Large time behavior of solutions to evolutionary PDEs with time-periodic coefficients

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Stability of periodic Orbits for ODEs

Consider $x(0) = x_0$ close to a periodic solution $x = x_p(t)$ of $\dot{x} = f(x)$ Decomposing $x = x_p(t) + y$, then $|y(0)| = |y_0|$ small and $y' = f(x_p + y) = A(t)y + N[y; t]$, where $A(t)$ is time-periodic and $N[y; t]$ contains nonlinearity. Define the fundamental matrix of the associated linear system $\Psi(t, \tau)$ with $\Psi(\tau, \tau) = I$. Then,

$$y(t) = \Psi(t, 0)y_0 + \int_0^t \Psi(t, \tau)N[y(\tau); \tau]d\tau \equiv \mathcal{N}[y](t)$$

If $|\Psi(t, \tau)z| \leq Ce^{-\alpha(t-\tau)}|z|$ for some $C, \alpha > 0$, then as before, introducing norm $||y|| = \sup_{t>0} e^{\alpha t}|y(t)|$. Easily shown $\mathcal{N}$ is contractive in the associated Banach space in a ball $\mathcal{B}_{2|y_0|}$, implying asymptotic stability of $x = x_p(t)$. In other words, asymptotic stability of the linearized linear problem $y_t = \mathcal{A}(t)y$ gives nonlinear stability of $x = x_p(t)$.

Floquet Problem for ODE

Consider $y' = A(t)y$, $A(t)$ is 2π periodic matrix, continuous in t. If $\Psi(t,\tau)$ is a fundamental solution matrix with $\Psi(\tau,\tau) = I$, then $A(t+2\pi) = A(t)$ implies $[\Psi(\tau+2\pi,\tau)]^{-1}\,\Psi(t+2\pi,\tau)$ is also a fundamental matrix. From uniqueness,

$$\Psi(t+2\pi,\tau) = \Psi(\tau+2\pi,\tau)\Psi(t,\tau)$$

Therefore, $\Psi(t+2n\pi,\tau) = [\Psi(\tau+2\pi,\tau)]^n\,\Psi(t,\tau)$. If eigenvalues of $\Psi(\tau+2\pi,\tau)$ have absolute value strictly less than one, then it follows $|\Psi^n(\tau+2\pi,\tau)| \le Ce^{-n\alpha}$ for $\alpha > 0$. If $t-\tau = 2\pi n + \gamma$, with $\gamma \in [0, 2\pi)$, it follows from continuity

$$|\Psi(t,\tau)| \le e^{-n\alpha}|\Psi(\gamma+\tau,\tau)| \le Ce^{-\alpha(t-\tau)}$$

The eigenvalues of $\Psi(\tau+2\pi,\tau)$, which is the same for any τ is referred to as *Floquet* multiplier.

Stability of time-periodic solutions to PDEs

The ODE examples generalizable to nonlinear evolutionary PDEs as well, except the role of $\mathbb{R}^n$ replaced by a Banach space of functions of spatial variable x and f replaced by some differential operator $\mathcal{F}$ in x, *i.e.* we have $u_t = \mathcal{F}[u]$. If $u_p(x,t)$ is a periodic solution, then writing $u = u_p + v$ results in

$$v_t = \mathcal{L}(t)v + \mathcal{Q}[v] \ , \ \text{where } \mathcal{L} \text{ is the linearization of operator } \mathcal{F}$$

If the linearized equation $v_t = \mathcal{L}(t)v$ is asymptotically stable, *i.e.* solution operator $\mathcal{S}$ with $v = \mathcal{S}(t,\tau)v_0$ satisfies $\|\mathcal{S}(t,\tau)v_0\| \le Ce^{-\alpha(t-\tau)}\|v_0\|$, then by writing the nonlinear equation as an integral equation, we can conclude nonlinear stability of of $u_p(x,t)$.

Thus, nonlinear stability of time-periodic solutions requires study of linear PDE in the form $v_t = \mathcal{L}(t)v$, where $\mathcal{L}(t)$ is some PDE operator in $x \in \mathbb{R}^d$ that is periodic in time.

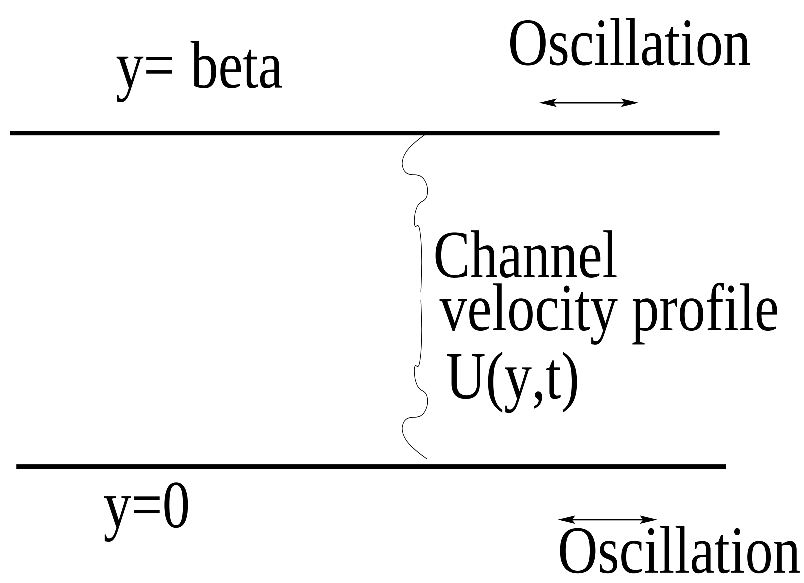
Stability of oscillatory pipe/channel flows

