Solution to Homework Set 5, Math 805

1. Prove Proposition 3.210 (page 74 Text) for the problem

$$\epsilon^2 \psi'' - U(x)\psi = 0 \quad , \quad \text{with } U(x) \equiv F^2(x) < 0 \quad , \text{for } \; x \in [a,b] \tag{1}$$

Let $F \in C^\infty[a,b]$ and assume $F \neq 0$ in $[a,b]$. Then, for small enough ϵ, there exists a fundamental set of solutions to (1) in the form

$$\psi_\pm = \Phi_\pm(x;\epsilon) \exp\left[\pm \epsilon^{-1} \int_a^x F(s) ds\right] \tag{2}$$

where $\Phi_\pm(x;\epsilon)$ is C^∞ in $\epsilon > 0$.

Solution: In this case, it is more convenient to define $F(x) = iG(x)$ where G is real valued. Then, the statement (2) becomes

$$\psi_\pm = \Phi_\pm(x;\epsilon) \exp\left[\pm i \epsilon^{-1} \int_a^x G(s) ds\right] \tag{3}$$

We define

$$S(x) = \int_a^x G(s) ds \tag{4}$$

As noted in the proof of Proposition 3.210 in the text, it is enough to show $\psi_\pm = \exp\left[\pm i\epsilon^{-1} R_\pm(x,\epsilon)\right]$ where $R_\pm$ is C^∞ in ϵ. If we substitute $\psi = \exp\left[\frac{w}{\epsilon}\right]$ into (1) and substitute $w' = F + \delta = iG + \delta$, then it results in equation (3.213) in the text, which now reads as

$$\delta' + 2i\epsilon^{-1} G\delta = -iG' - \epsilon^{-1}\delta^2 \tag{5}$$

We note in (5) that balancing $2i\epsilon^{-1}G\delta$ and $-iG'$ is formally consistent and leads us to formal conclusion $\delta \sim -\frac{\epsilon G'}{2G}$. This is proved in what follows since θ is the ensuing will be shown to be $O(\epsilon^2)$. We decompose

$$\delta = -\frac{\epsilon G'}{2G} + \theta, \tag{6}$$

to get the following equation for θ:

$$\theta' + 2i\epsilon^{-1}G\theta - \frac{G'}{G}\theta = -\epsilon^{-1}\theta^2 + \epsilon \left\{ \left[\frac{G'}{2G}\right]' - \left[\frac{G'}{2G}\right]^2 \right\} \tag{7}$$

Noting that integrating factor is

$$\exp\left[2i\epsilon^{-1}\int_a^x G(s)ds - \log G\right] = \frac{1}{G(x)} \exp\left[2i\epsilon^{-1} S(x)\right]$$

We obtain the integral equation

$$\theta = \theta^{(0)} + \mathcal{N}[\theta] \equiv \mathcal{M}[\theta], \tag{8}$$

where

$$\theta^{(0)} = \epsilon \int_a^x \exp\left[\frac{2i}{\epsilon}(S(t)-S(x))\right]\frac{G(x)}{G(t)}\left\{\left[\frac{G'}{2G}\right]' - \left[\frac{G'}{2G}\right]^2\right\}(t)dt \tag{9}$$

$$\mathcal{N}[\theta](x) = -\epsilon^{-1}\int_a^x \exp\left[\frac{2i}{\epsilon}(S(t)-S(x))\right]\frac{G(x)}{G(t)}\theta^2(t)dt \tag{10}$$

It is convenient to define

$$R(t) = \frac{1}{G^2(t)}\left\{\left[\frac{G'}{2G}\right]' - \left[\frac{G'}{2G}\right]^2\right\}(t) \tag{11}$$

Then, it follows that

$$\theta^{(0)} = \frac{\epsilon^2 G(x)}{2i}\int_a^x R(t)d\left[\exp\left\{\frac{2i}{\epsilon}(S(t)-S(x))\right\}\right] \tag{12}$$

On integration by parts, it is clear that

$$|\theta^{(0)}| \le C_0\epsilon^2 \tag{13}$$

where C_0 depends on lower bounds on G and upper bounds on G and its first three derivatives as well as the interval length $b-a$.

