Quasi-solution approach to Nonlinear Problems

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Basic Idea

Nonlinear problems, written as $\mathcal{N}[u] = 0$, are difficult to analyze unless nonlinearity is "weak" or has special structure

However, if we find some u_0 with $\mathcal{N}[u_0] = R$ small and Initial/Boundary Conditions approximately satisfied, then $E = u - u_0$ satisfies

$$LE = -R - \mathcal{N}_1[E] \ ,$$

where $L = \mathcal{N}_u$ and $\mathcal{N}_1[E] = \mathcal{N}[u_0 + E] - \mathcal{N}[u_0] - \mathcal{L}E$

If L has suitable inversion for given small intial/boundary conditions and nonlinearity $\mathcal{N}_1$ is regular, then a contraction mapping argument can often be employed in a suitable space to analyze the weakly nonlinear problem:

$$E = -\mathcal{L}^{-1}R - \mathcal{L}^{-1}\mathcal{N}_1[E]$$

Remarks

This kind of inversion regularly employed in other contexts–for instance in determining error bounds for $|u - u_0|$ in perturbation problems of the type: $\mathcal{N}[u; \epsilon] = 0$ when $\mathcal{N}[u_0; 0] = 0$

What does not seem to be recognized until recently is how to determine quasi-solution u_0 in a general, efficient and systematic manner.

Recently, there has been some work (Costin, Huang, Schlag, 2012), (Costin, Huang, T., 2012), (Costin, T., 2013), (T. 2013) in a number of different nonlinear ODE and integro-differential equation contexts. I will describe how computation typically based on orthogonal polynomials and exponential asymptotics (when domain extends to ∞) may be used to construct u_0 and bounds on E obtained.

Some Applications of quasi-solution approach

1. Dubrovin conjecture for P-1, $y'' = 6y^2 + z$: *For the unique solution of P-1 satisying* $y(z) = i\sqrt{\frac{z}{6}}\,(1 + o(1))$ *as* $e^{-i\pi/5}z \to +\infty$, *the sector* $\arg z \in \left(-\frac{3}{5}\pi, \pi\right)$ *is singularity free*. **Problem arises in characterizing small dispersion effects on gradient blow-up for focussing NLS (Dubrovin, Grava, Klein, 2008)**

2. Find solution f to Blasius similarity equation in $(0, \infty)$:

$$f''' + ff'' = 0\,, \quad f(0) = 0 = f'(0), \quad \lim_{x\to\infty} f'(x) = 1$$

3. Existence of 2-D water waves of permanent form (Involves a nonlinear integro-differential equation)

Note: Nonconstructive proofs exist in cases 2. and 3.; not in 1

Blasius similarity problem

Blasius (1908) derived the two point BVP ODE:

$$f''' + f f'' = 0 \text{ in } (0, \infty) \ \text{ with } f(0) = 0 = f'(0) \ , \ \lim_{x \to \infty} f'(x) = 1$$

as similarity solution to fluid Boundary layer equations.

Generalization include $f(0) = \alpha$, $f'(0) = \gamma$

Much work (Topfer, 1912, Weyl, '42, Callegari & Friedman '68, Hussaini & Lakin '86, others. Existence and uniqueness known.

Related problem:

$$F''' + F F'' = 0 \text{ in } (0, \infty) \ \text{ with } F(0) = 0 = F'(0) \ , F''(0) = 1$$

If $\lim_{x \to \infty} F'(x) = a > 0$, then $f(x) = a^{-1/2} F\left(a^{-1/2} x\right)$.

Though this transformation, the two point BVP is turned into an initial value problem; though convenient, transformation not needed for quasi-solution approach.

Definitions

Let

$$P(y)=\sum_{j=0}^{12}\frac{2}{5(j+2)(j+3)(j+4)}p_j y^j \tag{1}$$

where $[p_0, ..., p_{12}]$ **are given by**

$$\left[-\frac{510}{10445149},-\frac{18523}{5934},-\frac{42998}{441819},\frac{113448}{81151},-\frac{65173}{22093},\frac{390101}{6016},-\frac{2326169}{9858},\frac{4134879}{7249},-\frac{1928001}{1960},\frac{20880183}{19117},-\frac{1572554}{2161},\frac{1546782}{5833},-\frac{1315241}{32239}\right] \tag{2}$$