In pipe or channel fluid flow, one may have either an oscillatory pressure gradient or wall oscillating along the the axis. A time-oscillatory solution $\mathbf{u} = (U, 0, 0)$ possible for Navier-Stokes

$$\mathbf{u}_t = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \epsilon \Delta \mathbf{u} \ , \ \ \nabla \cdot \mathbf{u} = 0 \ ,$$

$$\text{Aternately} \ \ \mathbf{u}_t = \mathcal{P} \left[-\mathbf{u} \cdot \nabla \mathbf{u} \right] + \epsilon \mathcal{P} \Delta \mathbf{u} \equiv \mathcal{F}[\mathbf{u}],$$

where $\mathcal{P}$ is the Hodge projection. Spatially, U depends only on transverse variables y in the channel and on r in a pipe.



Orr-Sommerfeld equation for linear stability

The linear Stability equations for this oscillatory flow in a channel for fixed axial wave number α:

$$2\partial_t[\partial_y^2 - \alpha^2]\psi - [\partial_y^2 - \alpha^2]^2\psi = -\frac{iU}{\epsilon}[\partial_y^2 - \alpha^2]\psi + \frac{iU_{yy}}{\epsilon}\psi,$$

where $0 < y < \beta$, ϵ is the reciprocal of Reynolds number. $U(y,t)$ is time-periodic, precise expression depending on flow.

Initial condition: $\psi(y,0) = \psi_0(y)$ and no slip wall BC implies:

$$\psi(0,t) = \psi_y(0,t) = 0 = \psi(\beta,t) = \psi_y(\beta,t)$$

Note the Orr-Sommerfeld equations may be written more abstractly as $\phi_t = \mathcal{L}(t)\phi$ for suitably defined ϕ and $\mathcal{L}(t)$

In particular, if $\beta = \infty$, if the lower-plate oscillates sinusoidally along x-direction (stream-wise), then $U(y,t) = e^{-y}\cos(t-y)$.

Orr-Sommerfeld in the form $\phi_t = \mathcal{L}(t)\phi$

If we introduce $\phi = (\partial_y^2 - \alpha^2)\psi$, then equation may be written as:

$$\partial_t \phi = \frac{1}{2}\left(\partial_y^2 - \alpha^2\right)\phi - \frac{iU}{4\epsilon}\phi + \frac{iU_{yy}}{4\epsilon}\mathcal{I}[\phi] \equiv \mathcal{L}(t)\phi$$

where operator $\mathcal{I} : L^2(0,\beta) \to H^2(0,\beta)$ is defined by

$$\mathcal{I}[\phi](y) = \frac{\sinh(\alpha y)}{\alpha \sinh(\alpha\beta)} \int_\beta^y \sinh[\alpha(\beta - y')]\phi(y')dy'$$

$$-\frac{\sinh(\alpha(\beta - y))}{\alpha \sinh[\alpha\beta]} \int_0^y \sinh(\alpha y')\phi(y')dy',$$

which incorporates $\mathcal{I}[\phi](0) = 0 = \mathcal{I}[\phi](\beta)$. For $\beta = \infty$,

$$\mathcal{I}[\phi](y) = \frac{e^{-\alpha y}}{\alpha} \int_\infty^y \sinh(\alpha y')\phi(y')dy' - \frac{\sinh(\alpha y)}{\alpha} \int_0^y e^{-\alpha y'}\phi(y')dy',$$

Earlier work on Orr-Sommerfeld Equations

Earlier numerical Orr-Sommerfeld investigations (Hall, '78), Hall ('03), (Blennerhasset-Bassom, '08) for different ranges of Reynolds number; quantitative agreement with experiment (Eckmann-Grotberg, 1991) not good.

Blennerhasset and Bassom ('08) concluded instability for $\epsilon \approx \frac{1}{700}$ based on numerics on the same recurrence relation. They suggest an inviscid instability mode

Based on U varying on relatively slow time scale, a quasi-steady calculation (Hall, '03) based on inviscid Rayleigh equation:

$$(U - c) \left[\partial_y^2 - \alpha^2 \right] \psi - U_{yy} \psi = 0 \ \text{, with } \ c = 2 i \epsilon \sigma$$

suggested stability.

Experiment (Merkli-Thomann, '75, Clemen-Minton, '77) suggests instability; not clear how theory applies.

Another problem: Ionization of Hydrogen Atom

Consider the time-dependent 3-D Schroedinger equation with an oscillatory potential added to Coulomb potential:

$$\psi_t = -i\left(-\Delta - \frac{b}{r} + V(t,x)\right)\psi \ , \ \ r = |x|, \ \ x \in \mathbb{R}^3, t \geq 0$$

A classic problem in this area since E. Fermi is whether or not, hydrogen atom necessarily ionizes in an oscillatory external field of arbitrary magnitude and frequency. Mathematically, the question is whether or not for any $a > 0$,
$\lim_{t\to+\infty} \int_{|x|<a} |\psi(x,t)|^2 dx = 0$.

