Borel Summability in PDE initial value problems

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Main idea

For an autonomous differential operator $\mathcal{N}$, consider

$$v_t = \mathcal{N}[v] \;, \; v(x,0) = v_0(x)$$

Formal small time expansion:

$$\tilde{v}(x,t) = v_0(x) + t v_1(x) + t^2 v_2(x) \; + \; ..,$$

where $v_1(x) = \mathcal{N}[v_0](x)$, $v_2 = \frac{1}{2}\left\{\mathcal{N}_v(v_0)[v_1]\right\}(x)$,**..**

Generically divergent if order of $\mathcal{N}$ is greater than 1

If Borel summable, obtain

$$v(x,t) = \sum_B \tilde{v},$$

solution to the PDE initial value problem for small enough time.

Depending on properties in the Borel plane, solution can be extended over longer time periods $[0,T]$.

Eg: 1-D Heat Equation (Lutz, Miyake & Schaefke)

$$v_t = v_{xx} \ , \ v(x,0) = v_0(x) \ \ , \ \ v(x,t) = v_0 + t v_1 + ..$$

Obtain recurrence relation

$$(k+1) v_{k+1} = v_k'' \ \ , \qquad \text{implies} \quad v_k = \frac{v_0^{(2k)}}{k!}$$

Unless v_0 entire, series $\sum_k t^k v_k$ factorially divergent.

Borel transform in $\tau = 1/t$: $V(x,p) = \mathcal{B}[v(x, 1/\tau))](p)$,
$V(x,p) = p^{-1/2} W(x, 2\sqrt{p})$, then $W_{qq} - W_{xx} = 0$

Obtain $v(x,t) = \int_{\mathbb{R}} v_0(y) (4\pi t)^{-1/2} \exp[-(x-y)^2/(4t)] dy$,

i.e. **Borel sum of formal series leads to usual heat solution.**

We seek applications of these simple ideas to more complicated PDEs, including 3-D Navier-Stokes

Background

Borel Summability for linear PDEs studied before (Balser, Miyake, Lutz, Schaefke, ..)

Sectorial existence for a class of nonlinear PDEs (Costin & T.)

Complex singularity formation for a nonlinear PDE (Costin & T.)

Navier Stokes is a nonlinear PDE governing fluid velocity $\mathbf{v}(\mathbf{x}, t)$:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \Delta v \ + \ \mathbf{f}$$

$$\nabla \cdot \mathbf{v} = 0 \ , \qquad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$$

Using PDE techniques, Leray (1930s) proved local existence, uniqueness for classical solutions and global existence for weak solutions. Global existence of classical solutions known in 2-D, not in 3-D. Literature extensive ((Constantin, Temam, Foias,...).

Background II

Global existence of classical solution or lack of it has fundamental implications to fluid turbulence.

Blow up of classical solution with finite energy $\|v_0\|_{L^2(\mathbb{R}^3)}$ implies $\|\nabla \times v(.,t)\|_\infty$ and $\|v(.,t)\|_{L^3(\mathbb{R}^3)}$ blow up (Beale *et al*, Sverak).

This becomes incompatible with the modeling assumptions in deriving Navier-Stokes. Hence other parameters not included in Navier-Stokes would become important in turbulent flow.

For the usual PDE techniques, key to global existence question is believed to be *a priori* energy bounds involving ∇v (Tao). None is available thus far.

This motivates alternate formulation of initial value problems for nonlinear PDEs that are not dependent on energy bounds at all. Borel methods and generalization via *Ecalle* sum allows this.

Illustration: Borel Transform for Burger's equation

Substitute $v = v_0(x) + u(x,t)$ into $v_t + v v_x = v_{xx}$ to obtain

$$u_t - u_{xx} = -v_0 u_x - u v_{0,x} - u u_x + v_1(x)$$

$$\text{where } \; v_1(x) = v_0'' - v_0 v_{0,x} \;\; , \;\; \text{and} \;\; , u(x,0) = 0$$

Inverse Laplace Transform in $1/t$ and Fourier-Transform in x:

$$p\hat{U}_{pp} + 2U_p + k^2\hat{U} = \hat{v}_1 - ik\hat{v}_0 \hat{*} \hat{U} - ik\hat{U} \overset{*}{*} \hat{U} \equiv \hat{G}(k,p) + \hat{v}_1,$$

$\hat{*}$ is Fourier convolution, $\overset{*}{*}$ Fourier-Laplace convolution. Hence

$$\hat{U}(k,p) = \int_0^p \mathcal{K}(p,p';k)\hat{G}(k,p')dp' + \hat{U}^{(0)}(k,p) \equiv \mathcal{N}\left[\hat{U}\right](k,p)$$

$$\mathcal{K}(p,p';k) = \frac{ik\pi}{z}\left\{z'Y_1(z')J_1(z) - z'Y_1(z)J_1(z')\right\}$$

$$z = 2|k|\sqrt{p}\;,\;\; z' = 2|k|\sqrt{p'}\;,\;\; \hat{U}^{(0)}(k,p) = 2\frac{J_1(z)}{z}\hat{v}_1(k)$$

