_j/\pi} \tag{12.3}$$

Using (12.1), it is clear that on this segment Γ'_1, arg $(z \; - \; z_1)$, as computed from the above formula comes out to be constant, arg $a + \; \gamma_1$. Evaluating (12.3) at $\zeta = \xi_2$, one obtains

$$z_2 - z_1 \; = \; a \; e^{i \; \gamma_1} \; \int_{\xi_1}^{\xi_2} d\zeta (\zeta - \xi_1)^{-\gamma_1/\pi} \; \Pi_{j=2}^{n} \;\; (\xi_j - \zeta)^{-\gamma_j/\pi} \tag{12.4}$$

On the other hand, if $\zeta \in \Gamma'_2$,

$$z \; = \; f(\zeta) \; = \; z_2 + a \; e^{i \; \gamma_1 \; + \; i\gamma_2} \; \int_{\xi_2}^{\zeta} d\zeta' \; (\zeta' - \xi_1)^{-\gamma_1/\pi} \;\; (\zeta' - \xi_2)^{-\gamma_2/\pi} \;\; \Pi_{j=3}^{n} \;\; (\xi_j - \zeta')^{-\gamma_j/\pi} \tag{12.5}$$

So, in this segment arg $(z - z_2) \; =$ arg $a + \gamma_1 + \gamma_2$ in accordance to what is should be (See Fig. 4) Evaluating (12.5) for $\zeta = \xi_3$, we obtain

$$z_3 \; - \; z_2 \; = \; a \; e^{i \; \gamma_1 \; + \; i\gamma_2} \; \int_{\xi_2}^{\xi_3} d\zeta \; (\zeta - \xi_1)^{-\gamma_1/\pi} \;\; (\zeta - \xi_2)^{-\gamma_2/\pi} \;\; \Pi_{j=3}^{n} \;\; (\xi_j - \zeta)^{-\gamma_j/\pi} \tag{12.6}$$

Generally, we can verify that on Γ'_k,

$$\text{arg} \;\; (z - z_k) \;\; = \;\; \text{arg} \;\; a \; + \; \sum_{j=1}^{k} \gamma_j$$

Now, if the lengths of the sides of the polygons are l_1, l_2, l_n, we must have from (12.4), (12.6) and similar relations for $z_{k+1} - z_k$, the following relations for $k = 1, ..(n-1)$

$$l_k \;\; = \;\; |a| \; \int_{\xi_k}^{\xi_{k+1}} d\zeta \; \Pi_{j=1}^{k} \; (\zeta - \xi_j)^{-\gamma_j/\pi} \;\; \Pi_{j=k+1}^{n} \;\; (\xi_j - \zeta)^{-\gamma_j/\pi} \tag{12.7}$$

Equations (12.7) constitute $n-1$ relations to determine $n+1$ parameters ξ_1, ξ_2, $..\xi_n$ and $|a|$, once ξ_1 is specified to correspond to z_1. The additional two degree (three "real" degrees of freedom including the freedom of specifying ζ_1 to correspond to z_1) can be used by specifying the image locations of two other points, say ξ_2 and ξ_3, or perhaps $|a|$ and $|\xi_2|$. The three degrees of freedom in this special case is a reflection of the three degrees of freedom one has from the general Riemann mapping theorem.

Seemingly, there is an additional relation

$$l_n = |a| \left(\int_{\xi_n}^{+\infty} d\zeta \ \Pi_{j=1}^{n} \ (\zeta - \xi_j)^{-\gamma_j/\pi} + \int_{-\infty}^{\xi_1} d\zeta \ \Pi_{j=1}^{n} \ (\xi_j - \zeta)^{-\gamma_j/\pi} \right) \tag{12.8}$$

However, (12.8) is not independent of (12.7). To see this we note that

$$\int_{-\infty}^{\infty} d\zeta \ f'(\zeta) = 0$$

for a contour that avoids the singular points ξ_j by going over them. To see this, we note that $f' \sim$ constant ζ^{-2} as $\zeta \to \infty$. Therefore,

$$z_1 - z_n = \left(\int_{\xi_1}^{\xi_n} + \int_{-\infty}^{\xi_1} \right) f'(\zeta) \, d\zeta = - \sum_{j=1}^{n-1} \int_{\xi_j}^{\xi_{j+1}} f'(\zeta) \, d\zeta$$

Therefore, relation (12.8), which is the magnitude of $z_1 - z_n$ cannot be independent of (12.7).

