Week 7 lectures

1. More discussion on the asymptotics of $\Gamma(x+1)$

Recall we were discussing using Watson's Lemma to calcualte

$$x! \equiv \Gamma(x+1) = x^{x+1} \int_1^\infty e^{-px} \left\{ s_2'(p) - s_1'(p) \right\} dp$$

where s_1 and s_2 are the inversions of the relation $p = s - \ln s$ over the s-intervals (0, 1) and $(1, \infty)$ respectively. We now determine the explicit asymptotic behavior of $s_{1,2}$ and its derivatives. We first note from Taylor expansion at $s = 1$, we obtain

$$p = 1 + \frac{1}{2}(s-1)^2 - \frac{1}{3}(s-1)^3 \; + \; (-1)^k \frac{(s-1)^k}{k} \; + \; ..$$

So, for branch $s_1(p)$, where $s \in (0, 1)$ corresponds to $p \in (1, \infty)$, we have

$$\sqrt{2}(p-1)^{1/2} = (1-s) \left\{ 1 - \frac{2}{3}(s-1) + \frac{2}{4}(s-1)^2 ... \right\}^{1/2} \tag{1.1}$$

Since $(p-1)^{1/2}$ is a regular analytic function of $(1-s)$ near $s = 1$ with derivative nonzero at $s = 1$, it follows that it's inverse is a regular analytic function of $(p-1)^{1/2}$ at $p = 1$ and posseses a convergent Taylor in the form

$$1 - s_1(p) = \sum_{k=1}^\infty a_k (p-1)^{k/2}$$

By plugging this into relation (1.1), we can determine a_k by equating different powers of $(p-1)^{1/2}$. For instance, the first few terms are

$$a_1 = \sqrt{2}, \; , \; a_2 = -\frac{2}{3} \; , \; a_3 \; = \; \frac{1}{9\sqrt{2}} \; , \; a_4 = \frac{2}{135} \tag{1.2}$$

For the branch $s_2(p)$ we must take the other root of the squareroot in the relation (1.1) since $s \in (1, \infty)$ corresponds to $p \in (1, \infty)$. Therefore, instead of (1.1), we have

$$\sqrt{2}(p-1)^{1/2} = (s-1) \left\{ 1 - \frac{2}{3}(s-1) + \frac{2}{4}(s-1)^2 ... \right\}^{1/2} \tag{1.3}$$

We obtain on inversion the convergent expansion

$$s_2(p) - 1 = \sum_{k=1}^\infty b_k (p-1)^{k/2}, \tag{1.4}$$

where the first few coefficients are

$$b_1 = \sqrt{2}, \ , \ b_2 = \frac{2}{3} \ , \ b_3 \ = \ \frac{1}{9\sqrt{2}} \ , \ b_4 = -\frac{2}{135} \tag{1.5}$$

We can prove that $b_k = a_k$ for odd k and $b_k = -a_k$ for even k (as is seen in the first few coefficients). So we obtain

$$s_2'(p) - s_1'(p) = \sum_{j=0}^{\infty} (2j+1) a_{2j+1} (p-1)^{j-1/2} \tag{1.6}$$

So, from Watson's Lemma (used after change of variable $p = 1 + q$),

$$\Gamma(x+1) = x^{x+1} \int_1^{\infty} e^{-px} \left(s_2' - s_1' \right) dp \sim x^{x+1} e^{-x} \sum_{j=0}^{\infty} (2j+1) a_{2j+1} \int_0^{\infty} e^{-qx} q^{j-1/2} dq$$

$$= x^{x+1} e^{-x} \sum_{j=0}^{\infty} (2j+1) x^{-j-1/2} a_{2j+1} \Gamma(j+1/2) = \sqrt{2\pi} x^{x+1/2} e^{-x} \left\{ 1 + \frac{1}{12x} \ + \ O(x^{-2}) \right\},$$

where we used $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$. The first term is the well-known Stirling Formula for $x!$.

