# Week 6 notes

## 1. Boundary Properties of Conformal Map

**Remark 1.1.** *From Riemann mapping theorem, there exists 1-1 analytic mapping (i.e. analytic homeomorphism) $f : D \to D_1$ for any simply connected domain $D$ for which $\partial D$ contains more than one point. We now seek to understand how $f$ behaves near the boundary. The following Lemmas address the issue of boundary behavior of $f$. I follow Prof. O. Costin's notes closely at this point.*

**Definition 1.2.** *A set of points of $\{z_n\}_{n=1}^{\infty} \in D$ is said to approach the boundary $\partial D$, if for any compact set $K \subset D$, there exists $n_0$ so that $z_n \notin K$ for $n > n_0$. A curve $\Gamma \in \partial D$, parameterized by $\Gamma : \{z : z = \gamma(t) : a < t < b\}$, is said to approach $\partial D$ if there exists $t_0 \in (a, b)$ so that $z = \gamma(t) \notin K$ for $t > t_0$.*

**Lemma 1.3.** *If $\{z_n\}_{n=1}^{\infty}$ or $\Gamma$ approaches $\partial D$, then $f(z_n)$ (or $f(\gamma(t))$) approaches $\partial D_1$.*

*Proof.* Let $K_1 \subset D_1$ be an arbitrary compact set. Then, since $f$ is continuous (since it is analytic), $K = f^{-1}(K_1) \subset D$ is compact. There exists $n_0$ so that $z_n \notin K$ for $n > n_0$. Therefore, because $f$ is 1-1, $f(z_n) \notin f(K) = K_1$. Therefore $\{f(z_n)\}_{n=1}^{\infty}$ approaches $\partial D_1$. Similar proof goes for $f(\gamma(t))$. ∎

**Corollary 1.4.** *If $f : D \to D_1$ is an analytic homeomorphism (1-1 analytic map), then if $\{z_n\} \in D$ approaches $D$, then $|f(z_n)| \to 1$ as $\{z_n\}$ approaches $D$.*

*Proof.* Choose $K_{1,n} = \left\{\zeta : |\zeta| \le 1 - \frac{1}{n}\right\}$. From previous Lemma, there exists for each $n$, an integer $m_n$ so that $f(z_j) \notin K_{1,n}$, *i.e.* $1 > |f(z_j)| > 1 - \frac{1}{n}$ for $j > m_n$. This gives the desired result. ∎

**Lemma 1.5.** *If $\partial D$ is a simple closed curve that is continuous (i.e., the graph of a continous 1-1 function from $[a, b]$ to $\partial D$, except at the end points, where $\gamma(a) = \gamma(b)$), and $h : D_1 \to D$ is an analytic homeomorphism, then $h$ extends to a continous homeomorphism between $\bar{D}_1$ and $\bar{D}$. Further if $\partial D$ is analytic, then $h$ can be analytically continued across $\bar{D}_1$.*

*Proof.* We will only outline of the proof and that too for the analytic case. Let $S : D_1 \to H^+$ be an explicit fractional linear map that maps the circle into the upper-half plane. Suppose the boundary is analytic. Then there exists an analytic function $g : [a, b] \to \partial D$ where $g$ is analytic and 1-1 except at the end points, where $g(a) = g(b)$. The composition $S \cdot h^{-1} \cdot g$ is analytic at points on one side of the real axis close to $(a, b)$ and approaches real value as $(a, b)$ is approached, from previous Lemma. From reflection principle this composition is analytic on $(a, b)$. Since $S$ and $g$ are locally invertible analytic functions, so must be $h$. ∎

**Remark 1.6.** *The lemma above helps in proving the Schwartz-Christoeffel mapping in an alternate manner: Define*

$$g(\zeta) = \ln \left\{ \frac{f'(\zeta)}{\prod_{j=1}^{n} (\zeta - \xi_j)^{-\gamma_j/pi}} \right\}$$

*We already know that* $\text{Im } g = \text{constant}$ *on real axis except possible at $\xi_j$, where one may have singularities. However, the explicit transformation $z_1 = (z - z_j)^{\pi/\alpha_j}$*