Solution to integral equation $\hat{U} = \mathcal{N}[\hat{U}]$

$$\text{We find } |\mathcal{K}(p, p'; k)| \le \frac{C}{\sqrt{p}} \ , \ \ C \text{ a constant}$$

$$\|\hat{F}(., p) \hat{*} \hat{G}(., p)\|_{L^1(\mathbb{R})} \le C \|\hat{F}(., p)\|_{L^1(\mathbb{R})} \|\hat{G}(., p)\|_{L^1(\mathbb{R})}$$

Define norm $\|.\|^{(\alpha)}$ for functions $F(p, k)$

$$\|F\|^{(\alpha)} = \int_0^\infty e^{-\alpha p} \|F(., p)\|_{L^1(\mathbb{R})} \, dp$$

$$\text{easily follows } \|F \overset{*}{*} G\|^{(\alpha)} \le C \|F\|^{(\alpha)} \|G\|^{(\alpha)}$$

$\mathcal{N}$ seen to be contractive for large α implies Burgers solution for

for $\mathrm{Re}\, \frac{1}{t} > \alpha$ in the form $v(x, t) = v_0(x) + \int_0^\infty e^{-p/t} U(x, p) dp$

Global classical PDE solution implied if $\|\hat{U}(., p)\|_{L^1(\mathbb{R}^3)}$ bounded.

Borel summability for analytic v_0 requires analyticity of $U(., p)$ for

$p \in 0 \cup \mathbb{R}^+$; proof a bit more delicate.

Incompressible 3-D Navier-Stokes in Fourier-Space

Consider 3-D N-S in infinite geometry or periodic box. Similar results expected for finite domain with no-slip BC using eigenfunctions of Stokes operator as basis. In Fourier-Space

$$\hat{v}_t + \nu |k|^2 \hat{v} = -i k_j P_k \left[\hat{v}_j \hat{*} \hat{v} \right] + \hat{f}(k)$$

$$P_k = \left(I - \frac{k(k \cdot)}{|k|^2} \right) \quad , \quad \hat{v}(k, 0) = \hat{v}_0(k)$$

where P_k is the Hodge projection in Fourier space, $\hat{f}(k)$ is the Fourier-Transform of forcing $f(x)$, assumed divergence free and t-independent. Subscript j denotes the j-th component of a vector. $k \in \mathbb{R}^3$ or $\mathbb{Z}^3$. Einstein convention for repeated index followed. $\hat{*}$ denotes Fourier convolution.

Integral equation for Navier Stokes in Borel plane

Substitute $\hat{v} = \hat{v}_0 + \hat{u}(k,t)$, into Navier-Stokes, inverse-Laplace Transform in $1/t$ and inverting as for Burger's equation obtain integral equation:

$$\mathrm{U}(\mathrm{k},p) = \int_0^p \mathcal{K}(p,p')\hat{\mathrm{R}}(\mathrm{k},p')dp' + \mathrm{U}^{(0)}(\mathrm{k},p),$$

$$\hat{\mathrm{R}}(k,p) = -ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{\mathrm{U}} + \hat{U}_j \hat{*} \hat{\mathrm{v}}_0 + \hat{U}_j \overset{*}{*} \hat{\mathrm{U}}\right]$$

$$\mathrm{U}^{(0)}(k,p) = 2\frac{J_1(z)}{z}\hat{\mathrm{v}}_1(\mathrm{k}), \quad \text{where}$$

$$\hat{\mathrm{v}}_1(k) = -|\mathrm{k}|^2\hat{\mathrm{v}}_0 - ik_j P_k\left[\hat{v}_{0,j} \hat{*} \hat{\mathrm{v}}_0\right] + \hat{\mathrm{f}}(\mathrm{k})$$

Some Results for Navier-Stokes (NS) in $\mathbb{R}^3$

Define $\|.\|_{\mu,\beta}$, for $\mu > 3$, $\beta \geq 0$:

$$\|v_0\|_{\mu,\beta} = \sup_{k\in\mathbb{R}^3} e^{\beta|k|}(1+|k|)^{\mu}|\hat{v}_0(k)|$$

Theorem 1: If $\|f\|_{\mu,\beta}, \|\hat{v}_0\|_{\mu+2,\beta} < \infty$, NS has unique solution with $\|\hat{v}(\cdot,t)\|_{\mu,\beta} < \infty$ for $\mathrm{Re}\,\frac{1}{t} > \alpha$, where α depends on $\hat{v}_0, \hat{f}$. Furthermore, $\hat{v}(\cdot,t)$ is analytic for $\mathrm{Re}\,\frac{1}{t} > \alpha$ and $\|\hat{v}(\cdot,t)\|_{\mu+2,\beta} < \infty$ for $t \in [0,\alpha^{-1})$. For $\beta > 0$, v is analytic in x with same analyticity width as v_0 and f.