Remark 1.1. *The reason why asymptotics of $F(p)$ near $p = 0$ matters in the large x asymptotics for $\int_0^\infty e^{-px} F(p) dp$ is because for x large e^{-px} decreases fast. So the only sizable contribution comes from the neighborhood of $p = 0$*

Remark 1.2. *Watson's Lemma is also valid when $\int_0^\infty e^{-px} F(p) dp$ is replaced by $\int_0^a e^{-px} F(p) dp$ for a independent of x since we may replace $F(p)$ in the first formula by $F(p) \chi_{[0,a]}(x)$, which is integrable, where $\chi_{[0,a]}$ is the characteristic function on $[0, a]$.*

Remark 1.3. *Watson's Lemma can also be used to determine asymptotics of $I(x) = \int_{-\infty}^{\infty} e^{-xt^2} f(t) dt$. This is done by writing*

$$\int_{-\infty}^{\infty} e^{-xt^2} f(t) dt = \left\{ \int_{-\infty}^{0} + \int_0^{\infty} \right\} e^{-xt^2} f(t) dt$$

We now change variables $t = -\sqrt{p}$ in the first integral and $t = +\sqrt{p}$ in the second to obtain

$$I(x) = \int_0^{\infty} e^{-px} \frac{dp}{2\sqrt{p}} \left\{ f(\sqrt{p}) + f(-\sqrt{p}) \right\}$$

Watson's Lemma can now be applied with

$$F(p) = \frac{f(\sqrt{p}) + f(-\sqrt{p})}{2\sqrt{p}}$$

2. Riemann Hilbert Problem:

Riemann-Hilbert problem refers to determination of a sectionally Holomorphic function (*i.e.* analytic everywhere off a curve) $\Phi(z)$ so that across a simple smooth curve $\mathcal{C}$, Φ jumps by a specified amount ϕ, *i.e.*

$$\Phi_+(t) - \Phi_-(t) = \phi(t) \text{ . for } t \in \mathcal{C} \tag{2.7}$$

where subscript $+$ and $-$ indicates limiting values of Φ as point t on the curve is approached from different sides. The treatment here is close to that in O. Costin's notes.

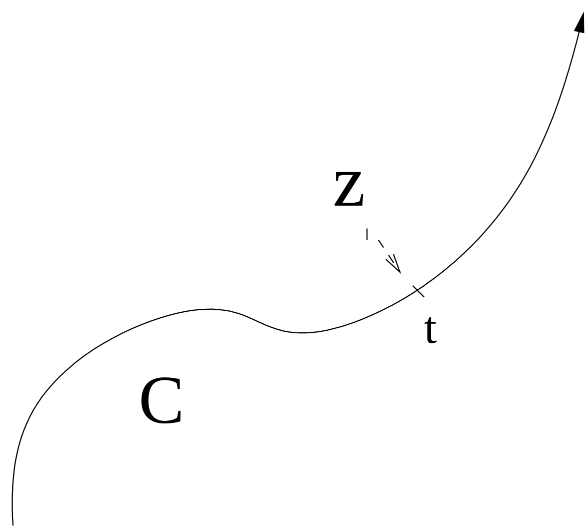

Figure 1. z approaching a point t on a simple smooth oriented curve $\mathcal{C}$ from the left $(+)$ side.

Remark 2.1. *Riemann-Hilbert problem arises in a wide variety of problems: such as solving integral equations of certain types, inverse scattering problems, nonlinear integrable systems, radon transforms that arise in tomography, just to name a few.*