No results of this kind available until recently, except for limiting V when perturbation theory may be applied (Fermi's golden rule, for instance). Also reliable numerical computation is challenging because of the dimensionality of the problem.

Relation of IVP with Floquet problem

In a general context, if

$$u_t = [\mathcal{A} + 2\cos t \;\; \mathcal{B}]\, u \;\; , u(x,0) = u_0,$$

where $\mathcal{A}$ and $\mathcal{B}$ are time-independent spatial operators.

If $\mathcal{A}^{-1}$ incorporates boundary or decay conditions at ∞, we can can write

$$\mathcal{A}^{-1} u_t = \left[\mathcal{I} + 2\cos t \;\; \mathcal{A}^{-1}\mathcal{B}\right] u$$

When *a priori* **exponential bounds in t exist, Laplace transform $U(.,p) = L[u(.,t)](p) \equiv \int_0^\infty e^{-pt} u(.,t) dt$ justified and found to satisfy**

$$(\mathcal{I} - K)\, U(.,p) = u_0, \tag{1}$$

where $K = p\mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{B}S_+ - \mathcal{A}^{-1}\mathcal{B}S_-$, where shift operators defined by $[S_- U](.,p) = U(.,p-i)$, $[S_+ U](.,p) = U(.,p+i)$

Quick note on Laplace transform

Note, if $U(p) = L[u](p)$, then

$$L\left[e^{it}u\right] = \int_0^\infty e^{-(p-i)t}u(t)dt = U(p-i)$$

Note

$$L\left[e^{-it}u\right] = \int_0^\infty e^{-(p+i)t}u(t)dt = U(p+i)$$

Therefore,

$$L\left[2\cos t \;\; u\right] = U(p+i) + U(p-i) \equiv \left[S_+U + S_-U\right](p)$$

Relation to Floquet problem- page II

If we define $p = \sigma + in$, $U(.,\sigma + in) = U_n$,

$$[\mathcal{A} - in]\,U_n - \sigma U_n - \mathcal{B}U_{n-1} - \mathcal{B}U_{n+1} = u_0 \qquad (2)$$

When $\mathcal{R}_n \equiv [\mathcal{A} - in]^{-1}$ exists,

$$U_n = \sigma\mathcal{R}_n U_n + \mathcal{R}_n\mathcal{B}U_{n-1} + \mathcal{R}_n\mathcal{B}U_{n+1} + \mathcal{R}_n u_0$$

We may define operator $\mathcal{K}$ acting on $\mathrm{U} = \{U_n\}_{n\in\mathbb{Z}}$ such that

$$[\mathcal{K}\mathrm{U}]_n = \sigma\mathcal{R}_n U_n + \mathcal{R}_n\mathcal{B}U_{n-1} + \mathcal{R}_n\mathcal{B}U_{n+1}$$

$$\text{Then,}\quad [\mathcal{I} - \mathcal{K}]\,\mathrm{U} = \mathrm{U}^0$$

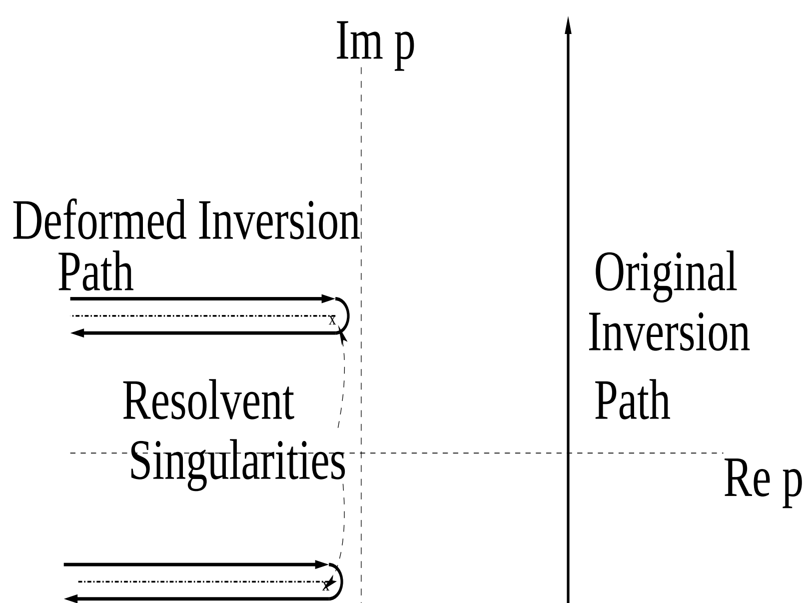
Fredholm applies when $\mathcal{K}$ is a compact operator on a Hilbert space, implying solvability iff only solution to Floquet problem $(\mathcal{I} - \mathcal{K})\mathrm{U} = 0$ is $\mathrm{U} = 0$, *i.e.* $u_t = [\mathcal{A} + 2\cos t\ \ \mathcal{B}]\,u$ **does not have nonzero $u(x,t) = \sum_{n=-\infty}^{\infty} e^{\sigma t} e^{int} u_n(x)$ for that σ.**

