1 Higher order WKB

Recall in the last class, we used formal arguments to show how n independent solutions to

$$\mathcal{L}y \equiv \epsilon^n y^{(n)} - U(x)y(x) = 0 \tag{1}$$

for $U(x) > 0$ has asymptotic behavior for small ϵ:

$$y(x) \sim g_j[1 + o(1)] \quad \text{where } g_j \equiv \exp\left[\omega_j \int^x [U(s)]^{1/n} ds\right] [U(x)]^{-(n-1)/2n} \tag{2}$$

where

$$\omega_j^n = 1 \quad \text{for} \quad j = 1, 2, ..n \tag{3}$$

We now wish to rigorously justify this result. One way to prove this result is to determine determine a common equation satisfied by every g_j. That equation, as will turn out is close to (1) for small ϵ. Then, using variation of parameter formula, we arrive at an integral equation representation for y and then prove contraction.

1.1 Determination of common equation satisfied by g_j

For simplicity, we will limit ourself for $n = 3$, though in the process it will be clear that the results are generalizable for any n. In that case, it is convenient to take $\omega_j = e^{i2j\pi/3}$. It is to noted that $Re\ \omega_{1,2} < 0$ and $\omega_3 = 1$. We will define $\beta = \frac{1}{\epsilon}$ and so $\beta >> 1$.

$$P(x) = \int_{x_0}^{x} [U(s)]^{1/n} ds \tag{4}$$

$$L(x) = [U(x)]^{(1-n)/(2n)} \tag{5}$$

Then, it is to be noted that

$$g_j(x) = L(x) \exp\left[\omega_j \beta P(x)\right] \tag{6}$$

It is convenient to define for $j = 1, 2, 3$:

$$m_{2,j} = \frac{g_j'}{\beta g_j} = \omega_j P + \frac{L'}{\beta L} \ ; \ m_{3,j} = \frac{g_j''}{\beta^2 g_j} = \left(\omega_j P + \frac{L'}{\beta L}\right)^2 + \beta^{-1}\left(\omega_j P' + \left[\frac{L'}{\beta L}\right]'\right) \tag{7}$$

and r_j so that

$$\mathcal{L}g_j = \beta^{-2} r_j g_j \tag{8}$$

Straight forward calculations (using maple) show that

$$r_j = \frac{7}{9}\frac{\omega_j\left(\frac{d}{dx}U(x)\right)^2}{(U(x))^{5/3}} - \frac{2}{3}\frac{\omega_j \frac{d^2}{dx^2}U(x)}{(U(x))^{2/3}} + \beta^{-1}\left[-\frac{28}{27}\frac{\left(\frac{d}{dx}U(x)\right)^3}{(U(x))^3} + \frac{4}{3}\frac{\left(\frac{d}{dx}U(x)\right)\frac{d^2}{dx^2}U(x)}{(U(x))^2} - \frac{1}{3}\frac{\frac{d^3}{dx^3}U(x)}{U(x)}\right] \tag{9}$$

It is to be noted that if we define matrix $\mathcal{M}$ so that

$$\mathcal{M} = \begin{bmatrix} g_1 & g_2 & g_3 \\ \beta^{-1} g_1' & \beta^{-1} g_2' & \beta^{-1} g_3' \\ \beta^{-2} g_1'' & \beta^{-2} g_2'' & \beta^{-2} g_3'' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix} \tag{10}$$

Then, it is convenient to define matrix Q_1 so that

$$\beta^{-1} Q_1 \equiv \left(\mathcal{M}' - \beta Q_0 \mathcal{M} \right) \mathcal{M}^{-1} \tag{11}$$

where

$$Q_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ U & 0 & 0 \end{bmatrix} \tag{12}$$

Using (8), (11) and (12), it is seen that

$$\beta^{-1} Q_1 \mathcal{M} = \mathcal{M}' - \beta Q_0 \mathcal{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta^{-1} r_1 & \beta^{-1} r_2 & \beta^{-1} r_3 \end{bmatrix} \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix} \tag{13}$$

From (10) and(13), it follows that

$$Q_1 \begin{bmatrix} 1 & 1 & 1 \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix} \tag{14}$$