Theorem 2: For $\beta > 0$, the NS solution v is Borel summable in $1/t$, i.e. there exists $U(x,p)$, analytic in a neighborhood of $\mathbb{R}^+$, exponentially bounded, and analytic in x for $|\mathrm{Im}\ x| < \beta$ so that $v(x,t) = v_0(x) + \int_0^\infty U(x,p)e^{-p/t}dp$. When $t \to 0$, $v(x,t) \sim v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$, where $|v_m(x)| \leq m! A_0 B_0^m$, with A_0, B_0 generally dependent on v_0, f.

Same results in $\mathbb{T}^3$. Further, when v_0, f_0 have finite Fourier modes, B_0 is independent of initial data and f_0.

Results on Navier-Stokes in $\mathbb{T}^3$

Define $\|.\|^{(\alpha)}$ so that

$$\|\hat{V}\|^{(\alpha)} = \int_0^\infty e^{-\alpha p} \|\hat{V}(., p)\|_{l^1(\mathbb{Z}^3)} dp$$

Theorem 3: If $\|\hat{v}_0\|_{l^1(\mathbb{Z}^3)}, \ \|\hat{f}\|_{l^1(\mathbb{Z}^3)} < \infty$ then there exists some $\alpha > 0$ so that integral equation $\hat{U} = \mathcal{N}\left[\hat{U}\right]$ has a unique solution for $p \in \mathbb{R}^+$ in the space of functions $\left\{\hat{U} : \|\hat{U}\|^{(\alpha)} < \infty\right\}$. Further,
$\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty \hat{U}(k,p) e^{-p/t} dp$ satisfies 3-D Navier-Stokes in Fourier-Space; corresponding $v(x,t)$ is a classical NS solution for $t \in (0, \alpha^{-1})$.

Remark 1: Classical PDE methods known to give similar results. However, in the present formulation, global PDE existence is a question of asymptotics of known solution to integral equation as $p \to \infty$. Sub-exponential growth implies global existence.

More Remarks on Theorem 3 for 3-D Navier-Stokes

Remark 2: Errors in Numerical solutions rigorously controlled. Discretization in p and Galerkin approximation in k results in:

$$\hat{U}_\delta(k, m\delta) = \delta \sum_{m'=0}^{m} \mathcal{K}_{m,m'} \mathcal{P}_N \mathcal{H}_\delta(k, m'\delta) + \hat{U}^{(0)}(k, m\delta)$$

$$\equiv \mathcal{N}_\delta \left[\hat{U}_\delta \right] \quad \text{for} \quad k_j = -N, ...N, \quad j = 1, 2, 3$$

$\mathcal{P}_N$ is the Galerkin Projection into N-Fourier modes. $\mathcal{N}_\delta$ has properties similar to $\mathcal{N}$. The continuous solution $\hat{U}$ satisfies $\hat{U} = \mathcal{N}_\delta \left[\hat{U} \right] + E$, where E is the truncation error. Thus, $\hat{U} - \hat{U}_\delta$ can be estimated using same tools as in Theorem 1.

Note: Similar control over discretized solutions to PDEs not available since truncation errors involve derivatives of PDE solution which are not known to exist beyond a short-time.

Extending Navier-Stokes interval of existence

Suppose $\hat{U}(.,p)$ is known over $[0,p_0]$ through Taylor series in p or otherwise, and computed $\|\hat{U}(.,p)\|_{l^1}$ is observed to decrease towards the end of this interval. Prior discussions show that any error in this computation can be rigorously controlled.

Results in the following page show that a more optimal Borel exponent $\alpha \leq \alpha_0$ may be estimated using the known solution in $[0,p_0]$, where α_0 is the initial α estimate in Theorem 1. This implies a longer interval $\left[0,\alpha^{-1}\right)$ for NS solution.

A longer existence time for NS is relevant to the global existence question for $f=0$, since it is known that there exists T_c so that any weak Leray solution becomes classical for $t > T_c$

Extending Navier-Stokes interval of existence -II

For $\alpha_0 \geq 0$, define

$$\epsilon = \nu^{-1/2} p_0^{-1/2} \ , \ a = \|\hat{v}_0\|_{l^1} \ , \ c = \int_{p_0}^{\infty} \|\hat{U}^{(0)}(.,p)\|_{l^1} e^{-\alpha_0 p} dp$$

$$\epsilon_1 = \nu^{-1/2} p_0^{-1/2} \left(2 \int_0^{p_0} e^{-\alpha_0 s} \|\hat{U}(.,s)\|_{l^1} ds + \|\hat{v}_0\|_{l^1} \right)$$