Now, consider properties of operator $\mathcal{N}$. From (10), it follows that

$$\Big|\mathcal{N}[\theta]\Big| \le C_1\epsilon^{-1}\|\theta\|_\infty^2, \tag{14}$$

where C_1 depends on upper and lower bounds of G and the interval length $b-a$. Further, it is clear from (10) that

$$\Big|\mathcal{N}[\theta_1]-\mathcal{N}[\theta_2]\Big| \le C_1\epsilon^{-1}\|\theta_1-\theta_2\|_\infty\|\theta_1+\theta_2\| \tag{15}$$

From property (13), it follows that for θ, θ_1 and θ_2, each satisfying for $\|\theta\|_\infty, \|\theta_1\|_\infty, \|\theta_2\|_\infty \le 2C_0\epsilon^2$, and for ϵ sufficiently small

$$\Big|\mathcal{M}[\theta]\Big| \le C_0\epsilon^2 + \epsilon^3 C_1 C_0^2 \le 2C_0\epsilon^2 \tag{16}$$

$$\Big|\mathcal{M}[\theta_1]-\mathcal{M}[\theta_2]\Big| = \Big|\mathcal{N}[\theta_1]-\mathcal{N}[\theta_2]\Big| \le 4C_1C_0\epsilon\|\theta_1-\theta_2\|_\infty \tag{17}$$

It immediately follows from (16) and (17) that for ϵ sufficiently small $\mathcal{M}$ maps a ball of size $2C_0\epsilon^2$ back to itself and is contractive in that ball. The rest of the proof involving C^∞ dependence on ϵ mirrors the one above as mentioned in the text since we can repeat the same arguments by pulling out more and more terms in the formal expansion in ϵ:

$$\delta = \frac{\epsilon G'}{2G} + \epsilon^2\delta_2 + .. + \epsilon^n\delta_n + \theta$$

and derive an equation for θ and apply contraction mapping theorem on a ball of size $O(\epsilon^{n+1})$.

2. Prove Proposition 3.210 (page 74 Text) for $F < 0$. We proceed as given in the text, except we use limits $\int_b^x$ instead of $\int_a^x$. This results, instead of (3.214) in the text, the following integral equation:

$$\delta = \int_x^b F'(s) e^{2[H(x)-H(s)]/\epsilon} ds + \frac{1}{\epsilon} \int_x^b \delta^2(s) e^{2[H(x)-H(s)]/\epsilon} ds =: \delta^{(0)} + \mathcal{N}[\delta], \tag{18}$$

where we now define

$$H(s) = -\int_a^x F(s) ds > 0 \tag{19}$$

We note from the above that $H(x) - H(s) \leq 0$ since $s \geq x$ Further from integration by parts we have

$$|\delta^{(0)}(x)| = \Big| \frac{\epsilon}{2} \int_x^b \frac{F'(s)}{-F(s)} d\left[e^{2[H(x)-H(s)]/\epsilon} \right] \leq \frac{3\epsilon A}{2B} \tag{20}$$

as before. and

$$|\mathcal{N}[\delta]| \leq \Big| \frac{\epsilon}{2} \int_x^b \frac{\delta^2}{-F(s)} d\left[e^{2[H(x)-H(s)]/\epsilon} \right] \leq \|\delta\|_\infty^2 \frac{\epsilon}{2B} \tag{21}$$

and

$$|\mathcal{N}[\delta_1] - \mathcal{N}[\delta_2]| \leq \Big| \frac{\epsilon}{2} \int_x^b \frac{\delta^2}{-F(s)} d\left[e^{2[H(x)-H(s)]/\epsilon} \right] \leq \|\delta_1 + \delta_2\|_\infty \|\delta_1 - \delta_2\|_\infty \frac{\epsilon}{2B} \tag{22}$$

It immediately follows that $\mathcal{M}$ is a contraction scheme as before. The rest of the proof parallels the one given in the text for case $F > 0$ with limits $\int_a^x$ replaced by $\int_b^x$.

3. Consider the two point boundary value problem

$$\epsilon y'' + a(x) y' + b(x) y = 0 \quad , \quad y(0) = A, \quad y(1) = B$$

for $a(x)$, $b(x)$ smooth with $a(x) > 0$ for $x \in [0, 1]$ and $0 < \epsilon << 1$. Show with appropriate transformation that WKB results produce the same results as formal boundary layer results.