Thus,

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{15}$$

where,

$$b_{3,2} = [(r_1 - r_2)(m_{32} - m_{33}) - (r_2 - r_3)(m_{31} - m_{32})]/\Delta, \tag{16}$$

$$b_{3,3} = -[(r_1 - r_2)(m_{22} - m_{23}) - (r_2 - r_3)(m_{21} - m_{22})]/\Delta, \tag{17}$$

$$b_{3,1} = r_3 - b_{3,2} m_{2,3} - b_{3,3} m_{3,3}, \tag{18}$$

with $\Delta = (m_{21} - m_{22})(m_{32} - m_{33}) - (m_{22} - m_{23})(m_{31} - m_{32})$ we have Thus, from (11) it follows that the matrix $\mathcal{M}$ satisfies the differential equaiton

$$\mathcal{M}' - \left(\beta Q_0 + \beta^{-1} Q_1 \right) \mathcal{M} = 0 \tag{19}$$

In particular by looking at the third row of this equation, it is seen that for any j, g_j satisfies the same third order homogeneous linear differential equation:

$$\mathcal{L}_w g_j := \beta^{-3} g_j''' - \beta^{-4} b_{33} g_j'' - \beta^{-3} b_{32} g_j' - \left[U(x) + \beta^{-2} b_{31} \right] g_j = 0 \tag{20}$$

Notice that $\mathcal{L} - \mathcal{L}_w$ is indeed small and scales atmost like β^{-2}. This can be used to prove a contraction mapping and thereby justify the WKB approximation.

2 Formal Boundary Layer Analysis

Example: Linear second order Equations with variable coefficients:

$$\epsilon\ u'' \ +\ a(x)\ u' \ +\ b(x)\ u \ =\ 0 \tag{21}$$

$$u(x_0) \ =\ A, \quad u(x_1) \ =\ B \tag{22}$$

Here $x_0 \ <\ x_1$, and prime denotes $\frac{d}{dx}$. The functions $a(x)$ and $b(x)$ are as smooth as necessary to carry out the steps below. Without loss of generality, we take $x_0 \ =\ 0$ and $x_1 \ =\ 1$.
Comment: We note that by applying transformation

$$u(x) \ =\ exp \left[-\frac{1}{2\epsilon} \int_0^x \ a(t)\ dt \right]\ w(x) \tag{18}$$

the (21) becomes

$$\epsilon^2\ w'' \ +\ \left(-\frac{a^2}{4} \ +\ \epsilon\ b \ -\ \frac{1}{2}\ \epsilon\ a' \right)\ w \ =\ 0 \tag{23}$$

In this form, the WKBJ method is immediately applicable and one can obtain approximate solutions as $\epsilon \ \rightarrow\ 0$. However, we like to directly find approximate solution to (21), though in a formal manner, using the so-called boundary layer method.
Solution to (21)-(22) for $a(x) \ >\ 0$

We assume there is an an outer region where for fixed x, as $\epsilon \ \rightarrow\ 0$, the solution to (21) has the asymptotic expansion:

$$u \ \sim\ \sum_{j=0}^{\infty} \ \epsilon^j\ u_j(x) \tag{24}$$

The equations for u_0 is

$$a(x) u_0' \ +\ b(x)\ u_0 \ =\ 0 \tag{25}$$

The general solution to this is of the form

$$u_0(x) \ =\ C\ \exp\ \left[\int_x^1 \ \frac{b(t)}{a(t)}\ dt \right] \tag{26}$$

for some constant C that is undetermined at this stage. There are now two boundary conditions in (22). Which one needs to be satisfied by u_0 depends on the location of the boundary layer (referred to as a thin region in x where solution changes dramatically). We do not know ahead of time where this layer is, if at all there is a boundary layer. Suppose, it is centered around some point x_d in the interval $[x_0,\ x_1]$. We rescale variable:

$$\tilde{x} \ =\ \frac{(x - x_d)}{\eta(\epsilon)} \tag{27}$$

Then, (21) becomes

$$\frac{\epsilon}{\eta^2}\ \frac{d^2 u}{d\tilde{x}^2} \ +\ \frac{1}{\eta}\ \left[a(x_d) \ +\ a'(x_d)\ \eta\ \tilde{x} \ +\ O(\eta^2)\ \right]\ \frac{du}{d\tilde{x}} \ +\ \left[b(x_d) \ +\ O(\eta) \right]\ u \ =\ 0 \tag{28}$$

