Two bubbles in Stokes Flow

Exact Solutions and Constraints

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Bubbles in 2-D Stokes Flow: Eqns and BCs

$$\mathbf{u} = (\psi_y, -\psi_x) \;\; \text{in} \;\; D$$

$$\nabla^4 \psi = 0.$$

On ∂D:

$$-p n_j + 2 e_{jk} n_k = \kappa n_j,$$

$$V_n = \mathbf{u} \cdot \mathbf{n},$$

p, e, κ, n: pressure, strain, curvature and normal. V_n: normal interface velocity

$$\mathbf{u} \sim (\beta(t) x, -\beta(t) y) + o(1) \quad \text{at} \;\; \infty$$

Background

Problem of interest to study of bubble coalescence

Exact solutions (singly connected domain):

Richardson (1968), Hopper (1990), Antanovskii (1994), Howison & Richardson (1994), Tanveer & Vasconcelos ('94,'95), Siegel,...

Rigorous general context results

Prokert (1995), Solonnikov (1999), ..

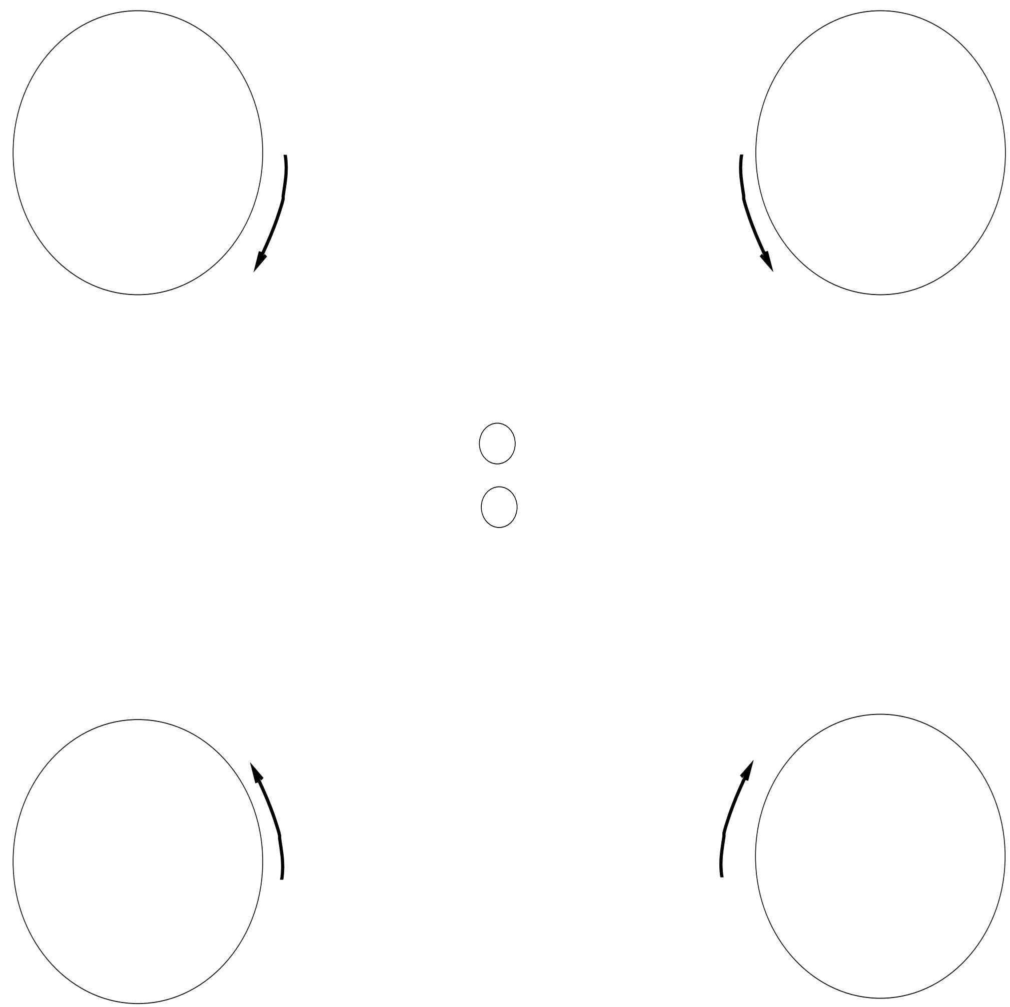
Methodology for multiply connected domains

Crowdy & Tanveer, '98, Richardson '99, Crowdy, '02, '03, ...