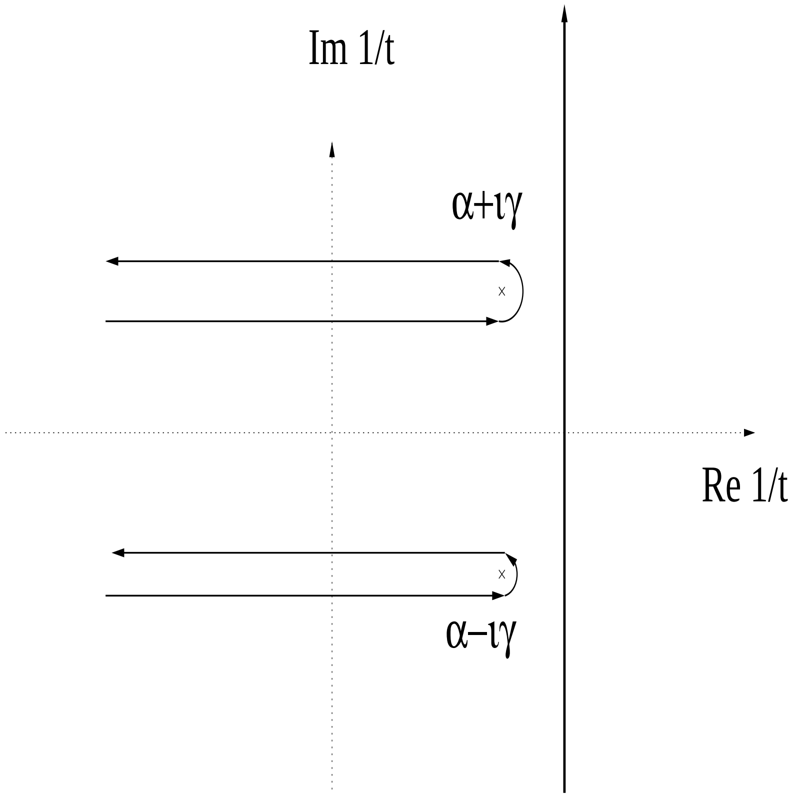
$$b = \frac{e^{-\alpha_0 p_0}}{\sqrt{\nu p_0} \alpha} \int_0^{p_0} \|\hat{U} \hat{*} \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{l^1} ds$$

Theorem 4: A smooth solution to 3-D Navier-Stokes equation exists on the interval $[0, \alpha^{-1})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Relation of optimal α to NS time singularities

$$\hat{U}(k,p) = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} e^{p/t} \left[\hat{v}(k,t) - \hat{v}_0(k) \right] d\left[\frac{1}{t}\right]$$



Rightmost singularity(ies) of NS solution $\hat{v}(k,t)$ in the $1/t$ plane determines optimal α. γ gives dominant oscillation frequency.

Generalized Laplace-transform representation

Since the Borel domain growth rate α relates to complex right-half $\frac{1}{t}$ NS singularities, the following representation for $n > 1$ is sought:

$$\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty e^{-q/t^n} \hat{U}(k,q) dq$$

Note $\hat{U}(.,p) \rightarrow \hat{U}(.,q)$ is an Ecalle' acceleration.

In order that $\hat{U}(.,q)$ has no growth for large q, unless there is a NS singularity for $t \in \mathbb{R}^+$, need to know *a priori* **that there is a singularity free sector in the right-half t-plane. This is proved to be true for $f = 0$ and we have the following result:**

Theorem 5: For $f = 0$, if NS has a global classical solution, then for all sufficiently large n, $U(x,q) = O(e^{-C_n q^{1/(n+1)}})$ as $q \rightarrow +\infty$, for some $C_n > 0$.

Numerical Solutions to integral equation

We choose the Kida initial conditions and forcing

$$\mathbf{v}_0(\mathbf{x}) = (v_1(x_1, x_2, x_3, 0), v_2(x_1, x_2, x_3, 0), v_3(x_1, x_2, x_3, 0))$$

$$v_1(x_1, x_2, x_3, 0) = v_2(x_3, x_1, x_2, 0) = v_3(x_2, x_3, x_1, 0)$$

$$v_1(x_1, x_2, x_3, 0) = \sin x_3 \left(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3\right)$$