Stability criteria

If Floquet problem has only zero solution for $\mathrm{Re}\,\sigma \geq 0$ in a Hilbert space where u_n decays sufficiently rapidly in n (recalling $p = \sigma + in$), then $u(x,t)$ decays in t since

$$u(x,t) = \int_{-i\infty}^{i\infty} e^{pt} U(x,p) dp$$

Since $U(.,p) = (I - \mathcal{K})^{-1} u_0 = \mathcal{R}_\sigma u_0$, the singularities of resolvent $\mathcal{R}_\sigma$ in σ determine the long-term behavior of $u(x,t)$



Floquet problem in ionization of hydrogen atom

Reference: O. Costin, J. Lebowitz, S.T, Comm. Math. Phys, 2010

$$\left(-\Delta - \frac{b}{r} - i\sigma + n\omega\right)\Phi_n = -i\Omega(|x|)\left[\Phi_{n+1} - \Phi_{n-1}\right]$$

With $\Phi_n = \frac{w_n(r)}{r} Y_{l,m}(\theta, \phi)$, equation for w_n:

$$\left[\frac{d^2}{dr^2} + \frac{b}{r} - \frac{l(l+1)}{r^2} + i\sigma - n\omega\right] w_n = -i\Omega[w_{n+1} - w_{n-1}]$$

$\Omega(r)$ assumed smooth and nonzero in support $r \le 1$. Also, self-adjointness gives $i\sigma \in \mathbb{R}$. Further, $w_n = 0$ for $r > 1$ for $n < 0$ as otherwise $\Phi_n = \frac{w_n(r)}{r} Y_{l,m}(\theta, \phi) \notin L^2(\mathbb{R}^3)$. This implies in particular $w_n(1) = 0 = w_n'(1) = 0$ for $n < 0$. Also, if $\{w_n(1), w_n'(1)\}_{n=0}^{\infty} = 0$, then a local Picard-type argument gives $w_n \equiv 0$.

Floquet problem asymptotics for Hydrogen atom

Define n_0 as the smallest positive integer for which either $w_{n_0}(1)$ or $w'_{n_0}(1)$ nonzero for assumed nonzero solution. Take the case $w_{n_0}(1) \neq 0$, taken 1 w.l.o.g. Find $\frac{\partial^j}{\partial \xi^j} w_{n_0-k}(1) = i^k \delta_{j,2k}$, where $\xi = \int_r^1 \sqrt{\Omega(s)} ds$. For ξ small, $w_{n_0-k} \sim \frac{i^k \xi^{2k}}{(2k)!}$.

Above suggests that for $r = O(1)$, for $k >> 1$,

$$w_{n_0-k} \sim \frac{i^k \xi^{2k}}{(2k)!} f(r),$$

Requiring $O(k)$ terms to vanish in the residual

$$R_k \equiv \frac{\mathcal{L}_k w_{n_0-k} - i\Omega \left[w_{n_0-k+1} - w_{n_0-k+1} \right]}{w_{n_0-k}},$$

$$\text{gives } f(r) = \Omega^{-1/4}(r) \Omega^{1/4}(0) \exp \left[\frac{1}{4} \int_1^r ds \frac{\omega \xi(s)}{\sqrt{\Omega(s)}} \right]$$

Hydrogen Floquet Problem asymptotics

The asymptotics $w_{n_0-k} \sim \frac{i^k \xi^{2k}}{(2k)!} f(r)$ invalid when $kr = O(1)$. We demand substitution of

$$w_{n_0-k} = \frac{i^k \xi^{2k}}{(2k)!} f(r) \frac{H(k\alpha r)}{H(k\alpha)}$$

gives $O(1)$ residual uniformly for $r \in (0,1]$. To leading order in k,

$$H(\zeta) \sim \sqrt{\frac{2}{\pi}} e^{\zeta} \zeta^{1/2} K_{l+1/2}(\zeta) \text{ where } K_{l+1/2} \text{ is a Bessel function,}$$

i.e. **any assumed nonzero solution is singular at $r = 0$. Therefore, Floquet problem has only $w_n = 0$ as acceptable solution for $\mathrm{Re}\,\sigma \geq 0$, implying hydrogen atom ionizes for assumed time-periodic compact potential. (Proofs appear in the cited paper).**

Elaboration of $\frac{\partial^j}{\partial \xi^j} w_{n_0-k}(1) = i^k \delta_{j,2k}$

From definition of n_0, $w_n(1) = 0 = w_n'(1)$ for $n < n_0$. Take $\Omega = 1$. Note

$$\left[\frac{d^2}{dr^2} + \frac{b}{r} - \frac{l(l+1)}{r^2} + i\sigma - (n_0 - 1)\omega\right] w_{n_0-1} = -i[w_{n_0} - w_{n_0-2}],$$

implying $w''_{n_0-1}(1) = -i$, $w_{n_0-1}(1) = 0 = w'_{n_0-1}(1)$. Replacing n_0 by $n_0 - 1$,

$$\left[\frac{d^2}{dr^2} + \frac{b}{r} - \frac{l(l+1)}{r^2} + i\sigma - (n_0 - 2)\omega\right] w_{n_0-2} = -i[w_{n_0-1} - w_{n_0-3}], \tag{2}$$

it follows $w''_{n_0-2}(1) = 0 = w'_{n_0-2}(1) = w_{n_0-2}(1)$. Differentiating (2) at $r = 1$, $w'''_{n_0-2}(1) = 0$. Second derivative of (2) at $r = 1$ gives $w^{(iv)}_{n_0-2}(1) = -iw''_{n_0-1}(1) = (-i)^2$. Note $\xi = 1 - r$ and induction gives $\partial_r^j w_{n_0-k}(1) = 0$ for $j < 2k$ and $= (-i)^k$ for $j = 2k$.