Define

$$t(x)=\frac{a}{2}(x+b/a)^2,\ I_0(t)=1-\sqrt{\pi t}e^t\operatorname{erfc}(\sqrt{t}),$$
$$J_0(t)=1-\sqrt{2\pi t}e^{2t}\operatorname{erfc}(\sqrt{2t}) \tag{3}$$

Main Results

$$q_0(t) = 2c\sqrt{t}e^{-t}I_0 + c^2 e^{-2t}\left(2J_0 - I_0 - I_0^2\right), \qquad (4)$$

Theorem: *Let F_0 be defined by*

$$F_0(x) = \begin{cases} \frac{x^2}{2} + x^4 P\left(\frac{2}{5}x\right) \text{ for } x \in [0, \frac{5}{2}] \\ ax + b + \sqrt{\frac{a}{2t(x)}}q_0(t(x)) \text{ for } x > \frac{5}{2} \end{cases} \qquad (5)$$

Then, there is a unique triple (a, b, c) close to

$(a_0, b_0, c_0) = \left(\frac{3221}{1946}, -\frac{2763}{1765}, \frac{377}{1613}\right)$ *in the sense that* $(a, b, c) \in \mathcal{S}$ *where*

$$\mathcal{S} = \Bigg\{(a, b, c) \in \mathbb{R}^3 : \sqrt{(a - a_0)^2 + \frac{1}{4}(b - b_0)^2 + \frac{1}{4}(c - c_0)^2} \leq \rho_0 := 5 \times 10^{-5}\Bigg\} \qquad (6)$$

with the property that F_0 is an approximation to true solution F to the IVP.

Main Results

More precisely,

$$F(x) = F_0(x) + E(x) \, , \tag{7}$$

where the error term E *satisfies*

$$\|E''\|_\infty \leq 3.5\times10^{-6} \, , \|E'\|_\infty \leq 4.5\times10^{-6} \, , \|E\|_\infty \leq 4\times10^{-6} \text{ on } [0, \frac{5}{2}] \tag{8}$$

and for $x \geq \frac{5}{2}$

$$\left|E\right| \leq 1.69 \times 10^{-5} t^{-2} e^{-3t} \, , \left|\frac{d}{dx}E\right| \leq 9.20 \times 10^{-5} t^{-3/2} e^{-3t}$$

$$\left|\frac{d^2}{dx^2}E\right| \leq 5.02 \times 10^{-4} t^{-1} e^{-3t} \tag{9}$$

Construction of quasi-solution F_0 for Blasius

Use numerical calculations and projection to Chebyshev basis on $\mathcal{I} = \left[0, \frac{5}{2}\right]$.

The residual $R = F_0''' + F_0 F_0''$ is a polynomial of degree 30; we project it to Chebyshev basis: $R(x) = \sum_{j=0}^{30} r_j T_j \left(\frac{4}{5}x - 1\right)$ and estimate $\|R\|_\infty \leq \sum_{j=0}^{30} |r_j| \leq 5 \times 10^{-7}$. Procedure generalizable to multi-variables using product space representation.

For interval $I = \left[\frac{5}{2}, \infty\right)$, any solution for which $F'(x) \to a > 0$ as $x \to \infty$ has the representation $F(x) = ax + b + G(x)$ where G is exponentially small.

Applying exponential asymptotics theory (Costin, '98),

$$G(x) = \sqrt{\frac{a}{2t(x)}} q(t(x)) \text{ , where } q(t) = \sum_{n=1}^{\infty} \xi^n Q_n(t) \text{ , where } \xi = \frac{ce^{-t}}{\sqrt{t}}$$

Two term trucation provides quasi-solution with small residual.

Analysis of Error term

To complete quasi-solution, need (a, b, c) approximately–through numerical matching.

Note: Quasi solution determined empirically; not unique. Anything that gives small residual R and approximately satisfies boundary/initial condition is a candidate.

Rigor needed in proving R is uniformly small and that $E = F - F_0$ satisfying

$$LE := E'' + F_0 E'' + E F_0'' = -R - E E'',$$

and small initial/boundary conditons has a small bound.

The analysis for E involves inversion of L subject to initial/boundary conditions and use of contraction arguments in appropriate spaces. It is detailed and matching arguments at $\frac{5}{2}$ are delicate but not out of the ordinary. We skip details.

