Week 9 lectures

1. Wiener-Hopf method

Remark 1.1. *For partial differential equations with constant coefficients involving two independent variables, when domain domain boundaries are either half-lines or a full line with different types of boundary conditions on two segments, the Wiener-Hopf method outlined in the next few classes is hopeful. The essential idea involves Fourier-Transforms, analytic continuation. In some ways, it has strong similarity to the Riemann-Hilbert problem. We don't have time for a general theory of Wiener-Hopf techniques, but will illustrate it through one or two concrete examples.*

1.1. Wiener-Hopf method in a PDE problem of steady cooling.

Mathemtical Problem: Find $\phi(x,y)$ satisfying:

$$\Delta \phi - \phi_x = 0 \text{ with} (x,y) \in \mathcal{D} \tag{1.1}$$

where $\mathcal{D}$ is the domain with half-line boundary $y = 0, \ x \geq 0$. We require

$$\phi(x,y) \to 0 \text{ as } x^2 + y^2 \to \infty \tag{1.2}$$

$$\phi(x,0) = e^{-ax} \text{ for } x \geq 0 \text{ with } a > 0 \tag{1.3}$$

We seek bounded solution ϕ with $\phi_y(x,0)$ integrable on $\mathbb{R}$. Further, from physical motivation, we seek solution so that

$$\lim_{x \to -\infty} e^{-\alpha x} \phi(x,0) = 0 \ \lim_{x\to\infty} e^{-\beta x} \phi_y(x,0^{\pm}) = 0 \tag{1.4}$$

for some α, β satisfying the restriction $1 \geq \alpha > \beta \geq 0$. The reason for the restriction will become clearer in the sequel.

Formal Solution Methodology:

Define Fourier

$$\Phi(\lambda,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x,y)\, e^{i\lambda x}\, dx \tag{1.5}$$

Fourier-transforming (1.1), with respect to x, we obtain

$$\Phi_{yy} - (\lambda^2 - i\lambda)\, \Phi = 0 \tag{1.6}$$

Also, Fourier-transforming (1.3), we obtain

$$\Phi(\lambda,0) = \frac{1}{\sqrt{2\pi}} \frac{1}{a - i\lambda} + U(\lambda) \tag{1.7}$$

where $U(\lambda)$ is the Fourier-transform of $u(x)$, defined as

$$u(x) = \phi(x,0) \text{ for } x < 0 \text{ and } = 0 \text{ otherwise} \tag{1.8}$$

From (1.4),

$$\lim_{x \,\to\, -\infty} e^{-\alpha x}\; u(x) \;=\; 0 \text{ for some } \alpha > 0 \tag{1.9}$$

From Fourier-Transform theory, this implies that

$$U(\lambda) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} dx\; e^{i\lambda x}\; u(x)$$

is an analytic function of λ for $Im\ \lambda \;<\; \alpha$. The necessary Fourier-inversion path in this case is achieved for a path within this region, connecting $Re\ \lambda \;=\; -\infty$ to $+\infty$.

The solution to (1.6) satisfying (1.7), as well as goes to zero for $|y| \;\to\; \infty$, in accordance with requirement (1.2), is given by

$$\Phi(\lambda, y) \;=\; \left[\frac{1}{\sqrt{2\pi}}\frac{1}{a - i\lambda} \;+\; U(\lambda)\;\right] \; exp\; [-|y|\sqrt{\lambda^2 - i\lambda}] \tag{1.10}$$

Here the cuts and branch for $\sqrt{\lambda^2 - i\lambda}$ is chosen as in Fig. 1, so that as $\lambda \;\to\; \pm\infty$ on the real axis, it behaves as $|\lambda|$. From (1.10), it follows that

(1.11)

$$F(\lambda) \;\equiv\; \Phi_y(\lambda,\, 0^+) - \Phi_y(\lambda,\, 0^-) \;=\; -2\; \left[\frac{1}{\sqrt{2\pi}}\frac{1}{a - i\lambda} \;+\; U(\lambda)\;\right] \sqrt{\lambda^2 - i\lambda}$$

We note from continuity of a solution $\phi(x, y)$ to (1.1), that for $x \;<\; 0$, $\phi_y(x, 0^+) \;=\; \phi_y(x, 0^-)$; hence

(1.12)

$$F(\lambda) \;=\; \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\, e^{i\lambda x}\, dx \text{ where } f(x) \;=\; \phi_y(x, 0^+) - \phi_y(x, 0^-) \;=\; 2\, \phi_y(x, 0^+)$$

Therefore from (1.4), $f(x)\; e^{-\beta x} \;\to\; 0$ as $x \;\to\; +\infty$ and $F(\lambda)$ is therefore, analytic $Im\ \lambda \;>\; \beta$ (See Fig. 1). It is to be noted that the Fourier-inversion path for $F(\lambda)$ has to be restricted to this region.