Solution: We introduce Liouville's transformation

$$y = e^{-\frac{1}{2\epsilon} \int_1^x a(t) dt} \psi \tag{23}$$

This results in

$$\epsilon^2 \psi'' = \left(\frac{a^2}{4} + \frac{\epsilon a'}{2} - \epsilon b \right) \psi \equiv U(x, \epsilon) \psi \tag{24}$$

The WKB result is that the two independent solutions for the ψ-equation above are

$$\psi_{1,2} = U^{-1/4} \exp \left[\pm \frac{1}{\epsilon} \int_a^x U^{1/2}(t) dt \right] [1 + O(\epsilon)] \tag{25}$$

We note that

$$U^{1/2}(x) = \frac{a(x)}{2} \left(1 - \frac{4\epsilon b}{a^2} + 2\epsilon \frac{a'}{a^2} \right)^{1/2} = \frac{a}{2} - \frac{\epsilon b}{a} + \frac{\epsilon a'}{2a} + O(\epsilon^2), \tag{26}$$

which implies

$$\frac{1}{\epsilon}\int_0^x U^{1/2}(t)dt = \frac{1}{2\epsilon}\int_0^x a(t)dt - \int_0^x \frac{b(t)}{a(t)}dt + \frac{1}{2}\log a + \text{constant} + O(\epsilon) \tag{27}$$

Therefore, upto an unimportant constant factor,

$$\psi_1 = \exp\left[\frac{1}{2\epsilon}\int_0^x a(t)dt - \int_0^x \frac{b(t)}{a(t)}dt\right][1+O(\epsilon)] \tag{28}$$

$$\psi_2 = \frac{a(0)}{a(x)}\exp\left[-\frac{1}{2\epsilon}\int_0^x a(t)dt + \int_0^x \frac{b(t)}{a(t)}dt\right][1+O(\epsilon)] \tag{29}$$

Therefore, noting the transformation between y and ψ, the two independent solutions y_1 and y_2 is of the form

$$y_1 = \exp\left[-\int_1^x \frac{b(t)}{a(t)}dt\right][1+O(\epsilon)] \tag{30}$$

$$y_2 = \frac{a(0)}{a(x)}\exp\left[-\frac{1}{\epsilon}\int_0^x a(t)dt + \int_0^x \frac{b(t)}{a(t)}dt\right][1+O(\epsilon)] \tag{31}$$

It is to be noted that if $x = O_s(1)$ (strictly order 1), then y_2 is exponentially small. General solution is

$$y = C_1 y_1 + C_2 y_2 \tag{32}$$

Boundary condition at $x = 1$ imply

$$B = C_1 + C_2\exp\left[-\frac{1}{\epsilon}\int_0^1 a(t)dt + \int_0^1 \frac{b(t)}{a(t)}dt\right](1+O(\epsilon)) \tag{33}$$

The boundary condition at $x = 0$ implies

$$A = C_1\exp\left[-\int_1^0 \frac{b(t)}{a(t)}dt\right][1+O(\epsilon)] + C_2 \tag{34}$$

It follows that

$$C_1 = B + O\left(e^{-\frac{1}{\epsilon}}\right), \quad C_2 = A - B\exp\left[\int_0^1 \frac{b(t)}{a(t)}dt\right] + O(\epsilon) \tag{35}$$

So, solution to the two point boundary value problem

$$y = \left[B + O(e^{-1/\epsilon})\right] y_1 + \left(A - B\exp\left[\int_0^1 \frac{b(t)}{a(t)}dt\right] + O(\epsilon)\right) y_2 \tag{36}$$

Note that this agrees with boundary layer analysis since y_2 is exponentially small outside the boundary layer and expression for y_1 agrees with the outer solution. Further, inside the boundary layer, we note that $\frac{a(0)}{a(x)} \sim 1$ and in the boundary layer where $x = O(\epsilon)$

$$y_2 \sim \left(\frac{a(0)}{a(x)}\right)\exp\left[-\frac{1}{\epsilon}\int_0^x a(t)dt + \int_0^x \frac{a(t)}{b(t)}dt\right] = \exp\left[-\frac{a(0)x}{\epsilon}\right][1+O(\epsilon)] = e^{-a(0)\tilde{x}}[1+O(\epsilon)]$$

4. Consider the two point boundary value problem

$$\epsilon^2 y'' - (1-x^2)y = -1, \qquad y(-1) = 0 = y(1)$$

for $x \in (-1,1)$. Carry out a formal inner-outer expansion for small ϵ upto $O(\epsilon)$ and show in this case we have boundary layers only at end points $x = \pm 1$ and no where else. Carry out appropriate matching to determine all unknown constants needed to determine solution upto and including $O(\epsilon)$. (**Hint:** In this inhomogeneous problem, you may have to rescale dependent variable as well in the boundary layers.)