Remark: The Schwarz-Christoeffel formula (12) simplifies by setting one $\xi_n = \infty$. In a formal procedure, it is seen that for fixed ζ, as $\xi_n \to \infty$ in a manner so that $a \, (-\xi_n)^{-\gamma_n/\pi} \to \tilde{a}$, then

$$z = z_1 + \tilde{a} \int_{\xi_1}^{\zeta} d\zeta' \ \Pi_{j=1}^{n-1} \ (\zeta - \xi_j)^{-\gamma_j/\pi} \tag{12.9}$$

It can be directly verified that (12.9) maps the real axis into the polygon under consideration, provided the subsidiary length constraints in (12.5) are satisfied. Eqn. (12.9) is simpler usually than the original expression as the factor $(\zeta - \xi_n)^{-\gamma_n/\pi}$ is entirely missing.

Proof of expression (12) being the map that maps the upper-half plane into the interior of an n-sided polygon with interior angles $\alpha_j = \pi - \gamma_j$ is possible by repeated use of Schwartz reflection principle. We sketch the outline of the proof briefly (see O. Costin notes or Nehari's book for details). Define the mapping function as f, *i.e.* $z = f(\zeta)$ maps the upper-half ζ plane into the interior of the polygon, with ξ_1, ξ_2, ... ξ_n corresponding to vertices z_1, z_2, ... z_n and define the expression in (12) as $\tilde{f}$. Assume the segment $\arg(z - z_1) = \phi_1$ on line segment Γ_1. Then, we can show that near $\zeta = \xi_1$

$$g_1(\zeta) \equiv \left\{ e^{-i\phi_1} \left[f(\zeta) - z_1 \right] \right\}^{\pi/\alpha_1}$$

is analytic in the upper-half ζ plane, continuous upto the boundary with vanishing imaginary part on the real axis segment that contains ξ_1 in the interior. Schwarz-reflection principle guarantees no singularity of $g_1(\zeta)$ at $\zeta = \xi_1$ as is true for

$$\tilde{g}_1(\zeta) \equiv \left\{ e^{-i\phi_1} \left[\tilde{f}(\zeta) - z_1 \right] \right\}^{\pi/\alpha_1}$$

and analytically continuable to the lower-half plane. We can repeat similar argument at each ξ_j and eventually arrive at the conclusion that $f'(\zeta)/\tilde{f}'(\zeta)$ is singularity free in the entire complex plane and is bounded at ∞, thereby leading to $f'(\zeta) = C\tilde{f}'$. With suitable normalization intrinsic in the the relation of a to length of polygonal sides, $C = 1$.

Lecture 13: More applications of conformal map

Example 1: Find a transformation that maps the interior of the rectangle with vertices at the points $z \ = \ \pm 1$, $z \ = \ \pm 1 \ + \ i \ c$ for $c \ > \ 0$ into the upper-half ζ plane

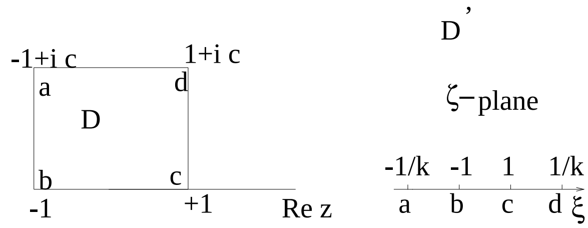

Figure 4: Domain D in the z and ζ plane

Solution: We choose the three degrees of freedom by requiring that $\zeta = \pm 1$ to correspond to $z = \pm 1$, and $\zeta = \pm \frac{1}{k}$ (with $0 \ < \ k \ < \ 1$ to correspond to $z = \pm 1 \ + \ i \ c$, respectively. Note that we have not specified k, as that would have corresponded to specifying four image points in the ζ-domain. Then, it is clear from Schwarz Christoeffel-formula that

$$z = -1 \ + \ b \ \int_{-1}^{\zeta} \frac{d\zeta}{\sqrt{1-\zeta^2} \ \sqrt{1-k^2 \ \zeta^2}} \tag{13.1}$$