Remark 2.2. *A particular case of Riemann-Hilbert problem is when $\mathcal{C}$ is the entire real line. In that case (or in any other case of closed contour $\mathcal{C}$) Φ_+ and Φ_- are two different analytic functions in the upper-half and lower-half plane (two sides of $\mathcal{C}$) with specified jump:*

$$\Phi_+(x) - \Phi_-(x) = f(x) \tag{2.8}$$

We may write this relation in terms of distribution

$$\frac{\partial \Phi}{\partial y} = f(x)\delta(y) \tag{2.9}$$

A generalization of this is the $\bar{\delta}$ (D-bar) problem, when we seek Φ satisfying

$$\frac{\partial \Phi}{\partial \bar{z}} = g(x,y) \tag{2.10}$$

in some region $\mathcal{D} \subset \mathbb{C}$, where

$$\frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left\{ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right\}$$

Definition 2.3. *A function ϕ is said to be Holder-Continuous on a smooth curve $\mathcal{C}$ of order $\lambda \in (0,1)$, if for any $x, y \in \mathcal{C}$,*

$$\phi(x) - \phi(y) = O\left(|x-y|^{\lambda}\right)$$

Let $\mathcal{C} = \{z : z = \gamma(t)\ 0 \le t \le 1\}$ be a simple smooth oriented curve. For a closed curve, we will always take the orientation to be counter-clockwise. At each point $\gamma(t)$, for $t \in (0,1)$, we draw a small enough circle so that the circle intersects $\mathcal{C}$ at just two points. The curve bisects the circle into two parts, each of which is a near semi-circle. If we approach $\mathcal{C}$ from the interior of a near semi-circle that lies to the left of $\mathcal{C}$, then we say that $\mathcal{C}$ has been approached from the left. Similarly, we define approach from the right. Limiting values of a function from the left and right of an oriented simple curve $\mathcal{C}$ will be always denoted by subscript $+$ and $-$ respectively. Note $\mathcal{C}$ may either be open or closed. In the latter case, $\gamma(0) = \gamma(1)$.

We consider the following function(s) $\Phi(z)$ for $z \notin \mathcal{C}$:

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\phi(s)}{s-z} ds \tag{2.11}$$

Now, note from Fubini's theorem, $\oint \Phi(z) dz = 0$ for any close contour in the z plane as long as this contour is at a finite distance from $\mathcal{C}$. So, from Moerara's theorem Φ is analytic off $\mathcal{C}$, *i.e.* Φ is a sectionally holomorphic function. Note in the case of closed contour $\mathcal{C}$, the above formula defines two distinct function $\Phi_+(z)$ in the domain on the left of $\mathcal{C}$ and $\Phi_-(z)$ in the domain to the right of $\mathcal{C}$.

We also notice from (2.11) that for any bounded curve $\mathcal{C}$, the asymptotic behavior

$$\Phi(z) \sim -\frac{1}{2\pi i z}\int_{C} \phi(s) ds \text{ as } z \to \infty \tag{2.12}$$

Theorem 2.1. *Plemelj formulae Assume ϕ is a holder continous function on a simple smooth compact curve $\mathcal{C}$, which may or may not be closed. Let $t \in \mathcal{C} \setminus \gamma(0) \cup \gamma(1)$ (i.e. not end points). Let z_n be a set of points approaching t from the left (or right). Then, let $\Phi_\pm(t) = \lim_{n\to\infty} \Phi(z_n)$, with Φ defined by (2.11). We have*

$$\Phi_\pm(t) = \pm\frac{1}{2}\phi(t) + \frac{1}{2\pi i}\fint_{\mathcal{C}} \frac{\phi(s)}{s-t} ds$$

Remark 2.4. *We consider the special case where $\mathcal{C}$ coincides with part of the real axis, [-1, 1] (see Fig. 2), without much loss of generality. More general smooth curves may be handled by introducing a smooth change in variable that parametrizes the curves. We will prove Plemelj formulae at the end of some preliminary lemmas.*

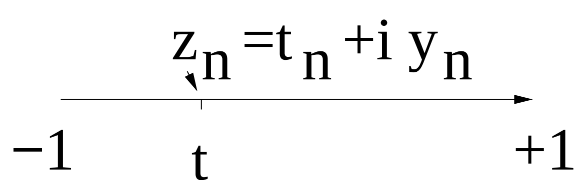

FIGURE 2. $z_n = t_n + iy_n$ approaching $t \in (-1, 1)$ from the left (top), with $t_n \to t$, $y_n \to 0^+$