We consider the possibility $1 \;\geq\; \alpha \;>\; \beta \;\geq\; 0$, which is checked *a posteriori*. Under these conditions, the inversion path for all the functions of interest (F and U) has to be restricted to the cross-shaded region in Fig. 1, that lies between the branch points 0 and i of $\sqrt{\lambda^2 - i\lambda}$. We will check at the end that these assumptions are indeed satisfied by the solution. We introduce the subscript + to denote any function analytic for $Im\ \lambda \;>\; \beta$ and $-$ subsecript to denote a function analytic for $Im\ \lambda \;<\; \alpha$. Then (1.12) can be rewritten as:

$$F_+(\lambda) \;=\; -2 \left[\left(\frac{1}{\sqrt{2\pi}}\frac{1}{a - i\lambda}\right)_+ \;+\; U_-(\lambda)\;\right] \sqrt{\lambda^2 - i\lambda} \tag{1.13}$$

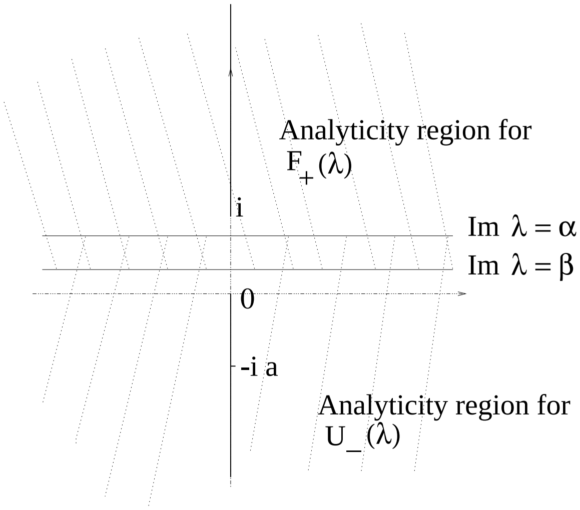

FIGURE 1. Branch cut for $\sqrt{\lambda^2 - i\lambda}$ and analyticity region of F and U

It is convenient to rewrite (1.13) as:

$$- \frac{F_+(\lambda)}{2\sqrt{\lambda}} = \frac{\sqrt{\lambda - i}}{\sqrt{2\pi}(a - i\lambda)} + \sqrt{\lambda - i}\, U_-(\lambda) \tag{1.14}$$

The first term on the right of (1.14) can be decomposed as the sum of a plus and minus function:

$$\frac{\sqrt{\lambda - i}}{\sqrt{2\pi}(a - i\lambda)} = \left[\frac{\sqrt{\lambda - i} - \sqrt{-ia - i}}{\sqrt{2\pi}(a - i\lambda)} \right]_- + \left[\frac{\sqrt{-ia - i}}{\sqrt{2\pi}(a - i\lambda)} \right]_+ \tag{1.15}$$

Consequently (1.14) becomes:

$$\left[-\frac{F(\lambda)}{2\sqrt{\lambda}} - \frac{\sqrt{-ia - i}}{\sqrt{2\pi}(a - i\lambda)} \right]_+ = \left[\frac{\sqrt{\lambda - i} - \sqrt{-ia - i}}{\sqrt{2\pi}(a - i\lambda)} + \sqrt{\lambda - i}\, U(\lambda) \right]_- \tag{1.16}$$

The function on the left of (1.16) is an analytic continuation of the function on the right, the equality holding for $\alpha > Im\ \lambda > \beta$. Therefore, this defines an an entire function $E(\lambda)$. Further since $\phi_y(x, 0)$ is integrable, i.e. in $\mathcal{L}_1$, as assumed earlier, from Riemann-Lebesgue Lemma, $F(\lambda) \to 0$ as $|\lambda| \to \infty$, at least when $Im\ \lambda > \beta$. Thus, on examination, the left hand side of (1.16) tends to zero as $|\lambda| \to \infty$ in this upper-half region. Also, since $u(x)$ is assumed bounded everywhere, $|\lambda\, U(\lambda)|$ must be bounded as $\lambda \to \infty$, with $Im\ \lambda < \alpha$. Thus, the right hand side of (1.16) also tends to zero as $|\lambda| \to \infty$, at least for $Im\ \lambda\ <\ \alpha$. These two decay conditions imply $E(\lambda)\ =\ 0$, i.e.