where $b \ > \ 0$, and arg $(1 \ - \ \zeta)$, arg $(1 \ - \ k \ \zeta)$ are each in $[-\pi, 0)$, while arg $(\zeta \ + \ 1)$ and arg $(1 \ + \ k \ \zeta)$ are in $[0, \ \pi]$. The length constraints become:

$$2 = \ b \ \int_{-1}^{1} \frac{d\zeta}{\sqrt{1-\zeta^2} \ \sqrt{1-k^2 \ \zeta^2}}$$

$$c \ = \ = b \ \int_{1}^{1/k} \frac{d\zeta}{\sqrt{\zeta^2-1} \ \sqrt{1-k^2 \ \zeta^2}}$$

Remark: If we introduce the elliptic integral of the first kind:

$$E(k, \zeta) \ = \ \int_{0}^{\zeta} \frac{d\zeta}{\sqrt{1-\zeta^2} \ \sqrt{1-k^2 \ \zeta^2}} \tag{13.2}$$

It is clear that in the previous example

$$z \ = \ b \ E(k, \ \zeta) \tag{13.3}$$

The inverse of this is known in the elliptic function theory to be the Jacobian elliptic function sn (Whittaker & Watson, "Modern Analysis", Chap. 22). In our case,

$$\zeta \ = \ sn \left(\frac{z}{b}, k \right) \tag{13.4}$$

This is not surprising at all since on on differentiation of expression of $z = E(k, \zeta)$, it follows that $\zeta(z)$ satisfies the first order nonlinear ODE

$$\left(\frac{d\zeta}{dz}\right)^2 = (1 - \zeta^2)(1 - k^2\zeta^2)$$

whose solution is known to be the Elliptic function.

Remark: In the asymptotic limit $k \to 0$, we obtain the semi-infinite strip obtained by letting $c \to \infty$. In that case, the mapping function simplies to

$$z = -1 + b \int_{-1}^{\zeta} (1 - \zeta^2)^{-1/2} d\zeta \;=\; -1 + b \sin^{-1} \zeta + b\,\frac{\pi}{2} \tag{13.5}$$

with choice of branch of $\sin^{-1}$ so that $\sin^{-1} 0 = 0$ the constraint on length translates to $2 = b\,\pi$. So,

$$z = f(\zeta) = \frac{2}{\pi} \sin^{-1} \zeta$$

It's inverse is

$$\zeta = h(z) = \sin \frac{\pi z}{2}$$

Recall that sometimes back, we showed directly the property that *sin* maps a semi-infinite strip into the upper-half plane.

Example 2: Let D be the domain outside the unit circle (See Fig. 5). Find a solution $\phi(x, y)$ that satisfies Laplace's equation in D', except at point $(x_0,\ y_0)$, which is a point vortex of strength p, and as $x \to \pm\infty$,

$$\frac{\phi(x, y)}{x} \to U \tag{13.6}$$

and on the unit circular boundary,

$$\frac{\partial \phi}{\partial n} \;=\; 0 \tag{13.7}$$

Definition: If $\phi(x, y) = \Re\Omega(x + iy)$, then a point vortex at $z = z_0$ of strength p is a singularity of $\Omega(z)$ such that $\Omega(z) - ip\log(z - z_0)$ is regular and single valued at $z = z_0$.

Solution: Let $\Omega(z)$ be the corresponding complex potential, with

$$\phi(x, y) \;=\; Re\,[\Omega\,(x + iy)\,] \;=\; Re\,[\,\phi + i\psi\,]$$

From the C-R conditions, (13.7) translates into

$$\frac{\partial \psi}{\partial \theta} \;=\; 0 \;\; ; \;\; \text{on integration} \;\; \psi \;=\; c \tag{13.8}$$

on $z = e^{i\theta}$, for some constant c, taken to be zero w.l.o.g. A point vortex at $z_0 = x_0 + iy_0$ implies