Now two distinguished limits are

$$\eta(\epsilon) \;=\; 1 \quad \text{and} \;\; \eta(\epsilon) \;=\; \epsilon \tag{29}$$

The first reproduces the original equation (21). The second choice, gives rise to the inner-equation:

$$\frac{d^2u}{d\tilde{x}^2} \;+\; \left[a(x_d) \;+\; a'(x_d)\,\epsilon\,\tilde{x} \;+\; O(\epsilon^2)\,\right] \frac{du}{d\tilde{x}} \;+\; \epsilon\;[b(x_d) \;+\; O(\epsilon)]\;u \;=\; 0 \tag{30}$$

Now, if we assume an inner expansion of the type:

$$u(x,\;\epsilon) \;\sim\; \sum_{j=0}^{\infty} \epsilon^j\; U_j(\tilde{x}), \tag{31}$$

then $U_0(\tilde{x})$ satisfies

$$U_0'' \;+\; a(x_d)\; U_0' \;=\; 0 \tag{32}$$

A general solution to (32) is of the form

$$U_0(\tilde{x}) \;=\; C_1 \;-\; C_0\; e^{-\tilde{x}\;a(x_d)} \tag{33}$$

It is to be noted that for $C_1 \;\neq\; 0$, for $\tilde{x} \;=\; (x - x_d)/\epsilon \;\rightarrow\; -\infty$, the solution (33) is dominated by an exponentially growing term $e^{-a(x_d)\;(x-x_d)/\epsilon}$ which is not present in the outer solution $u_0(x)$; hence there is no possibility of an overlap domain to the left of $x = x_d$. Thus, consistency eliminates choice of $x_d \;\neq\; 0$. From now on, we continue with the choice $x_d \;=\; 0$. Thus, if there should be a boundary layer, it must be on the left end point 0. The leading order outer solution (26) must therefore satisfy the boundary condition on the right, i.e. $u_0(1) \;=\; B$. This is possible when $C = B$ in the expression (26). From the leading order inner solution (33), now rewritten with $x_d \;=\; 0$, it follows htat boundary condition $u(0) \;=\; A$ translates into $U_0(0) \;=\; A$ and therefore constant C_1 has to be chosen so that

$$U_0(\tilde{x}) \;=\; A \;+\; C_0\;[1 \;-\; e^{-a(0)\;\tilde{x}}] \tag{34}$$

Leading order outer solution (26) and inner solution (34) match in an overlap domain $\epsilon \;<<\; x \;<<\; 1$, provided

$$C_0 \;+\; A \;=\; B\;exp\;\left[\int_0^1 \frac{b(t)}{a(t)}\;dt\;\right] \tag{35}$$

Next time, we will consider higher order approximations.

3 Higher Order Approximation and Matching

Recall the ansatz for the outer

$$u \;\sim\; \sum_{j=0}^{\infty} \epsilon^j\; u_j(x) \tag{36}$$

Plugging into the equation, it follows that for $j \geq 1$,

$$a(x)u_j' + b(x)u_j = -u_{j-1} \tag{37}$$

We determined last time that boundary layer can only occur at $x = 0$, so the boundary conditions at $x = 1$ have to be imposed on the outer-solution; *i.e.*

$$u_0(1) \;=\; B, \quad u_j(1) \;=\; 0 \quad \text{for } j \;\geq\; 1 \tag{38}$$

As found earlier,

$$u_0(x) \;=\; B\; exp\;[\int_x^1 \frac{b(t)}{a(t)}\; dt] \tag{39}$$

Plugging this into the right hand side of (37) for $j = 1$, and using the method of integrating factors, we obtain

$$u_1(x) \;=\; \left\{exp\left[\int_x^1 \frac{b(t)}{a(t)}\; dt\right]\right\} \;\left\{\int_x^1 \;\left[\frac{u_0''(t)}{a(t)}\; exp\;\left(-\int_t^1 \;\frac{b(s)}{a(s)}\; ds\right)\right]\; dt\;\right\} \tag{40}$$