Lemma 2.5. *Define $\Psi_n(s) = \frac{\phi(s)-\phi(t_n)}{s-z_n}$, where $z_n = t_n + iy_n \equiv t + \epsilon_n + iy_n$, where $\epsilon_n \to 0$ and $y_n \to 0^+$ (or $y_n \to 0^-$, when approach is from the right) as $n \to \infty$. If ϕ is Holder continuous, then*

$$\lim_{n\to\infty} \int_{-1}^{1} \Psi_n(s) ds = \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - t} ds$$

Proof. We note that

$$\int_{-1}^{1} \Psi_n(s) ds = \int_{-2}^{2} \Psi_n(s)\chi(s) ds,$$

where χ is the characteristic function for the interval $[-1, 1]$, *i.e.* $\chi(s) = 1$ in this interval and 0 outside. Then,

$$\begin{aligned}\int_{-1}^{1} \Psi_n(s) ds &= \int_{-2}^{2} \frac{\phi(s) - \phi(t + \epsilon_n)}{s - t - \epsilon_n - iy_n}\chi(s) ds \\ &= \int_{-2}^{2} \frac{\phi(\sigma + \epsilon_n) - \phi(t + \epsilon_n)}{\sigma - t - iy_n}\chi(\sigma + \epsilon_n) d\sigma\end{aligned}$$

However, from Holder Condition,

$$|\phi(\sigma+\epsilon_n)-\phi(t+\epsilon_n)| \le C|\sigma-t|^{\lambda},$$

and since $\frac{1}{|\sigma-t-iy_n|} \le \frac{1}{|\sigma-t|}$ we have

$$|\int_{-1}^{1} \Psi_n(s)ds| \le C\int_{-2}^{2} |\sigma-t|^{\lambda-1} d\sigma < \infty,$$

since $\lambda > 0$. From dominated convergence theorem, it follows that

$$\lim_{n\to\infty}\int_{-1}^{1} \Psi_n(s)ds = \int_{-2}^{2} \frac{\phi(\sigma)-\phi(t)}{\sigma-t}\chi(\sigma)d\sigma = \int_{-1}^{1} \frac{\phi(\sigma)-\phi(t)}{\sigma-t}d\sigma$$

∎

Lemma 2.6. *For $t \in (-1,1)$ with $y_n \to 0^{\pm}$, $\epsilon_n \to 0$ and $t_n = t+\epsilon_n$,*

$$\lim_{n\to\infty}\int_{-1}^{1} \frac{1}{s-t_n-iy_n}ds = \fint_{-1}^{1} \frac{1}{s-t}ds \pm \pi i$$

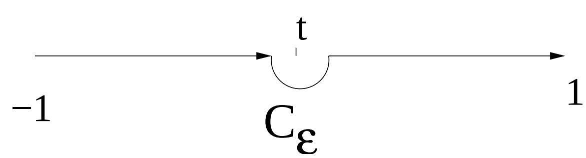

FIGURE 3. Deformed contour for evaluation of $\lim_{n\to\infty}\int_{-1}^{1}\frac{1}{s-t_n-iy_n}ds$, when $y_n \to 0^+$

Proof. Since the integrand is analytic in s when $y_n > 0$, we deform the contour to the one shown in Fig. 3, where $\epsilon > 0$ is chosen small enough so that $-1 < t-\epsilon < t+\epsilon < 1$. We get

$$\int_{-1}^{1} \frac{1}{s-t_n-iy_n}ds = \left\{\int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1}\right\} \frac{ds}{s-t_n-iy_n} + \int_{C_\epsilon} \frac{ds}{s-t_n-iy_n}$$