$$F(\lambda)\ =\ -\frac{2\sqrt{\lambda}\ \sqrt{-ia-i}}{\sqrt{2\pi}(a-i\lambda)} \tag{1.17}$$

$$U(\lambda)\ =\ -\ \frac{\sqrt{\lambda-i}-\sqrt{-ia-i}}{\sqrt{2\pi}\ \sqrt{\lambda-i}\ (a-i\lambda)} \tag{1.18}$$

From (1.10) and (1.18), we obtain

$$\Phi(\lambda,y)\ =\ \frac{\sqrt{1\ +\ a}\ e^{-i\pi/4}}{\sqrt{2\pi}(a-i\lambda)\sqrt{\lambda-i}}\ exp[-|y|\ \sqrt{\lambda^2-i\lambda}] \tag{1.19}$$

From Fourier-inversion,

$$f(x)\ =\ -i\ \frac{e^{-i\pi/4}\sqrt{1+a}}{\pi}\ \int_{-\infty}^{\infty}\frac{\sqrt{\lambda}\ e^{-i\lambda x}}{\lambda\ +\ i\ a}\ d\lambda \tag{1.20}$$

From use of Jordan's Lemma, it is clear that by closing the contour with an upper-half semi-circle, $f(x)\ =\ 0$ for $x\ <\ 0$. Further, for $x\ >\ 0$, by use of Jordan's Lemma, it is clear that

$$f(x)\ =\ -i\ \frac{e^{-i\pi/4}\sqrt{1+a}}{\pi}\ \int_C\frac{\sqrt{\lambda}\ e^{-i\lambda x}}{\lambda\ +\ i\ a}\ d\lambda$$

where the contour C is shown in Fig. 2.

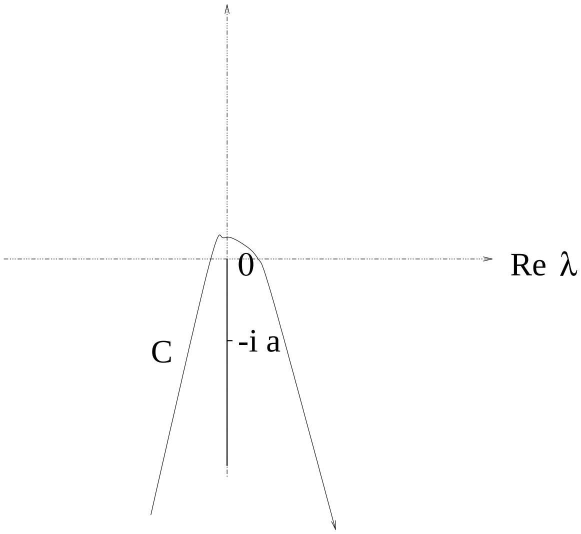

FIGURE 2. Contour C for evaluation of $f(x)$ in (19)

To evaluate $f(x)$, we write it alternately in the form
(1.21)

$$f(x) \;=\; -i\,\frac{e^{-i\pi/4}\sqrt{1+a}}{\pi}\,\left[\int_C \frac{e^{-i\lambda x}}{\sqrt{\lambda}}\,d\lambda -\; i\;a\;e^{-ax}\;\int_C \frac{e^{-i(\lambda\;+\;i\;a)\;x}}{\sqrt{\lambda}\;(\lambda\;+\;i\;a)}\,d\lambda\right]$$

Define

$$I_1(x) \;=\; \int_C \frac{e^{-i\lambda x}}{\sqrt{\lambda}}\,d\lambda \tag{1.22}$$

$$I_2(x) \;=\; \int_C \frac{e^{-i(\lambda+ia)x}}{\sqrt{\lambda}(\lambda\;+\;ia)}\,d\lambda \tag{1.23}$$

Then

$$f(x) \;=\; -i\,\frac{e^{-i\pi/4}\sqrt{1+a}}{\pi}\;\big[I_1(x)\;-\;i\;a\;e^{-ax}\;I_2(x)\big] \tag{1.24}$$