*transforms straightens out the boundary segments* $\Gamma_{j-1}$ *and* $\Gamma_j$ *into a single straight line. We know from above result that the mapping function* $z_1(\zeta)$ *must be regular at* $\zeta = \xi_j$*, i.e.*

$$z_1 = a_1(\zeta - \xi_j)\,[1 + a_2(\zeta - \xi_j)\ +\ ..]$$

*implying*

$$z - z_j = a_1^{\alpha_j/\pi}\ (\zeta - \xi_j)^{\alpha_j/pi}\,[1 + \text{locally regular function of } (\zeta - \xi_j)]$$

*Therefore,*

$$f'(\zeta) = \tilde{a}\ (\zeta - \xi_j)^{-\gamma_j/pi}\,[1 + \text{locally regular function of } (\zeta - \xi_j)]$$

*Therefore,* $g(\zeta)$ *is locally regular at* $\zeta = \xi_j$*. This allows us to argue that since* $\mathrm{Im} g \to \text{consant}$ *as real* $\zeta$*-axis is approached and hence from maximum principle of harmonic functions* $\mathrm{Im} g = \text{constant}$ *everywhere, implying* $g$ *is a constant. Hence Schwartz-Christoeffel formula follows.*

## 2. Asymptotic series: basic definitions and properties

**Remark 2.1.** *This part of the notes follow Professor O. Costin's notes closely.*

**Remark 2.2.** *The idea in asymptotics is to replace a function by a sequence of simpler functions that approximates it progressively better.*

**Example 1:** Suppose we seek to understand the behavior of $f(x) = e^x \int_x^\infty \frac{e^{-t}}{t} dt$ as $x \to +\infty$. If we integrate by parts once,

$$f(x) = \frac{1}{x} - e^x \int_x^\infty \frac{e^{-t}}{t^2} dt$$

We expect that for large $x$, $f(x)$ behaves like $\frac{1}{x}$ to the leading order. This turns out to be true. However, if we want better approximation, we can integrate by parts several more times, to generate

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - 6e^x \int_x^\infty \frac{e^{-t}}{t^4} dt$$

The formal series obtained in this manner is

$$\tilde{f}(x) \equiv \sum_{k=1}^\infty (-1)^k \frac{k!}{x^{k+1}}$$

This is definitely not convergent; nonetheless, for any fixed truncation

$$\tilde{f}^{(N)} \equiv \sum_{k=0}^N (-1)^k \frac{k!}{x^{k+1}}$$

does approximate $f(x)$ better and better with increasing $N$ in some sense that will be made clear.

**Definition 2.3.** *An asymptotic expansion is a formal series*

$$\hat{f}(t) = \sum_{k=0}^\infty f_k(t)$$

*for which*

$$\lim_{t \to t_0} \frac{f_{k+1}(t)}{f_k(t)} = 0,$$

*i.e.* $f_{k+1}(t) = o\left(f_k(t)\right)$ *as* $t \to t_0$*.*

**Remark 2.4.** *The point* $t_0$ *can be at a finite point or at* $\infty$*. Further, unlike convergent series, direction of approach of* $t_0$*, i.e. value of* $\arg(t - t_0)$ *matters. The series above can only be valid, say for* $t \to t_0^+$ *and not for other directions of approach. Notationally, we will use variable* $x$ *in cases when we seek behavior at* $\infty$ *and* $z$ *when the the behavior at 0 is sought.*

**Definition 2.5.** *A function* $f(t)$ *is asymptotic to the formal series* $\tilde{f}(t)$ *as* $t \to t_0^+$*,* $f \sim \tilde{f}$*, if for any integer* $N \geq 0$*,*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) \equiv f(t) - \tilde{f}^{(N)} = o(\tilde{f}_N),$$

*or*

$$f(t) - -\tilde{f}^{(N)} = O(\tilde{f}_{N+1}),$$

*i.e. there exists some constant* $C$ *so that* $|f(t) - \tilde{f}^{(N)}(t)| \leq C|\tilde{f}_{(N+1)}(t)|$ *as for* $|t - t_0|$ *sufficiently small.*

**Remark 2.6.** *We note that the series in Example 1 is an asymptotic series at* $+\infty$*. To prove this, we note that*

$$|f(x) - \tilde{f}_N(x)| = |e^x \int_x^{\infty} e^{-t} t^{N+1} dt| x^{-N-1} e^x \int_x^{\infty} e^{-t} dt = x^{-N-1}$$