Basic ideas of the proof

Define $\mathcal{L}_k = \frac{d^2}{dr^2} + \frac{b}{r} - (n_0 - k)\omega + i\sigma - \frac{l(l+1)}{r^2}$

$m_k(r) = \frac{\xi^{2k}}{(2k)!} f(r) \frac{H(k\alpha r)}{H(k\alpha)}$

Note $\mathcal{L}_k w_{n_0-k} = -i\Omega \left[w_{n_0-k+1} - w_{n_0-k-1} \right]$

Define $j_k = \xi \left[\mathcal{L}_k m_k - \Omega m_{k-1} \right] / m_k$. Explicit calculation shows $|j_k| \le C$ and $|j_k'(r)| \le C_2 + C_1 r^{-2} k^{-1}$

Define h_k so that $w_{n_0-k} = i^k m_k h_k(r)$. The object is to show $h_k(r) \sim 1$ for $r \in [0,1]$. We use inversion of $\mathcal{L}_k$ to obtain integral equation $h_k = \mathcal{A}_k h_{k-1} + \mathcal{H}_k h_{k+1}$

Can show $\|\mathcal{A}_k f\|_\infty \le \left(1 + \frac{c}{k^2}\right) \|f\|_\infty$, $\| \frac{d}{dr} \left[\mathcal{A}_k f \right] \|_\infty \le c_* k \|f\|_\infty$

Further, for any $\epsilon_1 > 0$, if $M_k = \sup_{r \in [\epsilon_1, 1]} |h_k'(r)|$ we can show for $k \ge \left[\frac{1}{\epsilon_1} \right] \equiv k_0$, $M_k \le M_{k-1} \left(\frac{k-1}{k} \right)^{1/2} + \frac{c_*}{k^2} + \frac{c_*}{k^3 \epsilon_1^2}$, implying $M_k \le \frac{C}{k^{1/2} \epsilon_1^{3/2}} + \frac{C \epsilon_1^{1/2}}{k^{1/2}}$. Separate argument near $r = 0$.

Idea of the proof-II

Also $\|\mathcal{H}_k f\|_\infty \leq \frac{c}{k^2}\|f\|_\infty$, $\|\frac{d}{dr}\left[\mathcal{H}_k f\right]\|_\infty \leq \frac{c_*}{k^2}\|f\|_\infty$

A separate argument based on Arzela Ascoli Theorem for $kr = O(1)$ on a subsequence using the *a priori* **inequality $|\frac{d}{dr}h_k| \leq Ck$**

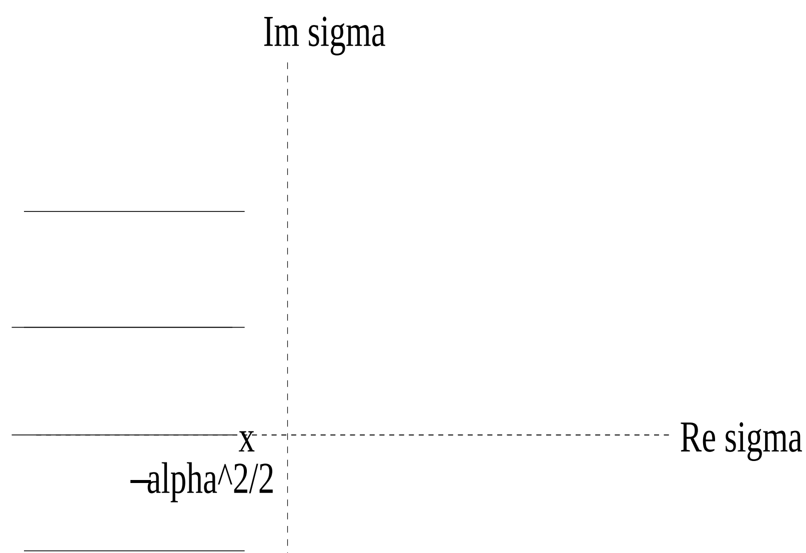
Floquet spectrum for oscillating plates

For the oscillating plate, the floquet problem becomes

$$\left(\partial_y^2 - \alpha^2 - 2\sigma - 2in\right)\Phi_n = \frac{iV}{2\epsilon}\left(1+\mathcal{I}\right)\Phi_{n+1} + \frac{iV^*}{2\epsilon}\left(1+\mathcal{I}\right)\Phi_{n+1},$$

$$\text{where } V(y) = a\left[e^{-(1+i)y} + e^{-(1+i)(\beta-y)}\right] \;, \psi = \mathcal{I}\phi$$