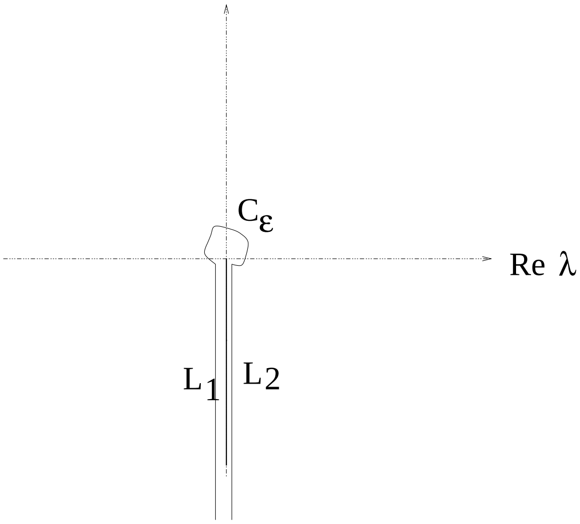

Figure 3. Contour for evaluation of I_1 in (1.22)

By deforming the contour C to coincide with the contour shown in Fig. 3, and noting that weak singularity at $\lambda = 0$ does not contribute anything to C_ϵ as $\epsilon \;\to\; 0$, we compute the contribution from L_1 and L_2 to find:

$$I_1(x) \;=\; 2\;e^{-i\pi/4}\;\int_0^\infty \frac{e^{-xr}}{\sqrt{r}}\;dr \;\;==\; \frac{2}{\sqrt{x}}\;\sqrt{\pi}\;e^{-i\pi/4} \tag{1.25}$$

To evaluate $I_2(x)$, we note that

$$I_2(0) \;=\; \int_C \frac{d\lambda}{\sqrt{\lambda}(\lambda \;+\; ia)} \;=\; 0 \tag{1.26}$$

by closing the contour from the top. Also, from (1.23),

$$I_2'(x) \;=\; -i\; e^{ax}\; I_1\;(x) \;=\; -\;\frac{2ie^{ax}}{\sqrt{x}}\;\sqrt{\pi}\; e^{-i\pi/4} \tag{1.27}$$

Therefore

(1.28)

$$I_2(x) \;=\; -\;2\;e^{i\pi/4}\;\sqrt{\pi}\int_0^x \frac{e^{ax'}}{\sqrt{x'}}\;dx' \;=\; -\;4\;e^{i\pi/4}\sqrt{\pi}\;\int_0^{\sqrt{ax}} e^{t^2}\;dt$$

Thus $f(x)$ is determined from (1.24). To determine what β is appropriate, we note that

(1.29)

$$\int_0^y e^{t^2}\;dt \;=\; \int_0^1 e^{t^2}dt + \int_1^y e^{t^2}\;dt \;=\; \int_0^1 e^{t^2}dt - \frac{1}{2}\,e + \frac{1}{2y}e^{y^2} + \int_1^y \frac{e^{t^2}}{2t^2}\;dt$$

On using L'Hospital rule, it is easy to show that as $y \;\to\; +\infty$, the ratio of the last term to the second last term on the right of (1.29) tends to zero. This implies that as $y \;\to\; \infty$,

$$\int_0^y e^{t^2}\;dt \;\sim\; \frac{1}{2y}e^{y^2} \tag{1.30}$$

On using the property (1.30) it follows that $I_2(x)$ tends to zero as $x \;\to\; \infty$ like $\frac{1}{\sqrt{x}}$, as does $I_1(x)$. Thus $f(x)$ goes to zero algebraically as $x \;\to\; +\infty$. So $\beta = 0$ is an appropriate choice.

Now consider choice of α by computing $u(x)$. Note

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\frac{-\sqrt{\lambda - i} + \sqrt{-ia - i}}{\sqrt{\lambda - i}(a - i\lambda)}d\lambda \tag{1.31}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-i\lambda x}}{\lambda + ia}d\lambda - \frac{1}{2\pi i}e^{-i\pi/4}(1+a)^{1/2}\int_{-\infty}^{\infty}\frac{e^{-i\lambda x}d\lambda}{(\lambda + ia)\sqrt{\lambda - i}} \equiv I_3(x) + I_4(x)$$

We note that for $x > 0$, by closing the contour with a lower half semi-circle as appropriate for use of Jordan's Lemma, and collecting residue at $\lambda = -ia$, we obtain

$$I_3(x) = -e^{-ax} \text{ for } x > 0 \tag{1.32}$$

On the other hand if $x < 0$, closing the contour with an upper-half semi-circle, it follows that $I_3(x) = 0$. Again, for $I_4(x)$, noting that the branch cut of $\sqrt{\lambda - i}$ was in the upper-half-plane, we may close the contour from below for $x > 0$, using Jordan's Lemma, and collect residue at $\lambda = -ia$, to obtain