A lot of numerical work

Pozrikidis, Kuiken, Kropinski,.....

Taylor's Four-Roll Experiment



Goursat Function Representation and Symmetry

Configuration retains mirror-symmetry about x & y-axis

Goursat function representation of flow:

$$\psi = Im\ [\bar{z} f(z,t) + g(z,t)]$$

$$u + iv = -f(z,t) + z\bar{f}'(\bar{z},t) + \bar{g}'(\bar{z},t)$$

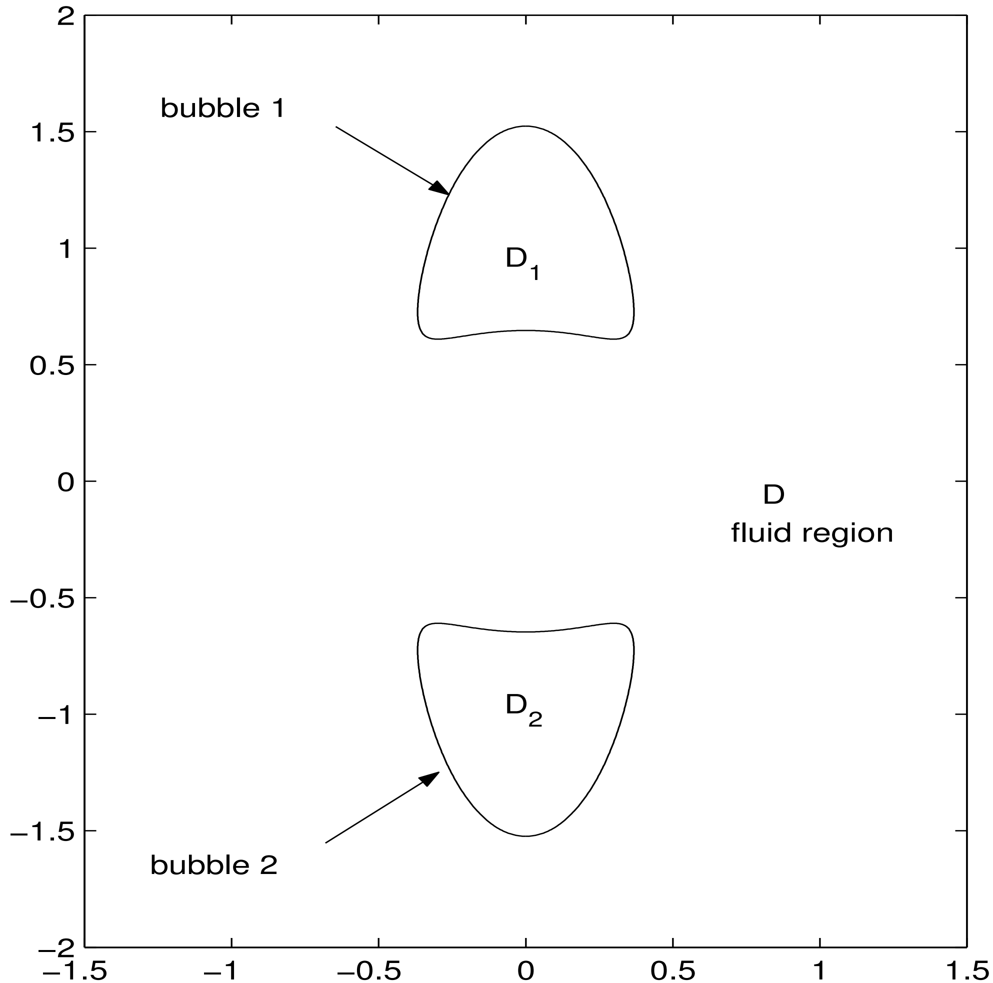
Stress BC on ∂D_1 and ∂D_2:

$$f(z,t) + z\bar{f}'(\bar{z},t) + \bar{g}'(\bar{z},t) = -i\frac{z_s}{2} + \mathcal{A}_1$$