As $n \to \infty$, since $\epsilon_n \to 0$, $t_n \to t$, we have due to uniform convergence,

$$\lim_{n\to\infty}\left\{\int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1}\right\} \frac{ds}{s-t_n-iy_n} = \left\{\int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1}\right\} \frac{ds}{s-t}$$

Also, for $y_n \to 0^+$, $t_n \to t$, from uniform convergence of integrand,

$$\lim_{n\to\infty}\int_{C_\epsilon} \frac{ds}{s-t_n-iy_n} = \int_{C_\epsilon} \frac{ds}{s-t} = \int_{-\pi}^{0} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}d\theta = \pi i$$

On the otherhand if $y_n \to 0^-$, we choose $\mathcal{C}_\epsilon$ to be small upper-semi-circular contour to get,

$$\lim_{n\to\infty} \int_{C_\epsilon} \frac{ds}{s - t_n - iy_n} = \int_{C_\epsilon} \frac{ds}{s-t} = \int_{\pi}^{0} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} d\theta = -\pi i$$

Therefore, for $y_n \to 0^\pm$,

$$\lim_{n\to\infty} \left\{ \int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1} \right\} \frac{ds}{s - t_n - iy_n} = \left\{ \int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1} \right\} \frac{1}{s-t} ds \pm \pi i$$

Taking the limit of $\epsilon \to 0$, we get the desired result. ∎

Proof of Theorem 2.1:

We note that

$$\int_{-1}^{1} \frac{\phi(s)}{s - z_n} ds = \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - z_n} ds + \phi(t) \int_{-1}^{1} \frac{ds}{s - z_n} ds$$
$$= \Psi_n + \phi(t) \int_{-1}^{1} \frac{ds}{s - t_n - iy_n}$$

Using Lemmas (2.5) and (2.6), we obtain the desired proof.

Remark 2.7. *With additional conditions on decay of ϕ, Plemelj formula can be extended to $\mathcal{C}$ which is not compact as in the following Lemma.*

Lemma 2.8. *Extension of Plemelj when $\mathcal{C} = \mathbb{R}$*

Assume $\mathcal{C} = \mathbb{R}$ (the real line). Assume that as $|s| \to \infty$, $\phi(s) \to L$ and that ϕ is Holder continous uniformly for any $x, y \in \mathbb{R}$ and that

$$|\phi(s) - L| = O(s^{-\mu}), \text{ as } |s| \to \infty \text{ for } \mu > 0$$

Then Theorem 2.1 holds.

Proof. is similar to that of 2.1 and is left as an exercise. ∎

Remark 2.9. *We note that Plemelj Formula immediately implies the jump formula:*

$$\Phi_+(t) - \Phi_-(t) = \phi(t)$$

Remark 2.10. *In the case, $\mathcal{C}$ a closed curve (i.e. separates the plane into two separate $+$ and $-$ regions),*

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\phi(s) ds}{s - z}$$

actually defines two distinct analytic functions $\Phi_+(z)$ and $\Phi_-(z)$ depending on whether z is on the plus (left) or minus side (right) of the oriented curve $\mathcal{C}$. These functions must be distinct because the Plemelj *formulae (9) demands that they are unequal, when each is continued to $\mathcal{C}$ itself.*

Remark 2.11. *Note that the analytic continuation of $\Phi(z)$ (or $\Phi_\pm(z)$ for closed curve) is possible across the curve $\mathcal{C}$ when $\phi(z)$ is analytic. This analytic continuation is given by*

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\phi(s)}{s-z}\,ds \;\pm\; \phi(z)$$

depending on whether the analytic continuation proceeded from the left of the curve $\mathcal{C}$ to its right, or the other way around.