(1.33)

$$I_4(x) = -\frac{1}{2\pi i} e^{-i\pi/4}(1+a)^{1/2} \int_{-\infty}^{\infty} \frac{e^{-i\lambda x} d\lambda}{(\lambda + ia)\sqrt{\lambda - i}} = e^{ax} \text{ for } x > 0$$

So, as expected we have for $x > 0$, $u(x) = I_3(x) + I_4(x) = -e^{ax} + e^{ax} = 0$. For $x < 0$, we rewrite

(1.34)

$$I_4(x) = -\frac{1}{2\pi i} e^{-i\pi/4}(1+a)^{1/2} e^{-ax} I_5(x), \text{ where } I_5(x) = \int_{-\infty}^{\infty} \frac{e^{-i(\lambda+ia)x} d\lambda}{(\lambda + ia)\sqrt{\lambda - i}}$$

Using Jordan's Lemma for $x < 0$, we deform the contour for I_5 into C shown in Figure 4, which allows us differentiation with respect to x inside the integral in λ (justifification involves use of Dominated convergence theorem to take the limit of divided difference inside the integral)

$$(1.35) \quad I_5'(x) = -i \int_C \frac{e^{-i(\lambda+ia)x} d\lambda}{\sqrt{\lambda - i}} = -ie^{(a+1)x} \int_C \frac{e^{-i(\lambda-i)x}}{\sqrt{\lambda - i}} d\lambda \text{ for } x < 0$$

The last integral is evaluated the same way as $I_1(x)$, except that $\lambda - i$ takes the place of λ, the branch cuts are at the top of $\lambda = i$ and that $x < 0$. Referring to Figure 4, $\lambda - i = re^{i\pi/2}$ on the right of the cut and $\lambda - i = re^{-3i\pi/2}$ on the left. Also, the C_ϵ evaluation gives zero as $\epsilon \to 0$ because of weak singularity at $\lambda = i$. Therefore, we obtain

(1.36)

$$I_5'(x) = -2ie^{(a+1)x} e^{i\pi/4} \int_0^{\infty} \frac{e^{rx}}{\sqrt{r}} dr = -2e^{-i\pi/4} e^{(a+1)x} (-x)^{-1/2} \sqrt{\pi}$$

Further, from Riemann Lebesgue Lemma as $x \to -\infty$, $I_5(x) \to 0$. Therefore, for $x < 0$

$$(1.37) \qquad I_5(x) == -2e^{-i\pi/4} \sqrt{\pi} \int_{-\infty}^{x} e^{(a+1)t} (-t)^{-1/2} dt$$

So, for $x < 0$, we obtain

$$(1.38) \qquad u(x) \sim -\frac{2i}{2\sqrt{\pi} i} e^{-ax} \int_{-\infty}^{x} e^{(a+1)t} (-t)^{-1/2} dt,$$

which on integration by parts gives $u(x) \sim \text{constant} \frac{e^x}{\sqrt{-x}}$ as $x \to -\infty$. Therefore, a choice of $\alpha = 1$ suffices since $\lim_{x \to -\infty} e^{-\alpha x} u(x) = 0$. Since prior calculations shows $\beta = 0$, it follows that indeed $0 \le \beta < \alpha \le 1$ as assumed and the calculations are consistent. We can also check by direct Fourier transformation of resulting $u(x)$ and $f(x)$ that we indeed obtain $U(\lambda)$ and $F(\lambda)$ for $\text{Im}\lambda < \alpha$ and $\text{Im}\lambda > \beta$ respectively and so Fourier-inversion formula applies.

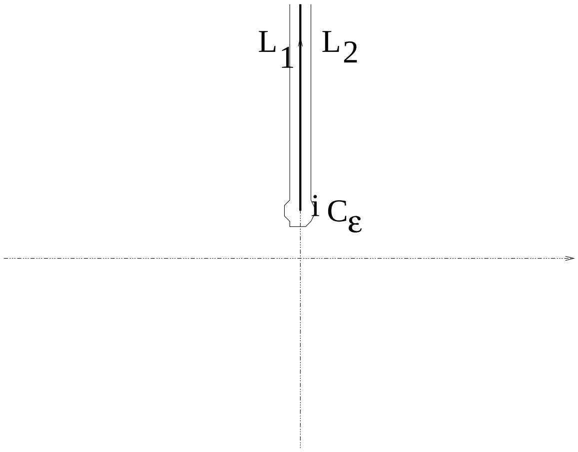

FIGURE 4. Contour C for computation of $I_5'(x)$ in (1.35)