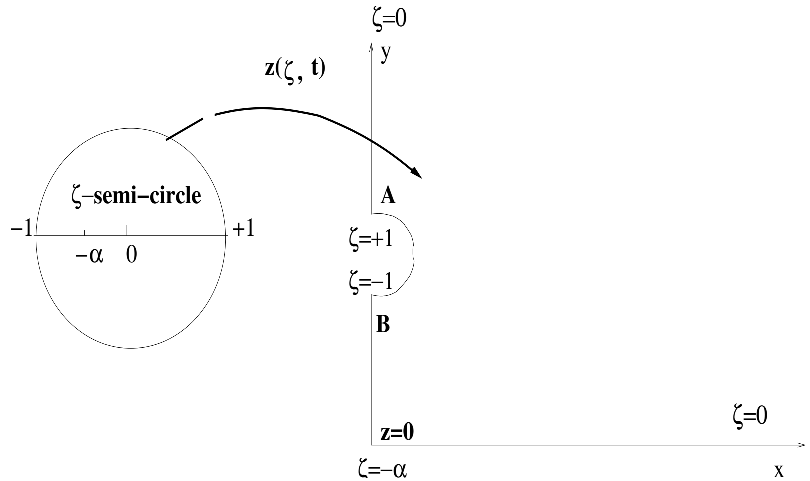
$$f(z,t) + z\bar{f}'(\bar{z},t) + \bar{g}'(\bar{z},t) = -i\frac{z_s}{2} + \mathcal{A}_2$$

Symmetry implies $\mathcal{A}_1 = \mathcal{A}_2^* = -\mathcal{A}_1^*$

Symmetric 2-bubble configuration



Conformal map from semi-circle to 1st Quad



$$z(\zeta,t) = i\zeta^{-1/2}(\zeta+\alpha)^{1/2}(1+\alpha\zeta)^{-1/2}h(\zeta,t)$$

$$h(\zeta,t) = \sum_{n=0}^{\infty} h_n(t)\zeta^n \text{ analytic for } |\zeta| < 1$$

$$h(\zeta,t) = \sum_{n=0}^{N} h_n(t)\zeta^n \ \text{ exact solution for any } \ N$$

Equation satisfied by $h(\zeta, t)$ in $|\zeta| > 1$

$$h_t = \zeta q_1 h_\zeta + q_2 h + q_3 \quad , \quad q_j \text{ analytic in } |\zeta| > 1$$

Theorem: If $h(\zeta, 0) = \sum_{n=1}^{N} h_n \zeta^n$*, then* $h(\zeta, t) = \sum_{n=1}^{N} h_n(t) \zeta^n$ *as long as solution exists with analytic shapes*

$$X_k = \alpha h_0 h_{k-2} + \sum_{j=0}^{N-k} \left[2(1+\alpha^2)(j+1) h_{j+1} \right.$$

$$\left. - \alpha(2j+1) h_j \right] h_{k+j} - \alpha \sum_{j=1}^{N-k+2} (2j-1) h_j h_{k+j-2}$$

Canonical variables: $X_k, k = 1, ..N + 2$

ODEs for X_k, α:

$$\dot{X}_n = -(n-1)\sum_{k=0}^{N-n} I_k X_{n+k} - 2\alpha h_0^2 \beta(t)\delta_{n,2}$$

$$\dot{X}_1 = -\frac{m(t)}{\pi}$$

$$I_0(t) = \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{|z_\zeta(e^{i\theta},t)|} \quad , \quad I_k(t) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(k\theta)d\theta}{|z_\zeta(e^{i\theta},t)|}$$

$$\dot{\alpha} = -\alpha\mathcal{I}(\alpha,t)$$

$$\mathcal{I}(\alpha,t) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\frac{1}{|z_\zeta(e^{i\theta},t)|}d\theta$$