Quasi-solution approach for general systems

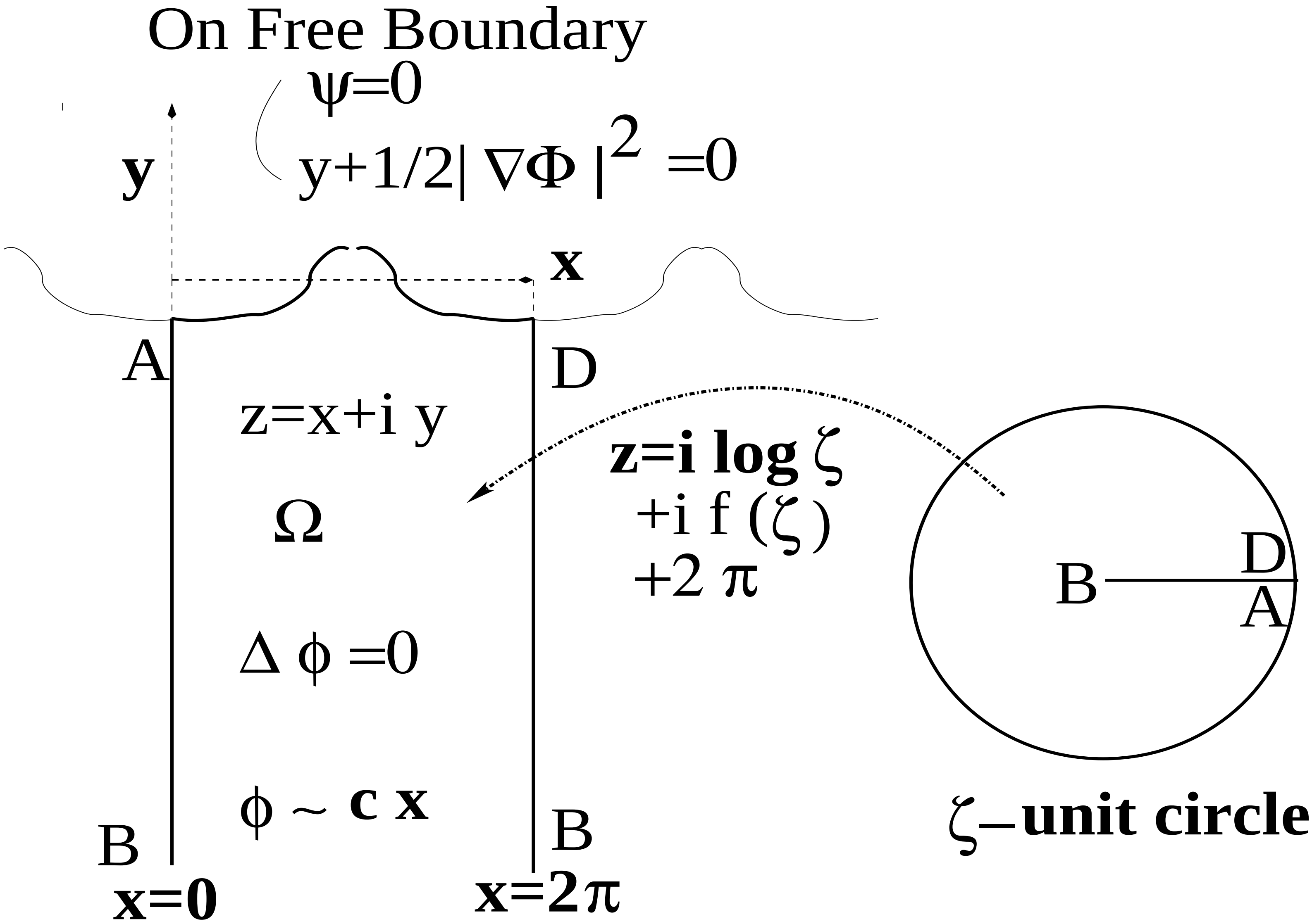
The procedure described is quite general. For proving Dubrovin conjecture for P-1: $y'' = 6y^2 + z$, we used a similar argument in the complex z-plane.