Theorem: For $0 < \beta < \infty$, the Floquet problem has only discrete spectrum. For $\beta = \infty$, discrete spectrum also, except for $\boldsymbol{\sigma} \in \left\{-\frac{\alpha^2}{2} + i\mathbb{Z} + \mathbb{R}^-\right\}$



Floquet problem for oscillating plate for $\beta = \infty$

For $\beta = \infty$, $V(y) = \frac{1}{2}e^{-y(1+i)}$, with Floquet problem:

$$\left(\partial_y^2 - \alpha^2 - 2\sigma - 2in\right)\Phi_n = \frac{iV(y)}{\epsilon}\Phi_{n-1} + \frac{2V}{\epsilon}\mathcal{I}[\Phi_{n-1}]$$

$$+\frac{iV^*(y)}{\epsilon}\Phi_{n+1} - \frac{2V^*}{\epsilon}\mathcal{I}[\Phi_{n+1}].$$

Let $\gamma_n = \sqrt{\alpha^2 + 2\sigma + 2in}$. Hall ('74) assumed

$$\Phi_n = \sum_{j,k,n} A_{j,k,n}e^{-(\gamma_n+k+ij)y} + \sum_{j,k} B_{j,k}e^{-(\alpha+k+ij)y},$$

constrained by $\mathrm{Re}\,\gamma_n + k > 0$, $\alpha + k > 0$. The recurrence relations for $A_{j,k,n}, B_{j,k}$ were solved numerically. Concluded $\mathrm{Re}\,\sigma < 0$ for $\epsilon \geq 1/200$. Note: more and more terms are needed for accuracy as $\epsilon \to 0^+$.

Further Laplace transform for $\beta = \infty$

Laplace Transform in y, which can be rigorously justified, gives

$$(s^2 - \lambda_n^2)\hat{\Phi}_n(s) = (s^2 - \lambda_n^2)\hat{\Phi}_n^{(0)}(s)$$

$$+\frac{i}{2\epsilon}\left(1 + \frac{2i}{[(s+1-i)^2 - \alpha^2]}\right)\hat{\Phi}_{n+1}(s+1-i)$$

$$+\frac{i}{2\epsilon}\left(1 - \frac{2i}{[(s+1+i)^2 - \alpha^2]}\right)\hat{\Phi}_{n-1}(s+1+i),$$

where

$$\hat{\Phi}_n^{(0)}(s) = \frac{\Phi_n'(0) + s\Phi_n(0)}{s^2 - \lambda_n^2},$$

$$\lambda_n^2 = \alpha^2 + 2\sigma + 2in$$

Contraction argument gives for large $\mathrm{Re}\, s$, unique solution

$\Phi(s) \sim \Phi^{(0)}(s)$

More on Floquet Problem for $\beta = \infty$

Convenient to introduce discretized variables

$$s_{k,j} = s+k-ij \ , \ \lambda_{n,k,j} = \lambda_n+k-ij \ , \ \Phi_{n,k,j}(s) = \Phi_n(s+k-ij)$$

Then, with

$$\beta^{(1)}_{n,k,j}(s) \equiv \frac{1}{s^2_{k,j} - \lambda^2_{n+j}} \left\{ 1 + \frac{2i}{\left[s^2_{k+1,j+1} - \alpha^2\right]} \right\},$$

$$\beta^{(-1)}_{n,k,j}(s) \equiv \frac{1}{s^2_{k,j} - \lambda^2_{n+j}} \left\{ 1 - \frac{2i}{\left[s^2_{k+1,j-1} - \alpha^2\right]} \right\},$$

$$\Phi_{n,k,j}(s) = \Phi^{(0)}_{n,k,j}(s) + \beta^{(1)}_{n,k,j}(s)\Phi_{n+1,k+1,j+1}(s)$$

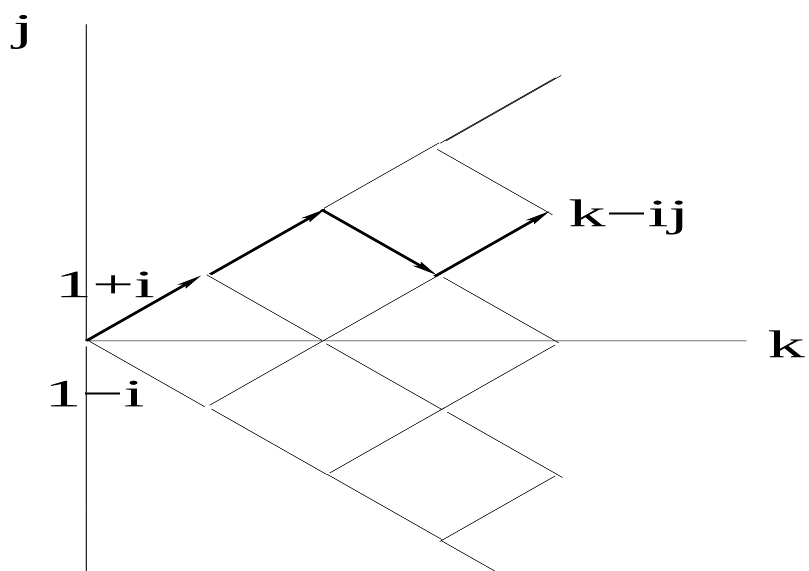
$$+\beta^{(-1)}_{n,k,j}(s)\Phi_{n-1,k+1,j-1}(s)$$