Alternative Cauchy Transform Approach

For $z \in D_k$, $k = 1, 2$, inside of one of two bubbles

$$C_k(z,t) = \frac{1}{2\pi i} \int_{\partial D(t)} \frac{\bar{z}' dz'}{z' - z}$$

Note: If bubble areas given, Cauchy Tranforms completely determine $D(t)$

Lemma: If domain $D(t)$ is invariant under transformation $z \longrightarrow z^*$ and $z \longrightarrow -z$, i.e. reflectionally symmetric about both x-y axis, then $\mathcal{A}_1 = \mathcal{A}_2 = 0$ implies
$C_1(z,t) = C_2(z,t)$

Invariance of meromorphic representation

Theorem: If $C(z,t)$ is a meromorphic function initially with a finite number of simple poles in D, then as long as solution exists,

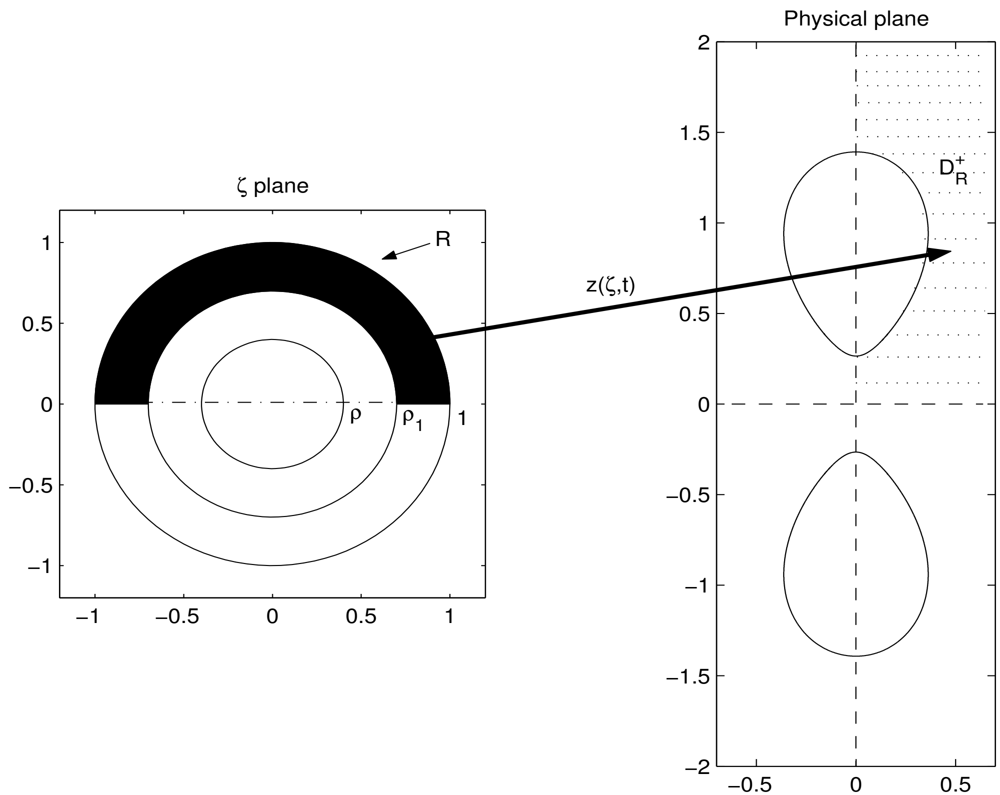
$$C(z,t) = A_\infty(t)z + \frac{A_0(t)}{z} + \sum_{j=1}^{N} \frac{2A_j(t)z}{z^2 - z_j^2(t)}$$

where $A_j(t) = A_j(0)$ for $j = 0....N$ and

$$\dot{A}_\infty - p_\infty(t)A_\infty = 2\beta(t) \;\; , \;\; \dot{z}_j = -2f(z_j(t),t)$$

Symmetry implies if z_j is complex, then there exists some $j' \neq j$ so that $z_{j'} = z_j^*$. Same applies to A_j. However, A_0, A_∞ must be real.