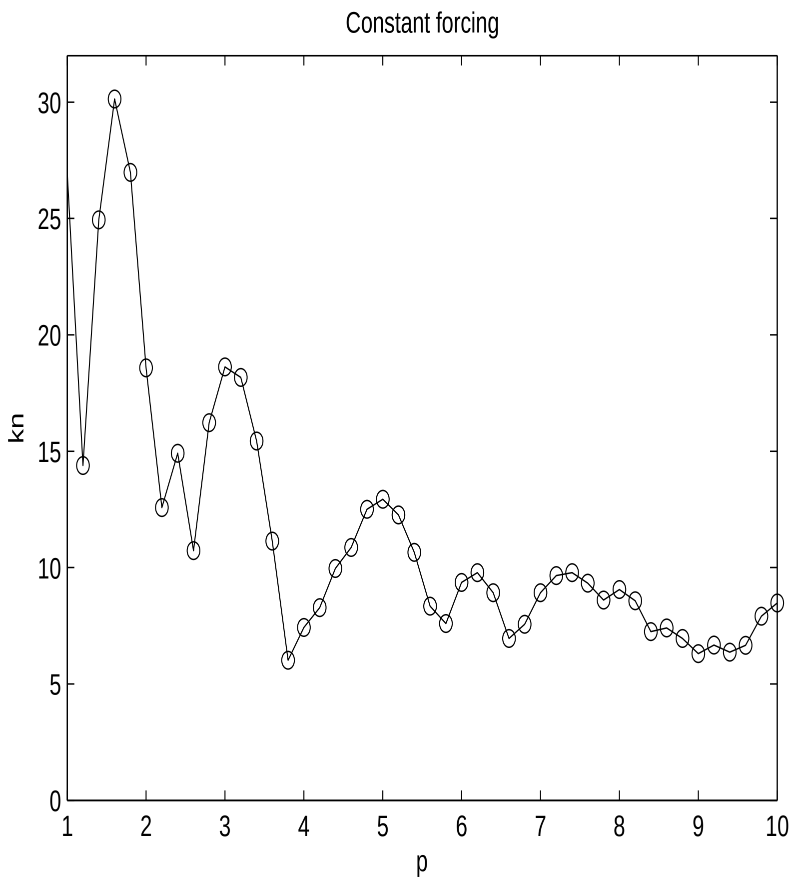
$$f_1(x_1, x_2, x_3) = \frac{1}{5} v_1(x_1, x_2, x_3, 0)$$

High Degree of Symmetry makes computationally less expensive

Corresponding Euler problem believed to blow up in finite time; so good candidate to study viscous effects

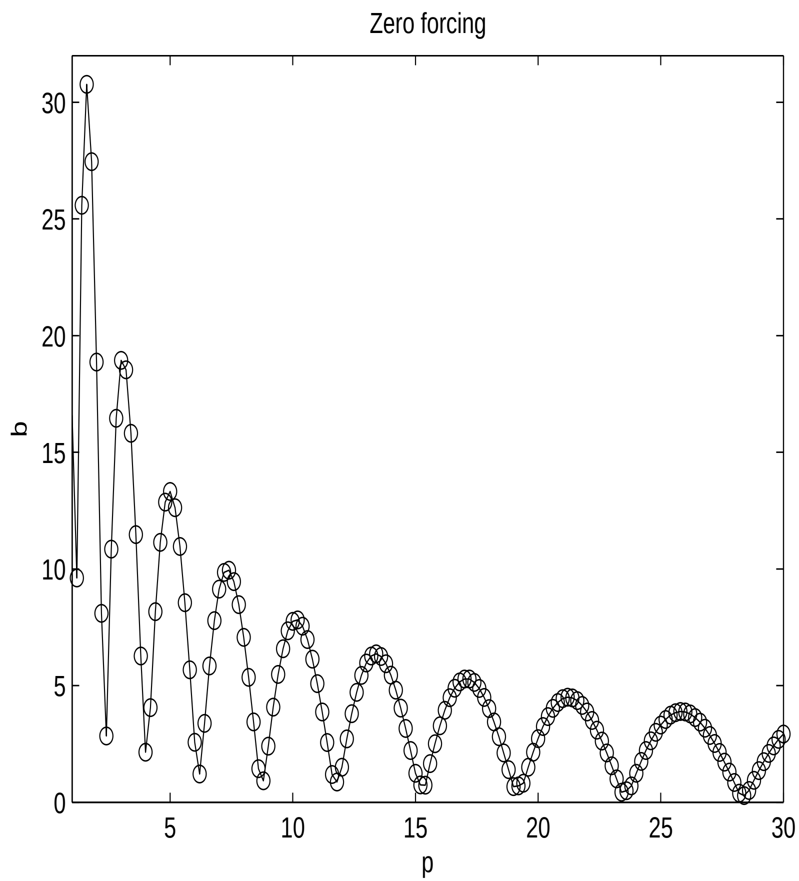
In the plots, "constant forcing" corresponds to $f = (f_1, f_2, f_3)$ as above, while zero forcing refers to $f = 0$. Recall sub-exponential growth in p corresponds to global N-S solution.