No *a priori* **limitation on the size of the system or the number of variables/parameters, though the method is most transparent for one independent variable. Error bound checks become more computer assisted with multi-variables and parameters.**

Rigorous error control does not need explicit Green's function; energy methods will do as long as residual E is small.

The accuracy can in principle be arbitrary, though the number of terms needed in the quasi-solution may become prohibitive for too high an accuracy. In some problems, inversion of L can give rise to large bounds; in some cases one may need make arguments in sub-domains to obtain refined estimates.

2-D symmetric steady water waves



Background

Extensive history of water waves for more then 200 years, starting with Laplace, Langrange, Cauchy, Poisson, Airy, Stokes, $\cdots$

Rigorous work for small amplitude waves by Nekrasov (1921), Levi-Civita (1924)

Large Amplitude Wave analysis by Krasovskii (1961), Keady and Norbury (1978), Amick and Toland (1981)

Numerical calculations also by a number of people including Longuet-Higgins and Cokelet, Schwartz in midseventies

Other variations include waves with nonzero vorticity, finite depth fluid, limiting cases lead to the KdV (work by Strauss, Bona, $\cdots$

Conformal Mapping approach

Steady symmetric 2-D water waves is equivalent to determining analytic function f in $|\zeta| < 1$ with property $1 + \zeta f' \neq 0$ in $|\zeta| \leq 1$ and satisfying

$$\mathrm{Re}\, f = -\frac{c^2}{2\left|1 + \zeta f'\right|^2} \quad \text{on} \quad |\zeta| = 1$$

The wave height $h = \frac{1}{2}\left[f(-1) - f(+1)\right]$. One seeks (f, c) for given h. For efficiency in representation, better to use

$$f = \sum_{j=0}^{\infty} f_j \eta^j \ , \quad \text{where} \ \ \eta = \frac{\zeta + \alpha}{1 + \alpha\zeta},$$

where $\alpha \in (0, 1)$ chosen in accordance to $\alpha = \frac{22}{27}h + \frac{3}{2}h^2 + 3h^3$ for $h \in (0, h_M)$, where $h_M \approx 0.4454\cdots$ correspond to Stokes highest wave that makes 120^o angle at the apex.

Representation in the η domain

$$\mathrm{Re}\, f = -\frac{c^2}{2\left|1 + q(\eta)\eta f'(\eta)\right|^2} \text{ on } |\eta| = 1 \text{ where}$$

$$q(\eta) = \frac{(\eta - \alpha)(1 - \alpha\eta)}{\eta(1 - \alpha^2)}$$

We define quasi-solution (f_0, c_0) so that f_0 is analytic in $|\eta| < 1$ and on $\eta = e^{i\nu}$, $R_0(\nu)$ and $R_0'(\nu)$ are small, where

$$R_0(\nu) = \left|1 + q(\eta)\eta f_0'(\eta)\right|^2 \mathrm{Re}\, f_0 + \frac{c_0^2}{2}$$

Also, require

$$(1 + \eta q f_0') \neq 0 \text{ , for } |\eta| \leq 1$$

Note f_0 is a polynonomial of order n, $R_0(\nu)$ is a polynomial in $\cos\nu$ of order $2n + 1$.

Change of dependent variables

$$w = -\frac{2}{3}\log c + \log\left(1 + \eta q f'\right) \text{ , implying } \left|1 + \eta q f'\right| = c^{2/3} e^{\mathrm{Re}\, w} \ ,$$

then w satisfies

$$\frac{d}{d\nu}\mathrm{Re}\, w + q^{-1} e^{2\,\mathrm{Re}\, w}\, \mathrm{Im}\, e^{w} = 0 \ \text{ for } \eta = e^{i\nu}$$

We note

$$w(\eta) = \sum_{j=0}^{\infty} b_j \eta^j \ , \ \text{where } b_j \text{ is real}$$