Mapping from annular region to $D(t)$



Note: $\zeta = \pm\rho_1 = \pm\sqrt{\rho(t)}$ **corresponds to** $z = \infty$, $z = 0$

Representation of Exact Solutions

Theorem: When $C(z,0)$ *is initially meromorphic with a finite number of poles, then*

$$z(\zeta,t) = iR(t)\left[\frac{P(-\zeta\sqrt{\rho}^{-1};\rho)P(-\zeta\sqrt{\rho};\rho)}{P(\zeta\sqrt{\rho}^{-1};\rho)P(\zeta\sqrt{\rho};\rho)}\right]L(\zeta,\eta_0,-1;\rho)$$
$$\times\left(\prod_{j=1}^{N} L(\zeta,\eta_j,\zeta_j;\rho)\right)$$

where

$$P(\zeta;\rho) = (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}),$$

$$L(\zeta,\eta_j,\zeta_j;\rho) = \frac{P(\zeta\sqrt{\rho}\eta_j;\rho)P(\zeta\sqrt{\rho}\eta_j^{-1};\rho)}{P(\zeta\sqrt{\rho}\zeta_j;\rho)P(\zeta\sqrt{\rho}\zeta_j^{-1};\rho)}.$$

Equations for parameters in Exact Solution

Lemma: Image of $\bar{\zeta}_j^{-1}$ under $z(\zeta, t)$ corresponds to z_j, the poles of $C(z, t)$. Further, condition $\dot{z}_j = -2f(z_j, t)$ translates to:

$$\frac{d}{dt}\left[\bar{\zeta}_j^{-1}\right] = -\bar{\zeta}_j^{-1}\mathcal{I}(\bar{\zeta}_j^{-1}, t)$$

where $$\mathcal{I}(\zeta, t) = \mathcal{I}^{+}(\zeta, t) - \mathcal{I}^{-}(\zeta, t) + I_c(t)$$

$$\mathcal{I}^{+}(\zeta, t) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{d\zeta'}{\zeta'}\left(1 - 2\frac{\zeta}{\zeta'}\frac{P'(\zeta\zeta'^{-1}; \rho)}{P(\zeta\zeta'^{-1}; \rho)}\right)\frac{1}{2|z_\zeta(\zeta', t)|},$$

$$\mathcal{I}^{-}(\zeta, t) = \frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{d\zeta'}{\zeta'}\left(1 - 2\frac{\zeta}{\zeta'}\frac{P'(\zeta\zeta'^{-1}; \rho)}{P(\zeta\zeta'^{-1}; \rho)}\right)\left(-\frac{1}{2\rho|z_\zeta(\zeta', t)|} - \frac{\dot{\rho}}{\rho}\right)$$

Equations for parameters in Exact Solution-Contd.

$$\mathcal{I}_c(t) = -\frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{d\zeta'}{\zeta'}\left(-\frac{1}{2\rho|z_\zeta(\zeta',t)|} - \frac{\dot{\rho}}{\rho}\right)$$

Further, $\rho(t)$, A_∞ satisfy

$$\frac{d}{dt}\sqrt{\rho} = -\sqrt{\rho}\mathcal{I}(\sqrt{\rho},t)$$

$$\beta(t) = \frac{A_\infty}{2} + A_\infty\left(\frac{\dot{a}}{a} + I(\sqrt{\rho},t) + \sqrt{\rho}I_\zeta(\sqrt{\rho},t)\right)$$

where a is the residue of $z(\zeta,t)$ at $\zeta = \sqrt{\rho}$

$$A_j(t) = A_j(0) \quad \text{for} \quad j = 0, \ldots N$$

Determination of parameters

Lemma: Residues of $C(z,t)$ *of* $A_j(t)$, $A_\infty(t)$ *determined by conformal mapping parameters* $\zeta_j(t)$, $\eta_j(t)$, $\rho(t)$ *and* $R(t)$ *from matching residues at* $\zeta = \zeta_j$ *of the equation*

$$\bar{z}(\zeta^{-1}, t) = C(z(\zeta,t),t) - E_1(z(\zeta,t)$$

where $E_1(z(\zeta,t),t)$ *to be analytic in* $\rho < |\zeta| < 1$.