*So,*

$$\lim_{x \to +\infty} |f(x) - \tilde{f}^{(N)}| \leq C|\tilde{f}_{N+1}|$$

**Remark 2.7.** *Asymptotic series are generally divergent. This is seen above in Example 1.*

**Remark 2.8.** *Not all convergent expansions are asymptotic.* $e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k! z^k}$ *is convergent for any* $z \neq 0$*, but is not asymptotic as* $z \to 0$*. This is because* $z^{-k-1}$ *is not smaller than* $z^{-k}$*.*

Generally, the functions involve $f_k$ are exponentials, power and logs. A power series only involves powers of $z = (t - t_0)$, i.e. $\tilde{f}(z) = \sum_{k=0}^{\infty} c_k z^k$. If any $c_k = 0$, then the previous definition of asymptotic series becomes invalid, because of division by zero. This motivates the following definition

**Definition 2.9.** *A function has an asymptotic power series if for any integer* $N \geq 0$

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1})$$

**Remark 2.10.** *The Taylor series of any analytic function* $f$ *is also its asymptotic power series as the point of expansion is approached. This follows from the observation that* $f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1})$*.*

**Remark 2.11.** *In the sense of the definition 2.9, of asymptotic power series,* $e^{-1/z^{\alpha}}$ *for* $\alpha > 0$ *is the zero series as* $z \to 0^+$*. However, this series is not very informative. In fact the power series is not always helpful.* $e^{-1/z^{\alpha}}$ *has no asymptotic simplification.*

**Lemma 2.12.** *Uniqueness of asymptotic power series If* $f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ *as* $z \to 0$ *(along some directions), then* $c_k$ *is unique.*

*Proof.* Assume there exists asymptotic series coefficient $\{d_k\}$, different from $\{c_k\}$ with $f(z) \sim \sum_{k=0}^{\infty} d_k z^k$. Contrary to what needs to be proved assume $k = k_0$ is the first smallest integer for which $d_k \neq c_k$. Then from definition of asymptotic power series,

$$\lim_{z \to 0} z^{-k_0} \left\{ f(z) - \sum_{k=0}^{k_0-1} c_k z^k \right\} = c_{k_0},$$

$$\lim_{z \to 0} z^{-k_0} \left\{ f(z) - \sum_{k=0}^{k_0-1} d_k z^k \right\} = d_{k_0}$$

Since $d_k = c_k$ for $k < k_0$, it follows that $c_{k_0} = d_{k_0}$, contradicting the hypothesis above. So, $d_k = c_k$ for any $k$. ▮

**Lemma 2.13.** *Algebraic properties of asymptotic power series If* $f \sim \sum_{k=0}^{\infty} c_k z^k$ *and* $g \sim \sum_{k=0}^{\infty} d_k z^k$ *as* $z \to 0$*, then*

$$a f(z) + b g(z) \sim a \tilde{f}(z) + b \tilde{g}(z)$$

$$f(z) g(z) \sim \tilde{f}(z) \tilde{g}(z)$$

*Proof.* Proof is left as an exercise. ▮

**Lemma 2.14.** *Let* $\tilde{f}(z) = \sum_{k=0}^{\infty} a_k z^k$*. The following property is satisfied by* $f(z)$*. There exists a sequence of non-negative integers* $\{p_n\}_{n=0}^{\infty}$ *so that so that for any* $n \geq 0$*,*

$$f(z) - \tilde{f}^{(p_n)}(z) = o(z^n) \ , \ \text{as } z \to 0$$

*then* $f \sim \tilde{f}$*.*

*Proof.* Take arbitrary $m \geq 0$. Choose $n$ sufficiently large so that $n > m$, $p_n > m$. Then,

$$f(z) - \tilde{f}^{(m)}(z) = f(z) - \tilde{f}^{(p_n)} + \tilde{f}^{(p_n)} - \tilde{f}^{(m)} = o(z^n) + o(z^m) = o(z^m),$$

since $\tilde{f}^{(p_n)} - \tilde{f}^{(m)}$ is a polynomial in $z$ with smallest power $z^{m+1}$. This is true for any $m$, hence $f \sim \tilde{f}$ ▮

2.1. **Integration and differentiation of asymptotic series.** Asymptotic power series can be safely integrated term by term (with proper limits); but this is generally not the case for differentiation. However, when we deal with functions in a complex sector, then differentiation of series is valid in any proper subsector, as we shall soon see.