Numerical solution to integral equation-plot-1



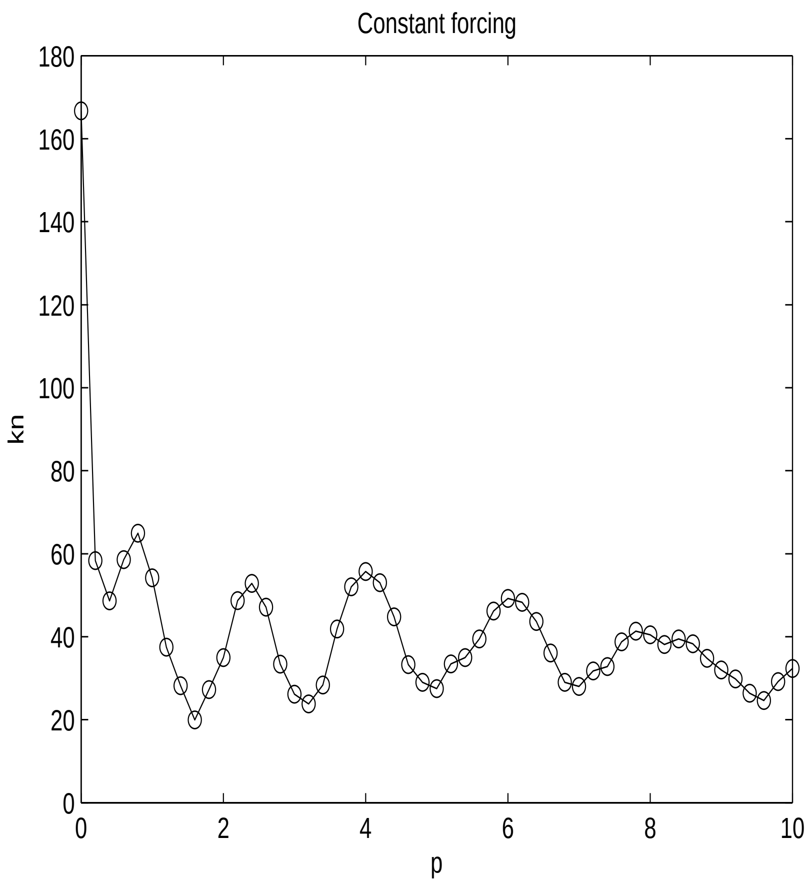
$\|\hat{U}(.,p)\|_{l^1}$ **vs.** p **for** $\nu = 1$**, constant forcing.**

Numerical solution to integral equation-plot-2



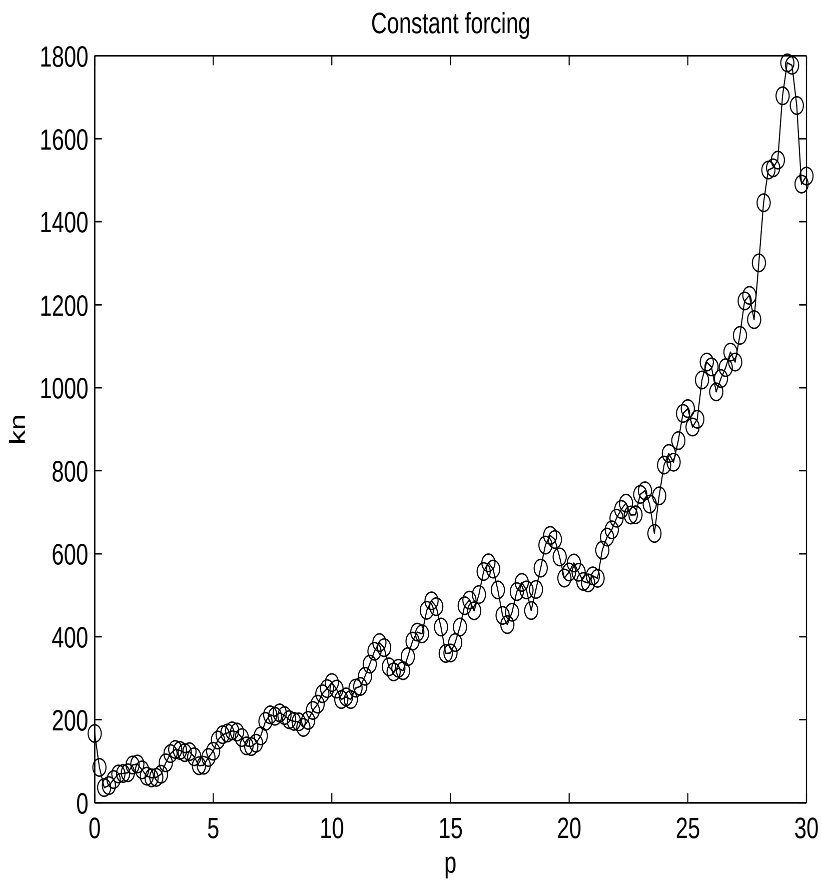
$\|\hat{U}(., p)\|_{l^1}$ **vs.** p **for** $\nu = 1$**, no forcing**