Simple example: Let $\mathcal{C}$ be a unit circle. Determine $\Phi_+(z)$ and $\Phi_-(z)$ so that for $t \in \mathcal{C}$, $\Phi_+(t) - \Phi_-(t) = 1$. We write

$$\Phi_\pm(z) = \frac{1}{2\pi i}\oint_{|s|=1} \frac{ds}{s-z}$$

for z inside and outside the unit circle, respectively. For $|z| < 1$, clearly, $\Phi_+(z) = 1$; while for $|z| > 1$, $\Phi_-(z) = 0$. Let $\mathcal{C}$ be a unit circle. Determine $\Phi_+(z)$ and $\Phi_-(z)$ so that for $t \in \mathcal{C}$, $\Phi_+(t) - \Phi_-(t) = 1$. We write

$$\Phi_\pm(z) = \frac{1}{2\pi i}\oint_{|s|=1} \frac{ds}{s-z}$$

for z inside and outside the unit circle. For $|z| < 1$, clearly, $\Phi_+(z) = 1$; while for $|z| > 1$, $\Phi_-(z) = 0$.
Another example: Consider the problem of finding a sectionally holomorphic function $\Phi(z)$ in $\mathbb{C} \setminus [-1, 1]$ satisfying

$$\Phi_+(x) - \Phi_-(x) = 1 \text{ for } x \in (-1, 1)$$

We know that

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{dt}{t-z} = \frac{1}{2\pi i}\ln\left\{\frac{z-1}{z+1}\right\}$$

will satisfy this condition. We can check directly. Approaching $x \in (-1, 1)$ from the top, $\ln(z-1) \to \ln(1-x) + i\pi$, while $\ln(z+1) \to \ln(1+x)$. Therefore, from the top,

$$\frac{1}{2\pi i}\ln\left\{\frac{z-1}{z+1}\right\} \to \frac{1}{2\pi i}\ln\left\{\frac{1-x}{1+x}\right\} + \frac{1}{2}$$

From the bottom, since $\ln(z-1) \to \ln(1-x) - i\pi$, it follows

$$\frac{1}{2\pi i}\ln\left\{\frac{z-1}{z+1}\right\} \to \frac{1}{2\pi i}\ln\left\{\frac{1-x}{1+x}\right\} - \frac{1}{2}$$

Therefore, indeed $\Phi_+(x) - \Phi_-(x) = 1$.

Remark 2.12. *The answer in the last example is not unique. We can add to a particular* $\Phi(z)$ *any analytic function of* z *with isolated singularities at end points. If we required* $\lim_{z\to z_e}(z-z_e)\Phi(z)=0$ *at each end point* z_e *and in addition required* $\lim_{z\to\infty}\Phi(z)=0$*, then the answer would be unique. To see this, let* $\Psi(z)$ *be another solution. Then* $\Phi(z)-\Psi(z)$ *is single valued at* $z=z_e$ *since each of* Ψ *and* Φ *jumps by the same amount* ψ*. Since the condition at* z_e *rules out simple pole, it follows* $\Phi-\Psi$ *is an entire function. Growth condition at* ∞ *implies* $\Phi-\Psi=0$.

3. Scalar Homogeneous Riemann-Hilbert (RH) Problem

In this case we seek to find sectionally holomorphic function $\Phi(z)$ so that on a simple smooth curve $\mathcal{C}$:

$$\Phi_+(t)=g(t)\Phi_-(t) \tag{3.13}$$

where $g(t)\neq 0$ on $\mathcal{C}$ and Holder continuous. By taking the log of both sides, this becomes a problem of determining $\Psi(z)$ so that $\Psi_+(t)-\Psi_-(t)=\ln g(t)$. This becomes a familiar problem as discussed in the last section. Once $\Psi(z)$ is determined, we have $\Phi(z)=e^{\Psi(z)}$. When $\mathcal{C}$ is closed there is additional complication since $\ln g(t)$ may not return to the same value as we traverse the closed path $\mathcal{C}$. We have to define some new concepts.