Solution: The ansatz for the outer expansion is as usual (since parameter now is ϵ^2):

$$y(x) \sim y_0 + \epsilon^2 y_2 \ + \ .. \tag{37}$$

Plugging in results in

$$-(1-x^2)y_0 = -1 \ \ , \quad \ -(1-x^2)y_j = -y''_{j-1} \tag{38}$$

So,

$$y_0(x) = \frac{1}{1-x^2} \quad , \quad y_2 = \frac{y''_0}{(1-x^2)} = \frac{2(1+3x^2)}{(1-x^2)^4} \tag{39}$$

This results in an outer expansion

$$y \sim \frac{1}{1-x^2} + \frac{2\epsilon(1+3x^2)}{(1-x^2)^4} + O\left(\frac{\epsilon^4}{(1-x^2)^7}\right) \tag{40}$$

Clearly this expansion cannot be valid near $x = \pm 1$ where we are imposing boundary conditions $y = 0$.

Consider firs a neighborhood of $x = -1$ and assume a boundary layer of thickness $O(\eta(\epsilon))$ and introduce inner variable.

$$x + 1 = \eta \tilde{x} \tag{41}$$

Of course the solution (40) has to match to the inner solution in some matching region that is a subset of $\eta << x + 1 << 1$. However, the outer solution (40) rewritten in terms of $\tilde{x}$ produces

$$y \sim \frac{1}{2\eta\tilde{x}}\left[1 + \frac{1}{2}\eta\tilde{x}) + O(\eta^2\tilde{x}^2)\right] + O\left(\frac{\epsilon^2}{\eta^4\tilde{x}^4}\right) \tag{42}$$

This suggests we should rescale y in the inner region at $x = -1$:

$$y = \frac{1}{\eta}U(\tilde{x}) \tag{43}$$

so that in the matching region (42) produces the condition

$$U(\tilde{x}) \sim \frac{1}{2\tilde{x}} + \frac{1}{4} + O\left(\eta\tilde{x}, \frac{\epsilon^2}{\eta^3\tilde{x}^4}\right) \tag{44}$$

However, introduction of rescalings (41) and (43) in (37) results in

$$\frac{\epsilon^2}{\eta^3}\frac{d^2U}{d\tilde{x}^2} - \tilde{x}\left[2 - \eta\tilde{x}\right]U = -1 \tag{45}$$

The only distinguished scale other than $\eta = 1$ is the choice

$$\eta = \epsilon^{2/3}, \tag{46}$$

in which case (45) gives rise to

$$U'' - \tilde{x}\left[2 - \epsilon^{2/3}\tilde{x}\right]U = -1 \tag{47}$$

This $\epsilon^{2/3}$ term is a regular perturbation term in (47) and so, we have in the inner region where $\tilde{x} = O(1)$,

$$U \sim U_0(\tilde{x} + \epsilon^{2/3}U_1(\tilde{x}) + O(\epsilon^{4/3}) \tag{48}$$

where

$$U_0'' - 2\tilde{x}U_0 = -1 \quad , \quad \text{with} \;\; U_0(0) = 0, \tag{49}$$

and for $j \geq 1$

$$U_j'' - 2\tilde{x}U_j = -\tilde{x}^2 U_{j-1} \quad , \quad \text{with } U_j(0) = 0 \tag{50}$$

Therefore, noting the homogeneous equation has solution $Ai(2^{1/3}\tilde{x})$, $Bi(2^{1/3}\tilde{x})$ with Wronksian $\mathcal{W} = \frac{2^{1/3}}{\pi}$, we obtain with $p = 2^{1/3}\tilde{x}$, the following expression:

$$U_0(\tilde{x}) = 2^{-2/3}\pi Ai(p)\int_0^p Bi(q)dq - 2^{-2/3}\pi Bi(p)\int_\infty^p Ai(q)dq + C_1 Ai(p) + C_2 Bi(p) \tag{51}$$

