Global Existence and Stability of translating bubbles in a Hele-Shaw cell

Saleh Tanveer

(Ohio State University)

Collaborator: J. Ye

Background

As mentioned before, $\epsilon = 0$ evolution when less viscous fluid pushes the more viscous fluid is ill-posed and generally not physically relevant. They do not appear as limits of $\lim_{\epsilon \to 0^+}$

With $\epsilon \neq 0$, local existence results available for general shapes (Duchon & Roberts).

Numerical evidence (Almgren, '95) suggests that global existence of smooth solutions not expected for arbitrary initial conditions in displacement of more viscous fluid.

Stability of steady states and global existence for initial conditions close to a steady state is a more achievable goal. Without injection or suction or external pressure gradient, a circular bubble in a quiescent fluid is a steady state. This is known to be asymptotically stable (Constantin & Pugh, '93, Escher & Simonett, '96, ..) for near circular shapes.

Global existence and stability of translating states

The general idea for global existence and stability study for disturbance u superposed on a steady state is fairly simple:

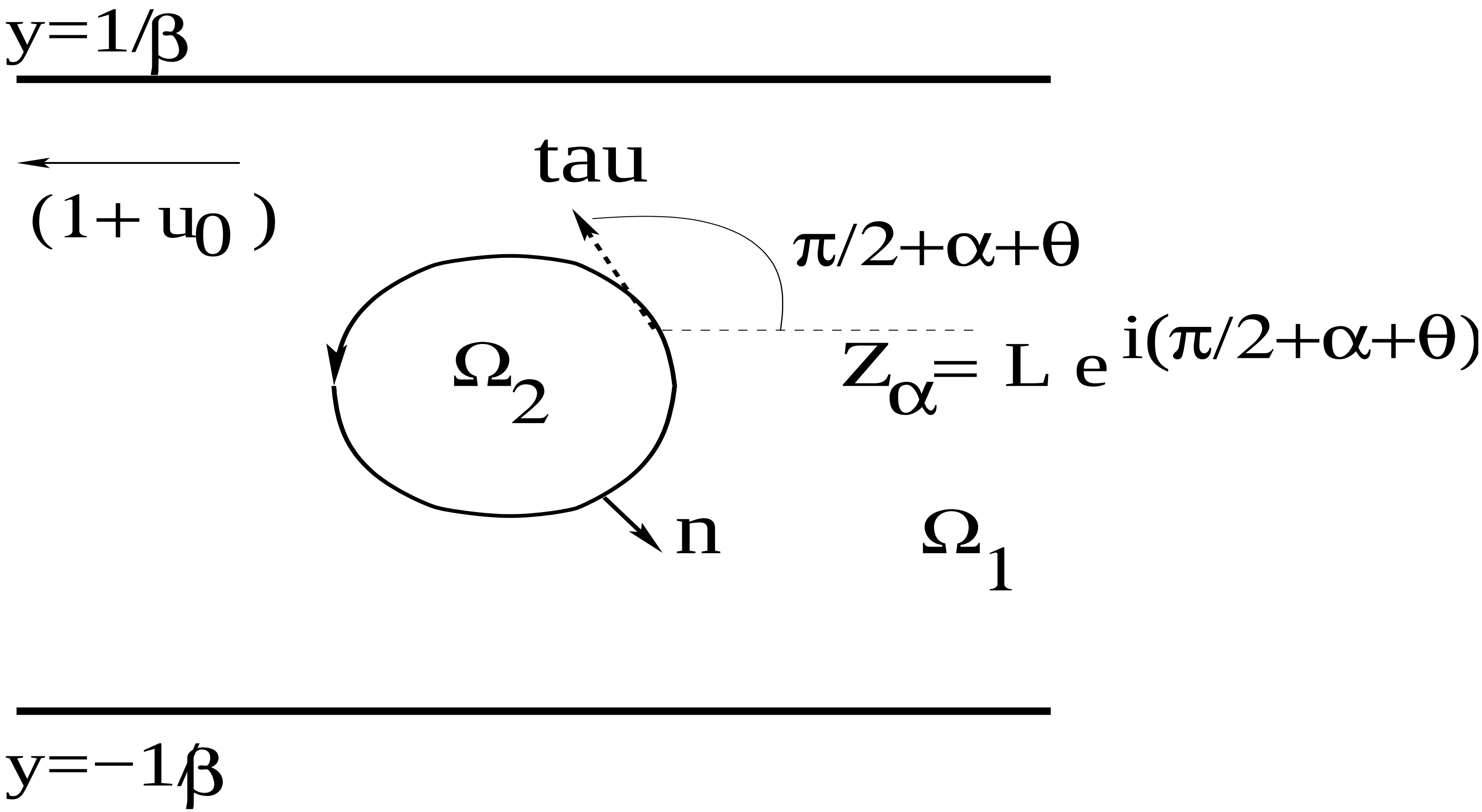
$$u_t - \mathcal{A} u = \mathcal{N}[u], \text{where } \mathcal{A} \text{ is linear time} - \text{independent}$$

$$u = e^{t\mathcal{A}} u_0 + \int_0^t e^{(t-\tau)\mathcal{A}} \mathcal{N}[u](\tau) d\tau$$

If $\|e^{t\mathcal{A}} f\| \leq C e^{-\lambda t} \|f\|$ for some suitable norm $\|.\|$, for $\lambda > 0$, then we can use contraction mapping in the class of functions with space time norm $\|f\|_\lambda = \sup_{t>0} e^{\lambda t} \|f(.,t)\|$ when $\|u_0\|$ is small enough.