Associated Homogeneous Equation and Solution

$$G^{(n)}_{k,j} = \beta^{(1)}_{n+j,k,j} G^{(n)}_{k-1,j-1} + \beta^{(-1)}_{n+j,k,j} G^{(n)}_{k-1,j+1} \ , \ \textbf{with} \ G^{(n)}_{0,0} = 1$$

Introduce $\tau = \{a_1, a_2, .., a_k\} \in \{-1, 1\}^k$ **with** $j_k \equiv a_1 + a_2 \ + \ .. + a_k$**. Then for** $|j| \leq k$**,**

$$G^{(n)}_{k,j}(s) = \sum_{\tau, j_k = j} \prod_{l=1}^{k} \beta^{(a_l)}_{n+j_{l-1}, l-1, j_{l-1}}(s)$$



$$G_{k,j}^{(n)}(s) = \sum_{\tau, j_k = j} \prod_{l=1}^{k} \frac{1}{\left((s+l-1+ij_{l-1})^2 - \lambda^2_{n+j_{l-1}} \right)}$$

$$\times \left[1 + \frac{2ia_l}{\left((s+l+ij_l)^2 - \alpha^2 \right)}, \right]$$

where

$$j_{l-1} = a_1 + a_2 + ..a_{l-1} \quad , \quad j_0 = 0 \quad , \quad \{a_1, a_2, ...a_k\} \in \{-1, 1\}^k$$

$$\lambda_n = \sqrt{\alpha^2 + 2\sigma + 2in}$$

Solution in terms of $\left\{\Phi_n(0), \Phi'_n(0)\right\}_{n\in\mathbb{Z}}$

It can be proved that

$$\hat{\Phi}_n(s) = \sum_{k=0}^{\infty} \left(\frac{i}{2\epsilon}\right)^k \sum_{j=-k,2}^{k} G_{k,j}^{(n)}(s)\Phi_{n+j,k,j}^{(0)}(s)$$

Requiring solution to be pole free at $s = \lambda_n$, $s = \alpha$ gives

$$\sum_{j\in\mathbb{Z}} a_{n,n+j}\Phi_n(0) + \sum_{j\in\mathbb{Z}} b_{n,n+j}\Phi'_n(0) = 0, \text{ for } n \in \mathbb{Z}$$

$$\sum_{j\in\mathbb{Z}} c_{n,n+j}\Phi_n(0) + \sum_{j\in\mathbb{Z}} d_{n,n+j}\Phi'_n(0) = 0, \text{ for } n \in \mathbb{Z},$$

$$\text{where } a_{n,n+j} = \sum_{k=|j|}^{\infty} \left(\frac{i}{2\epsilon}\right)^k \frac{\alpha_{k,j} G_{k,j}^{(n)}(\alpha)}{\alpha_{k,j}^2 - \lambda_{n+j}^2},$$

Similarly expressions for $b_{n,n+j}$, $c's$, $d's$. Note $\left|G_{k,j}^{(n)}\right| \le \frac{C}{k!}$.

Asymptotics for $G_{k,j}^{(n)}$ for $|j| << k$

for $|n| << k$, $k >> 1$, $\sigma << \frac{1}{\epsilon}$

We note that

$$\beta^{(a_l)}_{n+j_{l-1},l-1,j_{l-1}}(s) = \frac{1}{(s+l-1)^2-\lambda_n^2}\left[1-\frac{2ilj_{l-1}+j_{l-1}^2+2ij_{l-1}}{(s+l-1)^2-\lambda_n^2}\right]^{-}$$

For $l >> 1$, if $j_{l-1} << l$, then we have

$$\beta^{(a_l)}_{n+j_{l-1},l-1,j_{l-1}}(s) = \frac{1}{(s+l-1)^2-\lambda_n^2}+\frac{2ilj_{l-1}}{[(s+l-1)^2-\lambda_n^2]^2}+O(\frac{j_l^2}{l^2})$$

$$G_{k,j}^{(n)}(s) \sim \frac{A(n)\Gamma(s-\lambda_n)\Gamma(s+\lambda_n)}{\Gamma(s+k-\lambda_n)\Gamma(s+k+\lambda_n)}\frac{k!}{\left(\frac{k-j}{2}\right)!\left(\frac{k+j}{2}\right)!}\left[1+A_1\frac{j}{k}+..\right]$$

Computational details in $G_{k,j}^{(n)}(s)$

To get results for $G_{k,j}^{(n)}$ as quoted, we need

$$S_{k,j;m} \equiv \sum_{l=1}^{k} \sum_{\tau, j_k = j} g(l) j_{l-1}^{m}.$$

$$\text{Note that} \;\; S_{k,j;m} = \sum_{l=1}^{k} f(l) \partial_{\beta}^{m}|_{\beta=0} \;\; T_{l,k,j}(\beta),$$

$$T_{l,k,j}(\beta) = \sum_{\tau, j_k = j} e^{\beta j_{l-1}}$$

$$\zeta(z;\beta) \equiv \sum_{j=-k}^{k} T_{l,k,j} z^{j} = \sum_{\tau} e^{a_1(\beta + \log z)} .. e^{a_{l-1}(\beta + \log z)} e^{a_l \log z} ... e^{a_k \log z}$$

$$\zeta = \left(z e^{\beta} + \frac{1}{z} e^{-\beta} \right)^{l-1} \left(z + \frac{1}{z} \right)^{k-l+1}$$