**Proposition 2.15.** *Assume* $f$ *is integrable near* $z = 0$(1) *Assume* $f(z) \sim \tilde{f}(z) = \sum_{k=0}^{\infty} c_k z^k$*. Then*

$$\int_0^z f(s) ds \sim \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1}$$

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(1) Integrability need not extend to a ball; integrability in a sector or even along a curve approaching 0 suffices when asymptotic relation is only desired along those directions.

*Proof.* This follows by noting that $|f(s) - \sum_{k=0}^{N} c_k s^k| \le \epsilon |s|^N$ and therefore,

$$| \int_0^z \left\{ f(s) - \sum_{k=0}^{N} c_k s^k \right\} ds| \le \frac{\epsilon |z|^{N+1}}{N+1}$$

∎

**Proposition 2.16.** *Differentiation in a strip*

*Assume* $f$ *is analytic in a strip* $S_a \equiv \{x : |x| > R, |\mathrm{Im} x| < a\}$ *and that* $f(x) \sim \tilde{f}(x) \equiv \sum_{k=0}^{\infty} c_k x^{-k}$*. Then, for any* $0 < a' < a$*, for* $|x| \to \infty$*,* $x \in S_{a'}$*,*

$$f'(x) \sim \tilde{f}'(x) = -\sum_{k=0}^{\infty} k c_k x^{-k-1}$$

*Proof.* We write $f(x) = \tilde{f}^{(N)}(x) + g_N(x)$. We know $g_N$ is analytic in $S_a$ and that $|g_N(x)| \le C|x|^{-N-1}$. for $x \in S_a$. Take $\delta = \frac{1}{2}(a - a')$. Then, for any $x \in S_{a'}$, take circular contour $C_\delta$ around $x$ of radius $\delta$. From geometry, it is clear that $C_\delta \subset S_a$. We have

$$g_N'(x) = \frac{1}{2\pi i} \oint_{C_\delta} \frac{g_N(x') dx'}{(x' - x)^2} dx'$$

So,

$$|g_N'| \le C \delta^{-1} (|x| - \delta)^{-N-1}$$

If we replace $N$ by $p_N = N + 1$ in the above argument, we have for any $N \ge 0$,

$$\left| f(x) - \tilde{f}^{p_N} \right| \le C |x|^{-p_N - 1} = C|x|^{-N-2},$$

implying from Lemma 2.14 that $f \sim \tilde{f}$ ∎

**Remark 2.17.** *Similar proof shows that if* $f \sim \tilde{f}$ *for for* $x \to \infty$ *for* $x \in S$*, where* $S$ *is an angular sector, then* $f' \sim \tilde{f}'$*.*

**Remark 2.18.** *Differentiation is generally not valid on the real domain. Here is a counter-example. Consider*

$$f(x) = e^{-x} \sin\left[e^x\right] + \frac{1}{x+1}$$

*as* $x \to +\infty$*. It may be shown that*

$$f(x) \sim \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} \ + \ .. \equiv \tilde{f}$$

*However, it is not true that* $f' \sim \tilde{f}'$ *because the first term on differentiation, produces an* $O(1)$ *term as* $x \to +\infty$*. Under more restrictive conditions on* $f$*, like the one shown in the Theorem below differentiation of asymptotic relation is permitted.*

**Theorem 2.1.** *Let* $f(x)$ *be continuously differentiable and* $f(x) \sim x^p$ *as* $x \to \infty$*, where* $p \ge 1$ *is a constant. Then* $f'(x) \sim p\, x^{p-1}$*, provided* $f'(x)$ *is nondecreasing for all sufficiently large* $x$*.*

*Proof.* We write $f(x) = x^p\,[1 + \eta(x)\,]$. For given $\epsilon \in (0,1)$, there exists $X$ so that $|\eta(x)| \leq \epsilon$ when $x > X$. $X$ can be chosen sufficiently large so that $f'(x)$ is nondecreasing when $x \in (X, \infty)$. Then

$$h\ f'(x) \leq \int_x^{x+h} f'(t)\ dt = f(x + h) - f(x)$$

$$= \int_x^{x+h} p\ t^{p-1}dt + (x+h)\ \eta(x+h) - x^p\ \eta(x) \leq hp(x+h)^{p-1} + 2\epsilon\ (x+h)^p$$