Note: If $\beta(t)$ specified, all together, $2N+2$ equations for $2N+2$ parameters ζ_j, η_j, ρ and $R(t)$

No freedom left in specifying area!

Alternatively, for specified area, $\beta(t)$ determined from the flow.

Computation for Special Cases

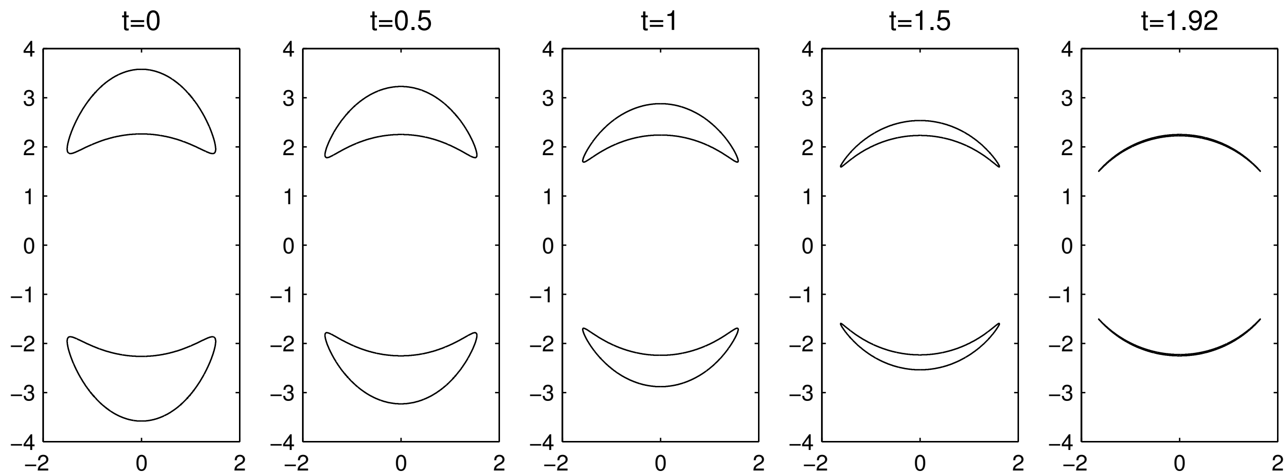
$$z(\zeta, t) = \frac{P(-\zeta/\sqrt{\rho}, \rho) P(-\zeta\sqrt{\rho}, \rho)}{P(\zeta/\sqrt{\rho}, \rho) P(\zeta\sqrt{\rho}, \rho)}$$

$$\times \left(R_2 + R_1 \frac{P(i\zeta\sqrt{\rho}, \rho) P(-i\zeta\sqrt{\rho}, \rho)}{P(-\zeta\sqrt{\rho}, \rho) P(-\zeta\sqrt{\rho}, \rho)} \right)$$

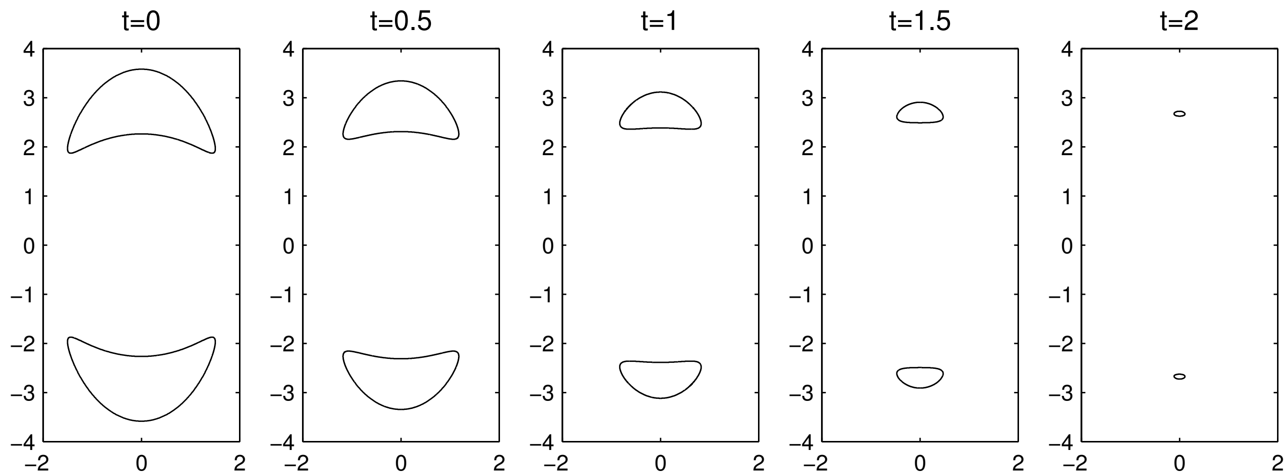
Note: Actually in the form of $N = 1$ exact solution