Since $q(\alpha) = 0$, it follows that $w(\alpha) = -\frac{2}{3}\log c$, $i.e$

$$-\frac{2}{3}\log c = \sum_{j=0}^{\infty} b_j \alpha^j$$

Quasi-solution under change of variable

Corresponding to the quasi-solution f_0, we define

$$w_0 = -\frac{2}{3}\log c_0 + \log\left(1 + \eta q(\eta) f_0'\right)$$

Then, we can check that w_0 satisfies

$$\frac{d}{d\nu}\operatorname{Re} w_0 + q^{-1} e^{2\operatorname{Re} w_0}\operatorname{Im} e^{w_0} = R(\nu) := -\frac{R_0'(\nu)}{c_0^2 - 2R_0} - \frac{4A(\nu)R_0(\nu)}{3(c_0^2 - 2R_0)}\,,$$

$$2A(\nu) = 3q^{-1}e^{2\operatorname{Re} w_0}\operatorname{Im}\left\{e^{w_0}\right\} = \frac{3}{c_0^2}\operatorname{Im}(\eta f_0')\left|1 + \eta q f_0'\right|^2,$$

$$h_0 = -\frac{(1-\alpha^2)}{2}\int_{-1}^{1}\frac{e^{w_0(\eta)-w_0(\alpha)} - 1}{(\eta-\alpha)(1-\alpha\eta)}d\eta\,, \text{with } h - h_0 \text{ small}$$

Weakly nonlinear formulation for $W = w - w_0$

$W = w - w_0 := \Phi + i\Psi$ **on** $\eta = e^{i\nu}$ **satisfies:**

$$\mathcal{L}\left[\Phi\right] := \frac{d}{d\nu}\Phi + 2A(\nu)\Phi + 2B(\nu)\Psi = \tilde{\mathcal{M}}[W] - R(\nu) =: r(\nu) ,$$

$$2B(\nu) = q^{-1}e^{2\,\mathrm{Re}\,w_0}\,\mathrm{Re}\left\{e^{w_0}\right\} = \frac{1}{c_0^2}\left[1 + q\eta f_0'\right]\left|1 + \eta q f_0'\right|^2,$$

$$\tilde{M}[W] := -\frac{2}{3}A(\nu)M_1 - 2B(\nu)M_2 ,$$

where $M_1 = e^{2\,\mathrm{Re}\;W}\;\mathrm{Re}\;e^{W} - 1 - 3\,\mathrm{Re}\;W\;, M_2 = e^{2\,\mathrm{Re}\;W}\;\mathrm{Im}\;e^{W} - \mathrm{Im}\,W$

Note

$$\Psi(\nu) = \frac{1}{2\pi}PV\int_0^{2\pi}\Phi(\nu')\cot\frac{\nu-\nu'}{2}d\nu'$$

Need inversion of $\mathcal{L}$ to obtain a weakly nonlinear integral equation for Φ

Water wave error: Function Spaces

Definition: **For fixed $\beta \geq 0$, define $\mathcal{A}$ to be the space of analytic functions in $|\eta| < e^{\beta}$ with real Taylor series coefficient at the origin, equipped with norm:**

$$\|W\|_{\mathcal{A}} = \sum_{l=0}^{\infty} e^{\beta l} \left| W_l \right| \ , \text{ where } W(\eta) = \sum_{l=0}^{\infty} W_l \eta^l$$

Define $\mathcal{E}$ to be the Banach space of real 2π-periodic even functions ϕ so that

$$\phi(\nu) = \sum_{j=0}^{\infty} a_j \cos(j\nu) \ , \text{with norm } \|\phi\|_{\mathcal{E}} := \sum_{j=0}^{\infty} e^{\beta j} |a_j|$$

Define $\mathcal{S}$ to be Banach space of real 2π- periodic odd functions

$$\psi(\nu) = \sum_{j=1}^{\infty} b_j \sin(j\nu) \ , \text{with norm } \|\psi\|_{\mathcal{S}} := \sum_{j=1}^{\infty} e^{\beta j} |b_j| < \infty$$

Control on error $W = w - w_0$

Define $\mathcal{E}_1$ subspace of $\mathcal{E}$ so that for $\Phi \in \mathcal{E}_1$,

$\Phi = a_0 + \sum_{j=2}^{\infty} a_j \cos(j\nu)$.