As determined last time, the inner variable choice is

$$\tilde{x} \;=\; \frac{x}{\epsilon} \tag{41}$$

and the inner equation becomes

$$\frac{d^2 u}{d\tilde{x}^2} \;+\; a(\epsilon\;\tilde{x})\;\frac{du}{d\tilde{x}} \;=\; -\; b(\epsilon\;\tilde{x})\; u \tag{42}$$

If we assume that a and b to be smooth near the origin, then the inner expansion takes the form:

$$u(x,\;\epsilon) \;\sim\; \sum_{j=0}^{\infty} \;\epsilon^j\; U_j(\tilde{x}) \tag{43}$$

More generally, if $a(x)$ and $b(x)$ are functions in $\mathbf{C}^n[0,1]$ space, then the asymptotic expansion (43) remains valid upto $j = n$ term. Making of the local expansion of a and b, we obtain as before:

$$U_0'' \;+\; a_0\; U_0' \;=\; 0 \quad \text{with} \;\; U_0(0) \;=\; A, \tag{44}$$

$$U_1'' \;+\; a_0\; U_1' \;=\; -\,a_1\;\tilde{x}\;\frac{dU_0}{d\tilde{x}} \;-\; b_0\; U_0 \;\;,\;\; U_1(0) \;=\; 0, \tag{45}$$

where $a_0 \;=\; a(0)$, $a_1 \;=\; a'(0)$, $b_0 \;=\; b(0)$. A general solution to (44) satisfying given boundary condition is of the form

$$U_0(\tilde{x}) \;=\; A \;+\; C_0\;(1 \;-\; e^{-a_0 \tilde{x}}) \tag{46}$$

In order for matching between U_0 and u_0 to occur, we must have

$$C_0 \;=\; k \;-\; A \quad \text{where} \;\; k \;=\; B\; exp\;\left[\int_0^1 \;\frac{b(t)}{a(t)}\; dt\right] \tag{47}$$

The solution to $U_1(x)$ can be obtained by solving (45), with given right hand side and we get

$$U_1(\tilde{x}) \;=\; -\frac{1}{2}\,a_1\;(A-k)\;\tilde{x}^2\; e^{-a_0\;\tilde{x}} \;-\; \frac{1}{a_0}\;(A\;-\;k)\;(a_1\;-\;b_0)\;\tilde{x}\; e^{-a_0\;\tilde{x}} \;+\; C_1(1-e^{-a_0\;\tilde{x}}) \;-\; \frac{b_0 k}{a_0}\;\tilde{x} \tag{48}$$

Matching $u_0 \ + \ \epsilon \, u_1$ with $U_0 \ + \ \epsilon \, U_1$ gives

$$C_1 \ = \ u_1(0) \tag{49}$$

Other terms cancel out exactly. The composite (uniformly valid) expanson to order ϵ is given by adding $u_0 \ + \ \epsilon \, u_1$ to $U_0 \ + \ \epsilon \, U_1$ and subtracting out the terms that match in the overlap domain. This is given by

$$u(x, \, \epsilon) \ \sim \ u_0(x) + \epsilon \, u_1(x) + (A - k) \, e^{-a_0 \, \tilde{x}} + \epsilon \left[-\frac{1}{2} \, a_1 \, (A-k) \, \tilde{x}^2 \ - \ \frac{1}{a_0} \, (A-k) \, \tilde{x} \ - \ u_1(0) \right] e^{-a_0 \, \tilde{x}} \tag{50}$$

Case B: $a(0) \ = \ 0$**,** $a(x) \ > \ 0$ **for** x **in (0, 1]**

Note that the arguments before can made again to show that there cannot be any boundary layer except possibly at $x \ = \ 0$. Thus, the outer solutions worked out above for u_0 and u_1 remain valid without any modifications. The inner equation near $x \ = \ 0$, however, is different both in the size of the boundary layers and the leading order equations that one gets. There are many possibilities, depending on the nature of zero of $a(x)$ at $x = 0$. We illustrate the possibilities in terms of a special case of (21):

$$\epsilon \, u'' \ + \ x^k \, u' \ + \ b \, u \ = \ 0 \ \ , \ \ b \ = \ \text{constant}, \ \ k \ > \ 0 \tag{51}$$