Definition 3.1. *Let* $\mathcal{C}:\{z:z=\gamma(t),\ t\in[0,1]\}$ *be a simple smooth closed oriented curve and* ϕ *be a differentiable function on* $\mathcal{C}$ *and with* $\phi\neq 0$ *on* $\mathcal{C}$*. Then the index of* ϕ *with respect to curve* $\mathcal{C}$ *is an integer defined by*

$$ind_{\mathcal{C}}\phi\equiv\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\phi'(s)}{\phi(s)}ds$$

Remark 3.2. *If* ϕ *is a meromorphic function, we know index is* $N-P$*, the difference of number of zeros and number of poles within* $\mathcal{C}$*. However,* $ind_{\mathcal{C}}\phi$ *is defined even if the function is not an analytic function any where since since* $\frac{1}{2\pi i}\left[\ln\phi\right]_{\text{jump}}$ *must be an integer if we follow the* $\ln$ *function on branch where it is continuous.*

Definition 3.3. *A function* Φ *has degree* k *at* ∞ *if there exists some constant* $C\neq 0$ *so that*

$$\Phi(z)=Cz^k+O(z^{k-1}) \text{ as } z\to\infty$$

A function Φ *is said to have a finite degree at* ∞ *if there exists some finite* m *so that* $\Phi(z)=o(z^m)$ *as* $z\to\infty$.

3.1. **Solution to homogeneous RH problem.**

Lemma 3.4. *Let $\mathcal{C}$ be a simple smooth closed oriented curve with $\text{ind}_{\mathcal{C}} g = k$ for some nonzero differentiable function g on $\mathcal{C}$. Assume $z = 0$ is a point inside $\mathcal{C}$. Then a solution to the scalar RH problem (3.13) is given by $\Phi_+(z) = e^{\Psi_+(z)}$, $\Phi_-(z) = z^{-k} e^{\Psi_-(z)}$ where*

$$\Psi_\pm(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(s)}{s - z} ds \text{ for } z \text{ inside and outside } \mathcal{C} \text{ respectively,}$$

where $f(t) = \log \left[t^{-k} g(t) \right]$.

Proof. Clearly, by taking the log of both sides of (3.13) we can write

$$\ln \Phi_+(t) - \ln \left[t^k \Phi_-(t) \right] = \ln \left[t^{-k} g(t) \right] = f(t)$$

Since the index of g is k, it follows that $f(t)$ is continuous on $\mathcal{C}$, since both $\log g$ and $\log t^k$ jump by $2\pi i k$ as we traverse the curve. Therefore, if we define $\Psi_+(z) = \ln \Phi_+(z)$ for z inside $\mathcal{C}$ and $\Psi_-(z) = \ln \left\{ z^k \Phi_-(z) \right\}$, it follows that

$$\Psi_+(t) - \Psi_-(t) = f(t)$$

We know a solution to this is given by

$$\Psi(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(s)}{s - z} ds$$

from which the Lemma follows. ∎

Remark 3.5. *If $z = 0$ is not inside C, we replace the $z^{\pm k}$, $t^{\pm k}$ in the the above formulae by $(z - z_0)^{\pm k}$ and $(t - z_0)^{\pm k}$ for some z_0 inside C.*

Remark 3.6. *The solution for $\Phi_\pm(z)$ in the last Lemma is not unique. We can clearly multiply each of $\Phi_\pm(z)$ by any entire function $E(z)$ and we still satisfy (3.13). However, if we require that Φ_- (whose domain includes ∞) has degree m at ∞, then the most general solution to (3.13) is*

$$\Phi_+(z) = P_{m+k}(z) e^{\Psi_+(z)} \ , \ \Phi_-(z) = P_{m+k}(z) z^{-k} e^{\Psi_-(z)}$$

where $P_{m+k}(z)$ is a polynomial of degree $m + k$, if $m + k \geq 0$ and zero otherwise. This is because as as $z \to \infty$, from (2.12), $\Psi_-(z) = O(1/z)$, and therefore $e^{\Psi_-(z)} \to 1$. Recalling that an entire function that grows no faster than z^{m+k} for $m + k \geq 0$ must be a Polynomial P_{m+k}, our claim follows.