Floquet Spectrum in the closed right-half plane

Use of Gamma function asymptotics and Euler-McLaurin summation converts the system of equation into a set of integral equations for which there is no nonzero solution for $\mathrm{Re}\,\sigma \geq 0$ for $|\sigma| \leq \frac{c}{\epsilon}$ for some small c.

Theorem: For $\beta = \infty$, the Floquet problem for oscillating plate has no spectrum in the region $\mathbf{Re}\,\boldsymbol{\sigma} \geq \mathbf{0}$ for $|\boldsymbol{\sigma}| \leq \frac{\mathbf{c}}{\boldsymbol{\epsilon}}$ for some small $\mathbf{c}$.

For $\sigma = O\left(\frac{1}{\epsilon}\right)$ a different asymptotic analysis is needed.

Further, for finite β, we use a Neumann series based on Volterra kind of integral equation, instead of explicit Laplace transform in y, though analysis is more complicated.

Other non-perturbative Floquet problems require somewhat different techniques, as exemplified in the following for the 3-D Schroedinger equation with time-periodic potential.

Conclusions

The Floquet spectral problem arises naturally in the linearized time-evolution equation for disturbance on a time-periodic solution. May be rigorously and constructively analyzed in a number of situations, including oscillating channel and pipe flows, 3-D Schroedinger equations, etc.

For Stokes layer problem $\beta = \infty$ problem, an intriguing connection revealed with calculation of expected value in some stochastic process. A continuum limit is identified as $\epsilon \to 0$ that reduces an infinite discrete system of linear equation into a system of integral equations for which the only solution is 0 for $\mathrm{Re}\, \sigma \geq 0$ when $\sigma << \frac{1}{\epsilon}$. Analysis for $\sigma = O(\frac{1}{\epsilon})$ is in progress

In some problems like the 3-D Schroedinger equation with a time-periodic compact potential added to Coulomb potential, the infinite set of differential-difference equations may be analyzed through rigorous WKB analysis.

An integral reformulation of 2-D channel IVP

If we introduce $\phi = (\partial_y^2 - \alpha^2)\psi$, then equation may be written as:

$$\partial_t \phi = \frac{1}{2}\left(\partial_y^2 - \alpha^2\right)\phi - \frac{iU}{4\epsilon}\phi + \frac{iU_{yy}}{4\epsilon}\mathcal{I}[\phi] \equiv \mathcal{L}(t)\phi$$

where operator $\mathcal{I} : L^2(0,\beta) \to H^2(0,\beta)$ is defined by

$$\mathcal{I}[\phi](y) = \frac{\sinh(\alpha y)}{\alpha \sinh(\alpha\beta)} \int_\beta^y \sinh[\alpha(\beta - y')]\phi(y')dy'$$

$$-\frac{\sinh(\alpha(\beta - y))}{\alpha \sinh[\alpha\beta]} \int_0^y \sinh(\alpha y')\phi(y')dy',$$

which incorporates $\mathcal{I}[\phi](0) = 0 = \mathcal{I}[\phi](\beta)$. For $\beta = \infty$,

$$\mathcal{I}[\phi](y) = \frac{e^{-\alpha y}}{\alpha} \int_\infty^y \sinh(\alpha y')\phi(y')dy' - \frac{\sinh(\alpha y)}{\alpha} \int_0^y e^{-\alpha y'}\phi(y')dy',$$

Integral reformulation-II

An operator $\mathcal{R}$ similar to similar to $\mathcal{I}$ can be defined as an inversion of $\left(\partial_y^2 - \alpha^2\right)$ such that for $\chi \in L^2(0, \beta)$, $\frac{d}{dy}\mathcal{I}\left[\mathcal{R}[\chi]\right]$ is zero at $y = 0$ and $y = \beta$. When $\beta = \infty$, replace by decay.

Evolution for ϕ may be written as:

$$\phi - \partial_t \mathcal{R}[\phi] = \frac{i}{2\epsilon}\mathcal{R}\left[U\phi\right] - \frac{i}{2\epsilon}\mathcal{R}\left[U_{yy}\mathcal{I}[\phi]\right]$$

Integration in time over $(0, t)$ results in an integral reformulation for rigorous justification of Laplace transform in t, and determining how Floquet spectrum relates to initial value problem.

Space integration of ψ equation gives $O(\frac{1}{\epsilon})$ growth rate, since

$$\frac{d}{dt}\left\{\|\psi_y\|^2 + \alpha^2\|\psi\|^2\right\} + \|\psi_{yy}\|^2 \le \frac{|U_y|_\infty}{2\epsilon\alpha}\left\{\|\psi_y\|^2 + \alpha^2\|\psi\|^2\right\}$$