If we now set $h = \epsilon^{1/2}\ x$, then we have

$$f'(x) \leq p\ x^{p-1} \left\{(1 + \epsilon^{1/2})^{p-1} + 2\ p^{-1}\ \epsilon^{1/2}\ (1 + \epsilon^{1/2}\ )^p \right\} \text{ for } x > X$$

Similarly, by taking an integral of $f'(t)$ between $x-h$ and $x$, and choosing $h = \epsilon^{1/2}\ x$, we obtain

$$f'(x) \geq p\ x^{p-1} \left\{(1 - \epsilon^{1/2})^{p-1} - 2\ p^{-1}\ \epsilon^{1/2}\right\} \text{ for } x > X/(1-\epsilon^{1/2})$$

From the last two equations, the theorem follows. ▮

### 2.2. **Asymptotics of Integrals.**

**Remark 2.19.** *We introduce the Watson's Lemma, which is one of the most important tools in the asymptotic analysis of integrals. Watson's Lemma concerns asymptotics of Laplace transform*

$$\{\mathcal{L}F\}\,(x) = \int_0^\infty F(p)e^{-px}$$

*as* $x \to \infty$*. Other variations of this problem are* $\int_0^a e^{-px}F(p)dp$*. However, on change of variables other problems involving integrals can be brought to either of these two forms. For instance,*

$$\Gamma(x+1) = \int_0^\infty e^{-t}t^x dt = x^{x+1}\int_0^\infty e^{-x(s-\ln s)}ds$$

$$= x^{x+1}\int_0^1 e^{-x(s-\ln s)}ds + = x^{x+1}\int_1^\infty e^{-x(s-\ln s)}ds$$

*We introduce the relation* $p = s - \ln s$*, which is 1-1 on each of the two intervals* $(0,1)$ *and* $(1,\infty)$*. We denote each of these inversions by* $s = s_1(p)$ *and* $s = s_2(p)$*. Then*

$$\Gamma(x+1) = x^{x+1}\int_0^\infty e^{-px}\left(s_2'(p) - s_1'(p)\right)dp, \tag{2.1}$$

*which is a problem that is addressed by Watson's Lemma*

**Lemma 2.20.** *Watson's Lemma If* $F \sim \mathbf{L}^1\left(\mathbb{R}^+\right)$ *and assume*

$$F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1+\beta_2-1}$$

*as* $p \to 0^+$ *for some constants* $\beta_1, \beta_2 > 0$*. Then,*

$$\mathcal{L}F \sim \sum_{k=0}^\infty c_k\Gamma(k\beta_1+\beta_2)x^{-\beta_1 k-\beta_2}$$

*as* $x \to \infty$ *along any ray in the complex right half* $x$*-plane.*

*Proof.* We only prove it for the case $F \sim p^{\beta}$ for $\beta > -1$, since induction allows us to prove the more general result. We note that

$$\int_0^{\infty} p^{\beta} e^{-px} = \Gamma(\beta+1) x^{-\beta-1}$$

Since $F \sim p^{\beta}$, it implies that for given $\epsilon$, there exists $a$ (independent of $x$) so that for $p \in [0, a]$ $a$, $|F(p) - p^{\beta}| \leq \epsilon |p|^{\beta}$. So,

$$| \int_0^{\infty} \left[ F(p) - p^{\beta} \right] e^{-px} dp| = \int_0^{a} \epsilon |p|^{\beta} e^{-\mathrm{Re}x} dp + \int_a^{\infty} \left\{ |F(p)| + |p|^{\beta} \right\} e^{-p\mathrm{Re}x} dp$$

$$\leq \epsilon \Gamma(\beta+1) \left[ \mathrm{Re}x \right]^{-\beta-1} + e^{-a\mathrm{Re}x} \left\{ \int_a^{\infty} |F(p)| dp + \int_a^{\infty} |p|^{\beta} e^{-(p-a)\mathrm{Re}x} dp \right\} \leq C\epsilon |x|^{-\beta-1}$$

$\blacksquare$

**Example:** For the Gamma function, problem, we can determine

$$s_2'(p) - s_1'(p) = \sum_{k=0}^{\infty} c_k p^{k/2-1/2}$$

where $c_0 = \sqrt{2}$, $c_1 = 0$, $c_2 = \sqrt{2}/6$ and therefore,

$$n! = \Gamma(n+1) \sim \sqrt{2\pi n} n^n e^{-n} \left( 1 + \frac{1}{12n} + \frac{1}{288n^2} + .. \right)$$