Numerical solution to integral equation-plot-3



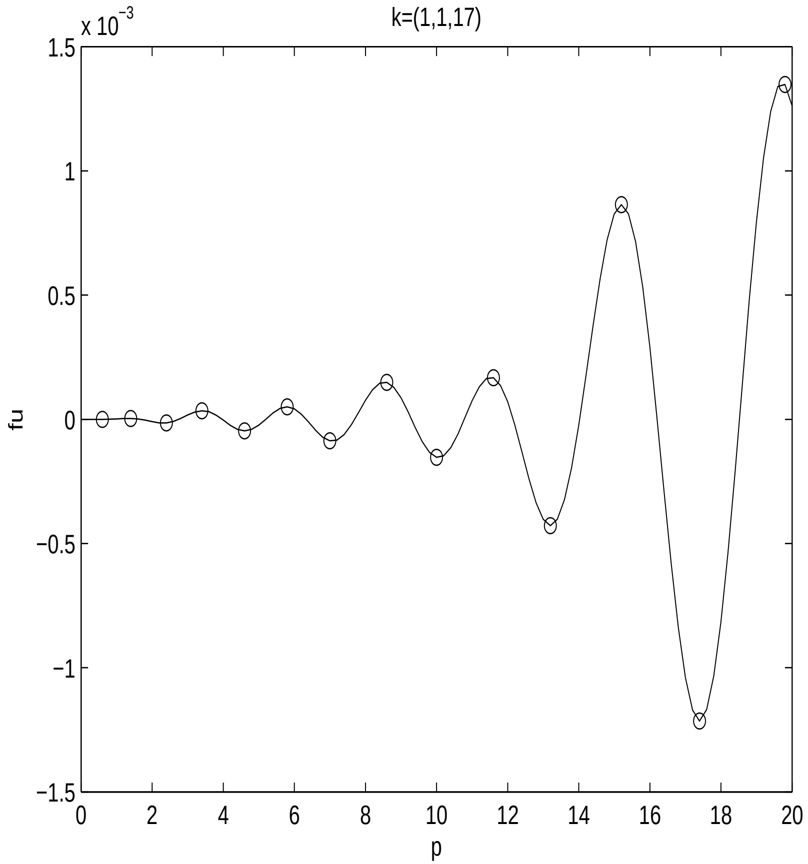
$\|\hat{U}(., p)\|_{l^1}$ **vs.** p **for** $\nu = 0.16$**, constant forcing**

Numerical solution to integral equation-plot-4



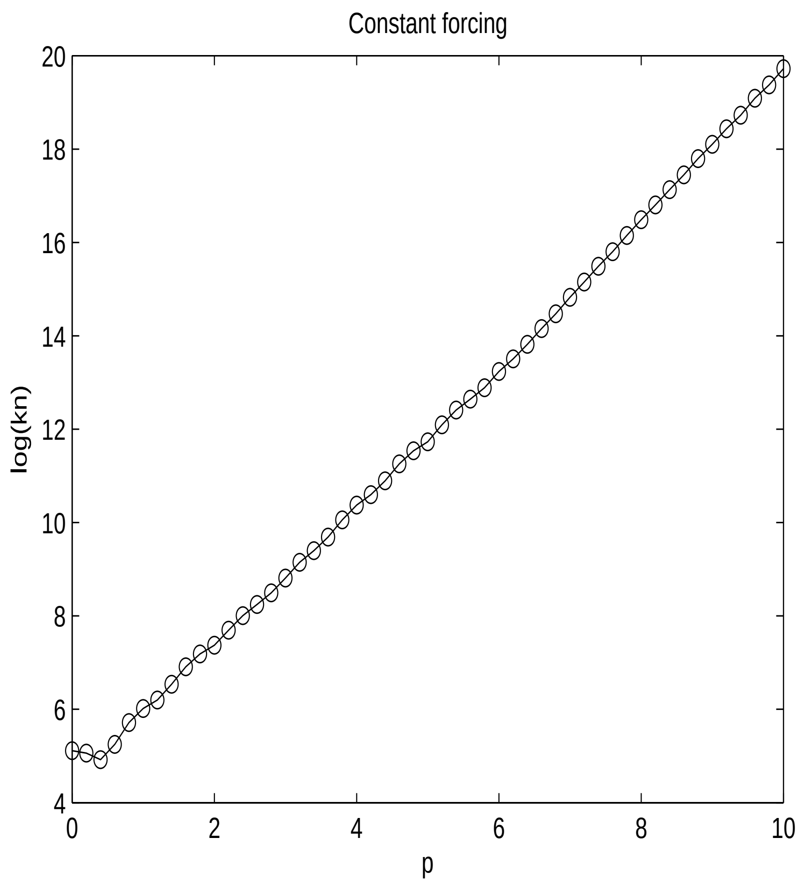
$\|\hat{U}(.,p)\|_{l^1}$ **vs.** p **for** $\nu = 0.1$**, constant forcing**