When certain conditions depending on quasi-solution hold, then the most general solution of $\mathcal{L}\Phi = r$ is given by

$$\Phi = \mathcal{K}r + a_1 G,$$

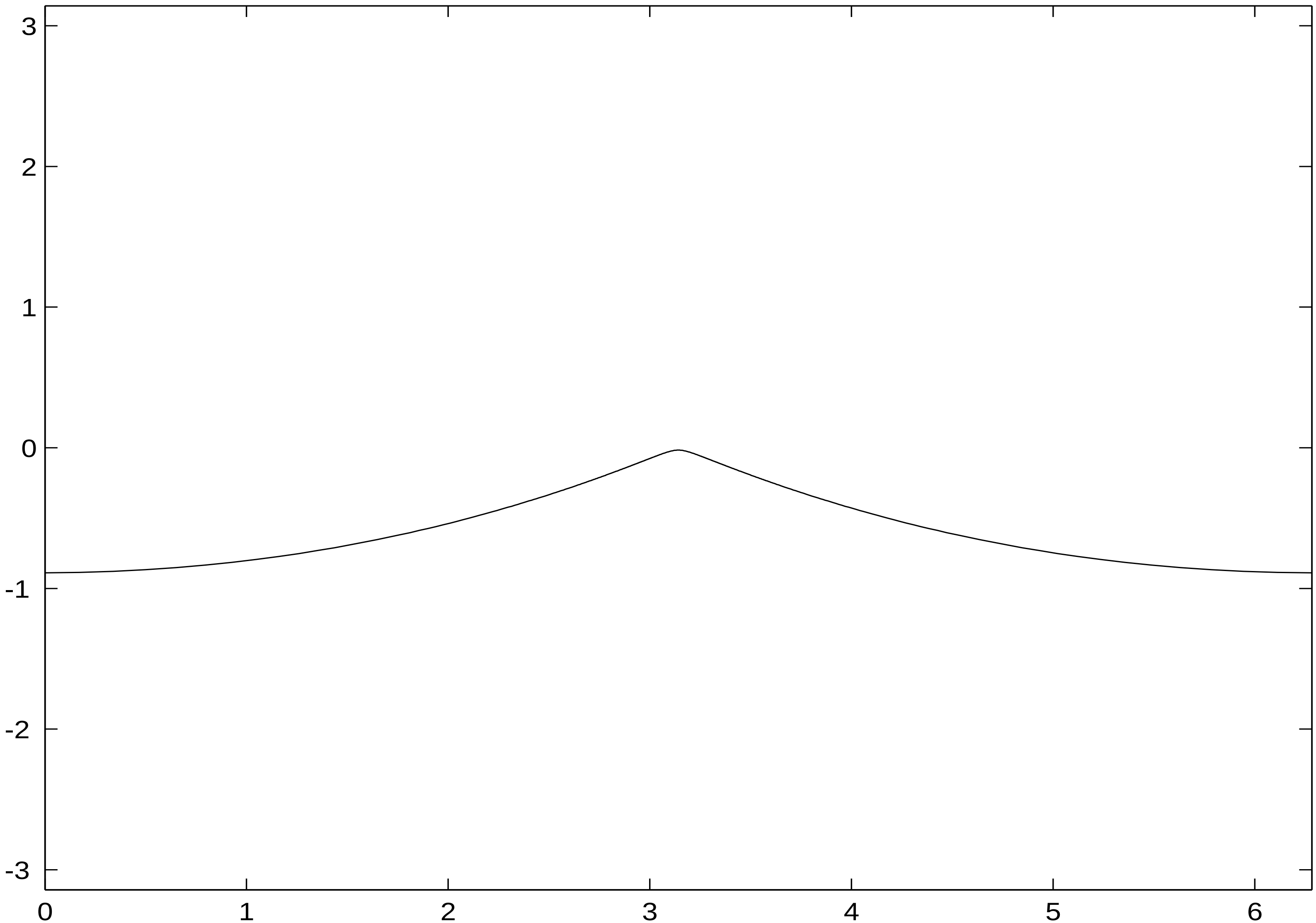
where $\mathcal{K} : \mathcal{S} \to \mathcal{E}_1$ is a bounded operator with norm M that may be estimated and $G \in \mathcal{E}$

Proving steady symmetric water waves equivalent to proving solution to weakly nonlinear integral equation:

$$\Phi = -\mathcal{K}[R] - \mathcal{K}\mathcal{M}[\Phi] + a_1 G =: \mathcal{N}[\Phi],$$

for each $a_1 \in (-\epsilon_0, \epsilon_0)$ small enough interval, height constraint determines a_1. For a range of height $h \in (0, h_M)$, we show $\mathcal{N}$ is contractive for chosen quasi-solutions (f_0, c_0)