$$u(0) \ = \ A \ \ \ , \ \ u(1) \ = \ B \tag{52}$$

Clearly since only nature of inner solutions to (51) is closely related to that of the inner solutions to (21) if case B applies and $a(x) \ \sim \ x^k$ as $x \ \rightarrow \ 0^+$. Thus the discussion here for (51) has general consequences for (21) as well, when $a(x)$ is zero at one of the end points and nonzero elsewhere in the closed interval [0, 1]. If introduce an inner variable

$$\tilde{x} \ = \ \frac{x}{\epsilon^l} \tag{53}$$

into (50), then it becomes

$$\frac{d^2 u}{d\tilde{x}^2} \ + \ \epsilon^{l(1 \, + \, k) - 1} \, \tilde{x}^k \, \frac{du}{d\tilde{x}} \ + \ \epsilon^{2l-1} \, b \, u \ = \ 0 \tag{54}$$

Different cases are possible, depending on the value of k.

If $0 \ < \ k \ < \ 1$ (we call this case B1), we choose

$$l \ = \ \frac{1}{1+k} \tag{55}$$

in (53). The first two terms in (54) are then order unity and the third term is $O(\epsilon^{(1-k)/(1+k)}) \ = \ o(1)$.

For $k \ > \ 1$ (call it case B2), the choice (55) is inappropriate as this makes the third term in (54) unbalanced, which is not possible. So, we choose instead

$$l \ = \ \frac{1}{2} \quad \text{for} \ \ k \ > \ 1 \tag{56}$$

in (53). Then the first and third term in (54) are order unity, while the second term is $o(1)$.

Finally, for $k = 1$ (called case B3), we choose $l = \frac{1}{2}$ again and in that case all three terms in (54) are $O(1)$ and need to be included.

Summarizing: Three distinct cases are *B1:* $0 < k < 1$, *B2:* $k > 1$ and *B3:* $k = 1$. We now study case *B1* in detail.

Inner and outer solutions for case B1:

From the equation for u_0 and u_1, we get

$$u_0'' = u_0 \left(bkx^{-1-k} + b^2 x^{-2k}\right) \tag{57}$$

Putting $u_1(x) = u_0(x)\, h(x)$ gives

$$h' = -b\,k\,x^{-1-2k} - b^2\,x^{-3k} \quad ; \qquad h(1) = 0 \tag{58}$$

Thus:

$$.u_0(x) = B\; exp\; \left[b(1-x^{1-k})/(1-k)\right] \tag{59}$$

$$u_1(x) = u_0(x) \left[\frac{b}{2}(x^{-2k}-1) - b^2\,(x^{1-3k} - 1)/(1-3k)\right] \tag{60}$$

The computations of $u_j(x)$ for $j > 1$ are in principle straight forward. A minor complication may occur in (60), when $k = 1/3$ as the last term is in an indeterminate form. In that case, we have to replace it by $-b^2\,u_0(x)\;ln\;x$.

The inner equation in this case is:

$$\frac{d^2u}{d\tilde{x}^2} + \tilde{x}^k\,\frac{du}{d\tilde{x}} + \zeta(\epsilon)\,b\,u = 0 \tag{61}$$

where

$$\tilde{x} = x\,\eta^{-1} = x\,\epsilon^{-1/(1+k)} \quad , \quad \zeta(\epsilon) = \epsilon^{(1-k)/(1+k)} \tag{62}$$

This suggests an expansion of the form

$$u \sim \sum_{j=0}^{\infty} \zeta^j\,U_j(\tilde{x}) \tag{63}$$

Note that the coefficients ζ^j are integral powers of the scaling factor η or of ϵ only for exceptional values of j and k. The $U_j(\tilde{x})$ satisfy

$$\frac{d^2U_0}{d\tilde{x}^2} + \tilde{x}^k\,\frac{dU_0}{d\tilde{x}} = 0 \quad , \quad U_0(0) = A \;, \tag{64}$$

and for $j > 0$,

$$\frac{d^2U_j}{d\tilde{x}^2} + \tilde{x}^k\,\frac{dU_j}{d\tilde{x}} = -b\,U_{j-1} \quad ; \quad g_j(0) = 0 \tag{65}$$