For given β, unknown parameters are R_1, R_2 and ρ

Exact solution for $\sigma = 1, \beta = 0.1$



Exact solution for $\sigma = 1, \beta = 0$



Numerical Solutions for $\mathcal{A}_1 \neq 0$

Can specify bubble area and β. In annular rep.:

$$z(\zeta, t) = \frac{ia}{\zeta - \sqrt{\rho}} + i \sum_{n=-\infty}^{\infty} a_n \zeta^n$$

$$F(\zeta, t) = \frac{iF_\infty}{\zeta - \sqrt{\rho}} + + i \sum_{n=-\infty}^{\infty} F_n \zeta^n$$

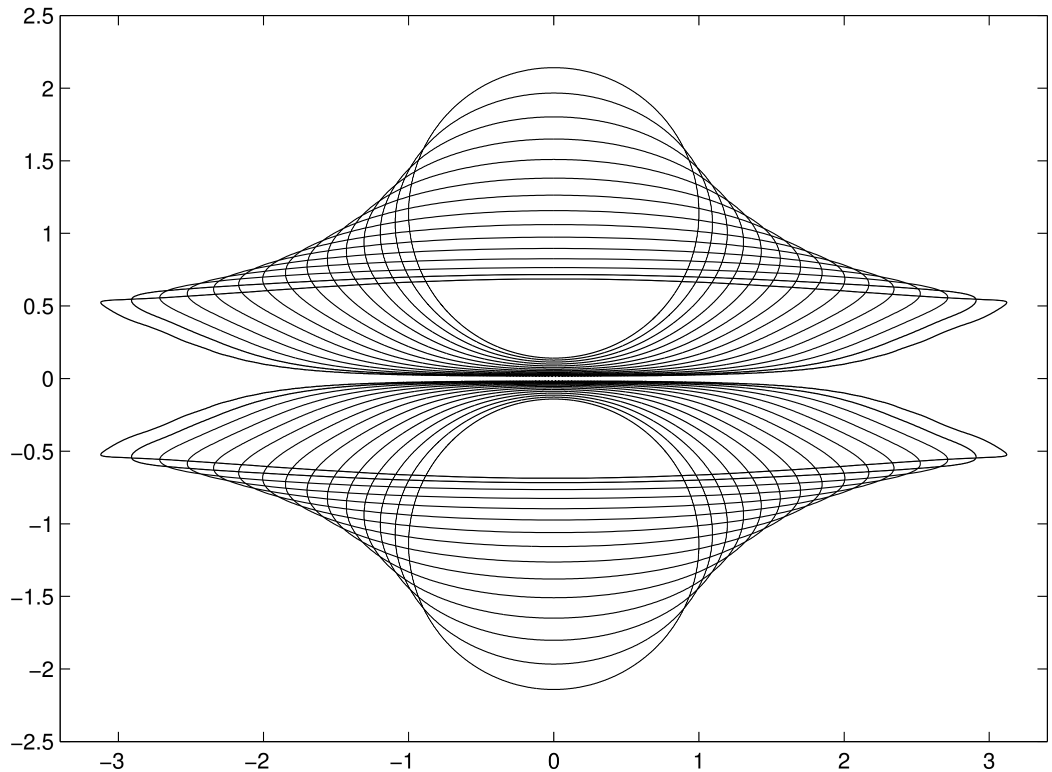
$$G(\zeta, t) = \frac{iG_\infty}{\zeta - \sqrt{\rho}} + + i \sum_{n=-\infty}^{\infty} G_n \zeta^n$$