Water Wave with $h = 0.4359$, highest speed



Accurate quasi-solution for $h = 0.4359$

In this case $c_0 = \frac{254979}{233294}$, $\alpha = \frac{7997989622533}{9000000000000}$. **Quasi-solution:**

$$\tilde{f}_0 = b_0 + \sum_{j=1}^{15} \frac{b_j}{j} \eta^j + \sum_{m=1}^{6} \lambda_m \gamma_m^{-1} \log\left(1 + \gamma_m \eta\right) \ , \text{ where}$$

$$\mathrm{b} = \left[-\frac{7491}{33875}, \frac{3496}{95411}, \frac{421}{16231}, \frac{991}{116428}, \frac{6053}{1113170}, \frac{939}{445538}, \frac{2921}{2444353}, \frac{325}{638894}, \frac{359}{1442979}, \frac{229}{2029023}, \frac{213}{4708117}, \frac{111}{5158825}, \frac{31}{4858465}, \frac{24}{7621883}, \frac{34}{64439691}, \frac{33}{123015796}\right]$$

$$\gamma = \left[\frac{30266}{33767}, -\frac{39823}{44724}, -\frac{36643}{43855}, \frac{9341}{11348}, -\frac{46141}{64708}, \frac{17251}{25880}\right]$$

$$\lambda = \left[-\frac{6067}{596979}, -\frac{42304}{88055}, \frac{1889}{11944}, -\frac{509}{48108}, \frac{5220}{59461}, -\frac{2169}{181300}\right]$$

Quasi-solution as function of h

Quasi-solution representation available as a function of height h as well. For smaller heights $h \leq \frac{3}{10}$ uniform expression involving polynomial of 15 th order in η and 5-th order in h.

For larger heights, better to use representation for small intervals in h in terms of low order polynomials in h

Rigorous error control possible for smaller heights uniformly in h; but for larger heights, the proof with h parameter becomes unwieldly. We can give good bounds for the worst case; *i.e.* largest height we tried, $h = 0.4359$.

Conclusion

1. With suitable quasi-solution u_0, many strongly nonlinear problems can be analyzed through weakly nonlinear analysis.
2. No *a priori* bar on the number of variables and/or parameters, as long as suitable bounds on inversion of Frechet derivative is possible; analysis most transparent for problems in one variable with no parameters. Otherwise, the error estimate calculation is more computer assisted.
3. ODE or systems of ODEs, including two point boundary value problems are easily amenable. Opens the opportunity for homoclinic-heteroclinic determination in higher dimension.
4. PDE similarity blow up or spectral analysis in 1+1 dimension amenable to our type of analysis.
5. Look forward to working with colleagues here.
6. Papers available online.

Linear Problem $\mathcal{L}\Phi = r$ for given a_1 and r

We seek solution $\Phi = \sum_{j=0}^{\infty} a_j \cos(j\nu) \in \mathcal{E}$ to $\mathcal{L}\Phi = r$ for given $a_1 \in (-\epsilon_0, \epsilon_0)$ and $r = \sum_{j=1}^{\infty} r_j \sin(j\nu)$. Equivalent to solving for $\mathrm{a} = (a_0, 0, a_2, a_3, \cdots) \in \mathrm{H}$ for given a_1 and $\mathrm{r} = (0, r_1, r_2, r_3, \cdots) \in \mathrm{H}$, where H is the weighted l^1 space with norm $\|\mathrm{g}\|_{\mathrm{H}} = \sum_{j=0}^{\infty} e^{\beta l} |g_l|$ and the following equations are satisfied:

$$a_0 + \sum_{l=2}^{\infty} \frac{a_l}{2A_1} \left(A_{l+1} - A_{l-1} + B_{l-1} - B_{l+1} \right) = \frac{r_1}{2A_1} + \frac{1}{2A_1} \left(1 - 2B_0 - A_2 \right.$$

$$\frac{2A_k}{l_k} a_0 + \sum_{l=2}^{k-1} \frac{a_l}{l_k} \left(A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k} \right) - a_k$$

$$+ \sum_{l=k+1}^{\infty} \frac{a_l}{l_k} \left(A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k} \right) = \frac{r_k}{l_k} - \frac{a_1}{l_k} \left(A_{k-1} + A_{k+1} + B_{k-1} \right.$$