The first equation is solved by

$$U_0(\tilde{x} = C_0\,G(\tilde{x}) + A \quad , \quad G(\tilde{x}) = \int_0^{\tilde{x}} exp\;\left(-t^{1+k}/(1+k)\right)\,dt \tag{66}$$

where C_0 is obtained by matching U_0 with u_0. By breaking up the integral for G into two parts: $\int_0^{\tilde{x}} = \int_0^{\infty} - \int_{\tilde{x}}^{\infty}$, and using integration by parts on the second integral, it is clear that the latter integral is exponentially small in $\tilde{x}$. Thus, it is clear that the matching condition becomes:

$$\lim_{\tilde{x} \to \infty} U_0(\tilde{x}) = C_0 \, (1+k)^{-k/(1+k)} \, \Gamma\left(1/(1+k)\right) + A = \lim_{x \to 0} u_0(x) = B \, e^{b/(1-k)} \quad (67)$$

This determines C_0. For $j > 0$, we find $U_j(\tilde{x})$ by the recursive formula:

$$U_j(\tilde{x}) = \int_0^{\tilde{x}} \exp\left(-s^{1+k}/(1+k)\right) \left\{ \int_0^s \left[-bU_{j-1}(t) \, exp \, \left(t^{1+k}/(1+k)\right) \, dt\right] \right\} ds + C_j \, G(\tilde{x}) \quad (68)$$

In order matching of the inner and outer variables, it becomes necessary to study the asymptotics of (68) for large $\tilde{x}$. This is a bit tedious. We first study the behavior of the inner-integral in (68) by breaking up the integral and integrating one of them by parts:

$$\int_0^1 U_{j-1}(t) \, exp \, \left(t^{1+k}/(1+k)\right) \, dt + \frac{U_{j-1}(s)}{s^k} \, exp \left[\frac{s^{1+k}}{1+k}\right] - \frac{U_{j-1}(1)}{s^k} \, exp \left[\frac{1}{1+k}\right]$$
$$- \int_1^s U'_{j-1}(t) \, exp \, \left(t^{1+k}/(1+k)\right) \, t^{-k} \, dt + \int_1^s k \, t^{-1-k} \, U_{j-1}(t) \, exp \, \left(t^{1+k}/(1+k)\right) \, dt \quad (69)$$

With this decomposition, it is clear that as $\tilde{x} \to +\infty$

$$U_1(\tilde{x}) \sim -b \, U_0(\infty) \, \frac{\tilde{x}^{1-k}}{1-k} + \tilde{C}_1 + O(\tilde{x}^{-k}) \, , \quad (70)$$

where

$$\tilde{C}_1 = C_1 \, G(\infty) - b \int_0^{\infty} ds \, e^{-s^{1+k}/(1+k)} \left\{ \hat{C} + s^{-k} \left[U_0(s) - U_0(\infty)\right] exp \, \left(s^{1+k}/(1+k)\right) \right.$$
$$\left. + \int_1^s exp\left(t^{1+k}/(1+k)\right) \left[-U'_0(t) \, t^{-k} + k \, t^{-k-1} \, \right] \, dt \right\} \quad (71)$$

where

$$\hat{C} = \int_0^1 U_0(t) \, exp \, \left[t^{1+k}/(1+k)\right] \, dt - U_0(1) \, exp \left[1/(1+k)\right] \quad (72)$$

Note that the term when $U_0(\tilde{x}) + \zeta \, U_1(\tilde{x})$ is rewritten in outer variables x, we obtain

$$U_0(\infty) - b \, U_0(\infty) \, \frac{x^{1-k}}{1-k} + \zeta(\epsilon) \, \tilde{C}_1 + O(\epsilon \, x^{-2k}) \quad (73)$$

In the outer asymptotic expansion $u_0(x) + \epsilon \, u_1(x) + ..$, it is clear from (59) and (60) that the terms that can match with (73) as as $x \to 0$ are

$$B \, exp \left[b/(1-k)\right] \left(1 - \frac{b}{1-k} \, x^{1-k}\right) + O\left(x^{2-2k}, \, \epsilon x^{-2k} \, \right) \quad (74)$$

There are no terms that corresponds to a constant times $\zeta(\epsilon)$. This means that matching requires that

$$\tilde{C}_1 = 0 \quad (75)$$

This translates into a condition on C_1, since C_1 and $\tilde{C}_1$ are related through (71).