F_∞, G_∞ related to p_∞ and β

Stress relations gives G_n, F_n in terms of a_n

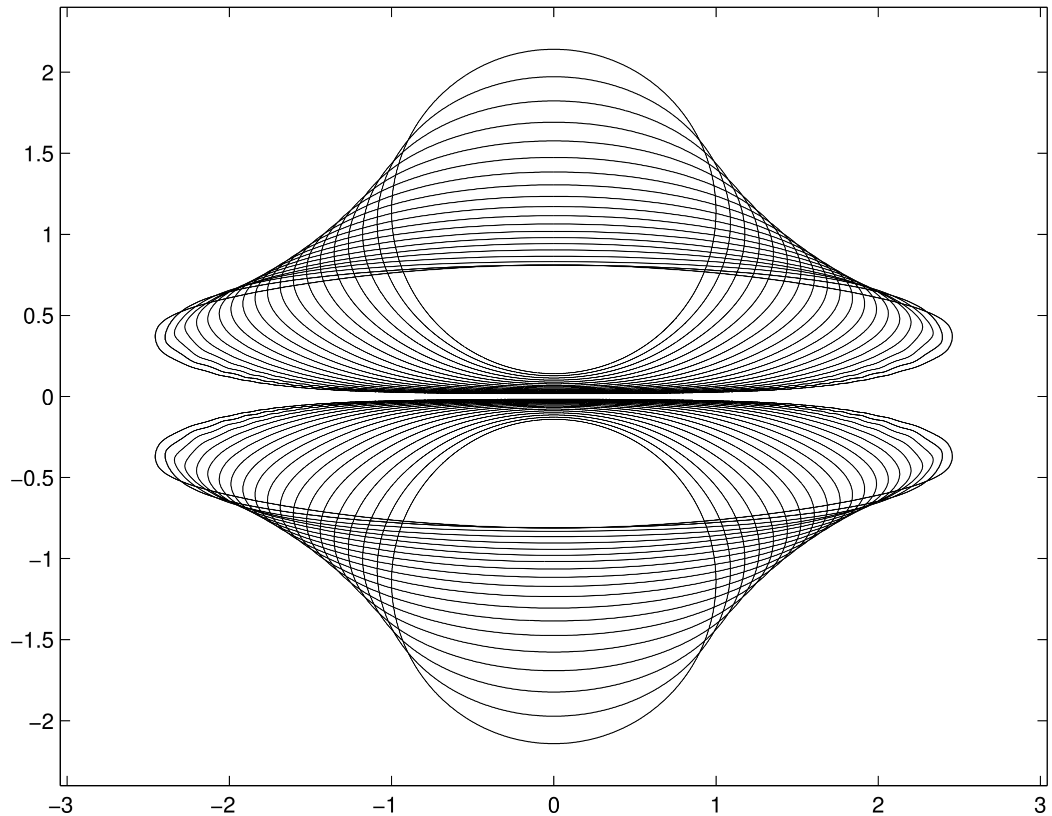
Truncation gives ODEs for $a_n(t)$, $a(t)$, $\rho(t)$

$\sigma = 0$, $\beta = 0.5$ evolution



Times shown: $t = 0, (0.1), 1.5$

$\sigma = 1$, $\beta = 0.5$ evolution



Times shown: $t = 0, (0.1), 1.9$

Conclusion

1. For a two-bubble configuration, there is a constraint between bubble area and straining rate at ∞ within class of exact solutions.
2. More general solutions outside this class determined numerically
3. Bubbles appear to come close indefinitely
4. For shrinking bubbles, bubble can shrink to a line or a point, depending on straining rate.
5. Mathematics of Cauchy Transform and conformal map is attractive for domains with higher connectivity, having certain rotational symmetries.