Numerical solution to integral equation-plot-5



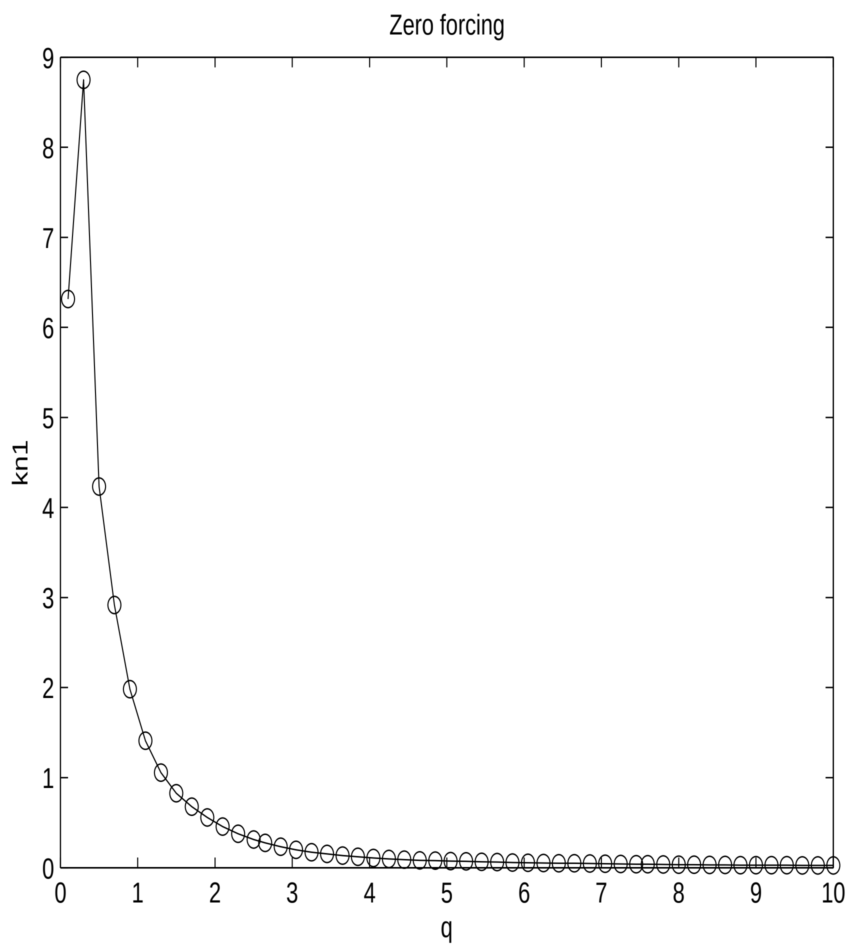
$\hat{U}(k,p)$ **vs.** p **for** $k = (1, 1, 17)$**,** $\nu = 0.1$**, no forcing.**

Numerical solution to integral equation-plot-6



$\log \|\hat{U}(.,p)\|_{l^1}$ **vs.** $\log p$ **for** $\nu = 0.001$**, constant forcing**

$\|\hat{U}(.,q)\|_{l^1}$ vs. q, $n = 2$, $\nu = 0.1$



Kida I.C. $v_1^{(0)} = \sin x_1 (\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3)$

Other components from cyclic relation:

$$v_1^{(0)}(x_1, x_2, x_3) = v_1^{(0)}(x_3, x_1, x_2) = v_3^{(0)}(x_2, x_3, x_1)$$

Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \ , \ c = \int_{q_0}^{\infty} \|\hat{U}^{(0)}(.,q)\|_{l^1} e^{-\alpha_0 q} dq$$

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \left(2 \int_0^{q_0} e^{-\alpha_0 s} \|\hat{U}(.,s)\|_{l^1} ds + \|\hat{v}_0\|_{l^1} \right)$$

$$b = \frac{e^{-\alpha_0 q_0}}{\sqrt{\nu} q_0^{1-1/(2n)} \alpha} \int_0^{q_0} \|\hat{U} \hat{*} \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{l^1} ds$$

Theorem 6: A smooth solution to 3-D Navier-Stokes equation exists in the $\|.\|_{l^1}$ space on the interval $[0, \alpha^{-1/n})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Remark: If q_0 is chosen large enough, ϵ, ϵ_1 is small when computed solution in $[0, q_0]$ decays with q. Then α can be chosen rather small.

Other problems where approach is applicable

- **Navier-Stokes with temperature field (Boussinesq approximation)**

- **Fourth order Parabolic equations of the type:**

$$u_t + \Delta^2 u = N[u, Du, D^2u, D^3u]$$

- **Magneto-hydrodynamic equation with certain approximations.**

- **For some PDE problems with finite-time blow-up, blow-up time related to exponent α of exponential growth of Integral equation as $n \rightarrow \infty$.**

Conclusions

We have shown how Borel summation methods provides an alternate existence theory for PDE Initial value problems like N-S.

With this integral equation (IE) approach, the PDE global existence is implied if known solution to IE has subexponential growth at ∞.

The solution to integral equation in a finite interval can be computed numerically with rigorously controlled errors.

Integral equation in a suitable accelerated variable q will decay exponentially for unforced N-S equation, unless there is a real time singularity of PDE solution.

The computation over a finite $[0, q_0]$ interval gives a refined bound on exponent α at ∞, and hence a longer existence time $[0, \alpha^{-1/n})$ to 3-D Navier-Stokes.

Approach is applicable to a wide class of other PDE initial value problems.