$$\Omega \;- i\,p\,\log\,(z - z_0) \tag{13.9}$$

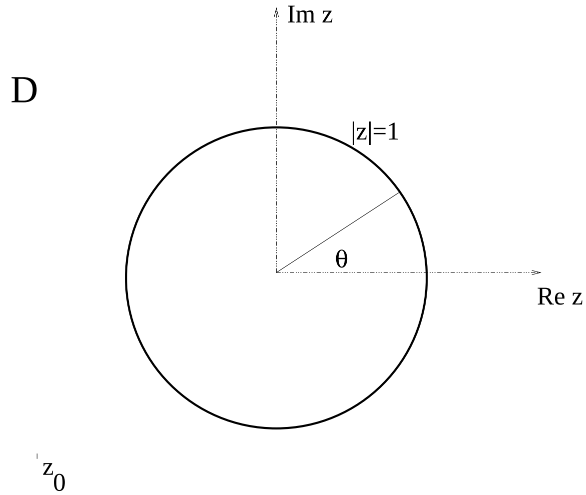

Figure 5: Potential problem outside a circle

is regular and single valued as $z \to z_0$ Since $\psi = 0$ on the boundary of $|z| = 1$, Schwarz reflection principle holds, if $\Omega(z)$ is continuous upto $|z| = 1$, as will be assumed. Thus, points $z = 0$ and $z = \frac{1}{z_0^*}$, which are image of $z = \infty$ and $z = z_0$ across $|z| = 1$ (i.e. via inversion), are possible singular points of $\Omega\ (z)$ inside $z = 0$. Thus, consider

$$g(z) \ = \ \Omega\ (z) \ - \ U\ [z \ + \ \frac{1}{z}] \ - \ i\ p\ \log\ \frac{z - z_0}{1 - z z_0^*} \ - \ i\ \Gamma\ \log\ z \qquad (13.10)$$

It is easily verified that $Im\ g \ = \ 0$ on $|z| = 1$ and that it has removable singularity at $z = z_0$ and $z = 1/z_0^*$. $g(z)$ is an entire function. Note $|g(z)|/|z|$ must be bounded from condition (13.6) and so $g(z)$ can at best be a linear function of z. But (13.7) determines the coefficient of z. Therefore, except for an additive constant, the solution must be of the form

$$\Omega\ (z) \ = \ \ U\ [z \ + \ \frac{1}{z}] \ + \ \ i\ p\ \log\ \frac{z - z_0}{1 - z z_0^*} \ + \ i\ \Gamma\ \log\ z \qquad (13.11)$$

The real part of this gives ϕ. Note that Γ is an arbitrary constant, and hence there is non-uniqueness for two dimensional Laplace's equation unless the total circulation $\Gamma + p$ at ∞ is specified.

Remark: The above can be used to find determine the solution to Laplace's equation outside an ellipse for instance (see Fig. 6), where a point vortex is located at $z_0 = (x_0, y_0)$ within the domain. Recall a suitable Joukowski's transformation maps the outside of an ellipse to the outside of a circle. Further, a point vortex remains a point vortex after a conformal transformation since if $[\Omega \ - \ ip\ \log\ (z - z_0)]$ has a removable singularity at $z = z_0$, $\chi(\zeta) \ - \ ip\ \log\ (\zeta - \zeta_0)$ also has a removable singularity at $\zeta = \zeta_0$, where $z = f(\zeta)$ and $\chi(\zeta) \ = \ \Omega(f(\zeta))$. This follows by noting that from Taylor expansion about ζ_0,

$$z - z_0 \ = \ f(\zeta) \ - \ f(\zeta_0) \ = \ (\zeta - \zeta_0)\ h(\zeta)$$

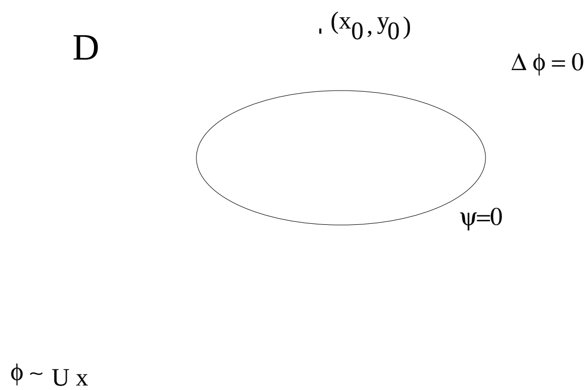

Figure 6: Harmonic ϕ outside an ellipse

where $h(\zeta)$ is analytic and nonzero at $\zeta = \zeta_0$.