Since $U_0(0) = 0$,

$$-2^{-2/3}\pi Bi(0)\int_\infty^0 Ai(q)dq + C_1 Ai(0) + C_2 Bi(0) = 0 \tag{52}$$

From the equaton for U_1, we have

$$U_1(\tilde{x}) = 2^{-4/3}\pi Ai(p)\int_\infty^p Bi(q)q^2 U_0(2^{-1/3}q)dq - 2^{-4/3}\pi Bi(p)\int_0^p Ai(q)q^2 U_0(2^{-1/3}q)dq + C_3 Ai(p) + C_4 Bi(p \tag{53}$$

with

$$-2^{-2/3}\pi Bi(0)\int_\infty^0 U_0(2^{-1/3}q)Ai(q)dq + C_3 Ai(0) + C_4 Bi(0) = 0 \tag{54}$$

Therefore, we now seek to determine the expression in the matching region where $\tilde{x}$ (and so p) is large. This is where the asymptotics of Ai and Bi become handy. We note that for large positive p, we recall

$$Ai(p) \sim \frac{1}{2\sqrt{\pi}}p^{-1/4}e^{-2p^{3/2}/3} \quad , \quad Bi(p) \sim \frac{1}{\sqrt{\pi}}p^{-1/4}e^{2p^{3/2}/3} \tag{55}$$

Therefore, it follows on integration by parts that for large p,

$$2^{-2/3}\pi Bi(p)\int_\infty^p Ai(q)dq \sim \frac{-2^{-2/3}}{2p} \sim -\frac{1}{4p} \tag{56}$$

Again because of the exponential largeness of Bi for large p the asymptotics of $\int_0^p Bi(q)dq$ is dominated by the end point p. Integration by parts gives

$$2^{-2/3}\pi Ai(p)\int_0^p Bi(q)dq \sim\sim \frac{2^{-2/3}}{2p} \sim \frac{1}{4p} \tag{57}$$

This implies that in the matching region where $\tilde{x} >> 1$,

$$2^{-2/3}\pi Ai(p)\int_0^p Bi(q)dq - 2^{-2/3}\pi Bi(p)\int_\infty^p Ai(q)dq = \frac{1}{2\tilde{x}} + O(\tilde{x}^{-2}) \tag{58}$$

Since $Bi(p)$ blows up exponentially as $p \to \infty$, it follows from above and from (51) and (58) that the matching occurs to (42) to the leading order only if $C_2 = 0$. Therefore, from (51) and (52), we now have

$$U_0(\tilde{x}) = 2^{-2/3}\pi Ai(p)\int_0^p Bi(q)dq - 2^{-2/3}\pi Bi(p)\int_\infty^p Ai(q)dq + C_1 Ai(p), \quad \text{where } p = 2^{1/3}\tilde{x} \tag{59}$$

where

$$-2^{-2/3}\pi Bi(0)\int_\infty^0 Ai(q)dq + C_1 Ai(0) = 0 \tag{60}$$

Now, we seek to determine higher order matching. Again through integration by parts, and using $U_0(\tilde{x}) \sim \frac{1}{2\tilde{x}}$ it follows that

$$\frac{\pi}{2^{4/3}}Ai(p)\int_0^p q^2 U_0(2^{-1/3}q)Bi(q)dq - \frac{\pi}{2^{4/3}}Bi(p)\int_\infty^p q^2 U_0(2^{-1/3}q)Ai(q)dq \sim \frac{\tilde{x}}{4} + O(\tilde{x}^2) \tag{61}$$

It follows that

$$\frac{\pi}{2^{4/3}}Ai(p)\int_0^p q^2 U_1(2^{-1/3}q)Bi(q)dq - \frac{\pi}{2^{4/3}}Bi(p)\int_\infty^p q^2 U_1(2^{-1/3}q)Ai(q)dq \sim \frac{\tilde{x}^2}{8} + O(\tilde{x}^2) \tag{62}$$

Therefore, the inner-expansion in the matching region behaves as

$$y(x) \sim \epsilon^{-2/3}U_0(\tilde{x}) + U_1(\tilde{x}) + \epsilon^{2/3}U_2(\tilde{x}) + .. = \frac{1}{2\epsilon^{2/3}\tilde{x}} + O(\epsilon^{-2/3}\tilde{x}^{-2})\frac{1}{4} + O(\tilde{x}^{-1}) + O(\epsilon^{2/3}\tilde{x}^2) \tag{63}$$

which matches with outer expansion (42).