Two limits accessible for finding properties of semi-group $e^{t\mathcal{A}}$. One is $0 < \epsilon << 1$, but involves exponential asymptotics (ongoing joint work with X. Xie). The other is a translating bubble, small compared to wall separation.

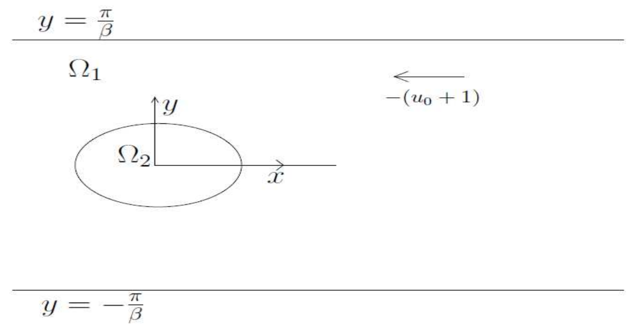
Geometry of the flow



Hele−Shaw Bubble evolution in the frame of the steady bubble.

Global existence results for Hele-Shaw Problem:

Define harmonic ϕ_1, ϕ_2 in Ω_1, $\Omega_2 \subset \mathbb{R}^2$,



$$\phi_1 \sim -(u_0+1)x + O(1), \text{ as } x \to \infty, \ \frac{\partial \phi_1}{\partial y}\left(x, \pm\frac{\pi}{\beta}\right) = 0, \qquad (1)$$

On $\partial\Omega_1 \cap \partial\Omega_2$:

$$(2+u_0)x + \phi_1 - \frac{\mu_2}{\mu_1}\phi_2 = \epsilon\kappa \text{ and } \frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} = v_n \qquad (2)$$

ϵ, n, v_n, $\frac{\mu_2}{\mu_1}$, $2+u_0$ denote surface tension, inwards normal, interface speed, viscosity ratio and steady bubble speed

Boundary Integral Formulation

We seek representation of the velocity of fluid 1 and 2 in the form

$$u_{1,2} - iv_{1,2} = -(u_0 + 1) + \frac{1}{2\pi i}\int_0^{2\pi} \gamma(\alpha')\mathcal{M}(z,\alpha')d\alpha',$$

where $\mathcal{M}(z,\alpha') = \frac{1}{z - Z(\alpha')}$ **and for** $\beta \neq 0$**,**

$$\mathcal{M}(z,\alpha') = \frac{\beta}{4}\coth\left[\frac{\beta}{4}(z - Z(\alpha'))\right] - \frac{\beta}{4}\tanh\left[\frac{\beta}{4}(z - Z^*(\alpha'))\right]$$

As the free-boundary is approached from fluid 1 and fluid 2 respectively,

$$u_{1,2} - iv_{1,2} = -(u_0 + 1) + \frac{1}{2\pi i} PV\int_0^{2\pi} \gamma(\alpha')\kappa(\alpha,\alpha')d\alpha' \pm \frac{\gamma(\alpha)}{2Z_\alpha(\alpha)},$$

where $\kappa(\alpha,\alpha') = \mathcal{M}\left(Z(\alpha),\alpha'\right)$

Boundary Integral Formulation of Hou *et al*

Normal interface speed $U = (u_1, v_1) \cdot \mathrm{n} = (u_2, v_2) \cdot \mathrm{n}$ **is**

$$U = (u_0{+}1)\cos(\alpha + \theta(\alpha) {+} \mathrm{Re}\left(\frac{Z_\alpha}{2\pi s_\alpha} PV \int_0^{2\pi} \kappa(\alpha, \alpha')\gamma(\alpha')d\alpha'\right),$$

Tangent speed as interface is approached for two fluids:

$$\partial_\alpha \phi_{1,2} = \mathrm{Re}\left[Z_\alpha\left(u_{1,2} - iv_{1,2}\right)\right] = (u_0 + 1)s_\alpha \sin(\alpha + \theta(\alpha))$$

$$+ \mathrm{Re}\left(\frac{Z_\alpha}{2\pi i} PV \int_0^{2\pi} \kappa(\alpha, \alpha')\gamma(\alpha')d\alpha'\right) \pm \frac{1}{2}\gamma(\alpha)$$

We use above relation in the α-derivative of interface relation $(2 + u_0)x + \phi_1 - \frac{\mu_2}{\mu_1}\phi_2 = \epsilon\kappa$ to obtain a Fredholm integral equation for γ for given $Z(\alpha, t)$.

Since $v_n = \frac{\partial \phi}{\partial n}$, a boundary point $Z(\alpha, t) = X(\alpha, t) + iY(\alpha, t)$ must have normal speed U, with arbitrary tangent speed T, *i.e.* **$(X_t(\alpha, t), Y_t(\alpha, t)) = U\mathrm{n} + T\tau$.**

Hou *et al* equal arclength choice

Hou *et al* '93 noted that if

$$T(\alpha,t) = \int_0^{\alpha} (1+\theta_\alpha(\alpha',t))d\alpha' - \frac{\alpha}{2\pi}\int_0^{2\pi}(1+\theta_\alpha(\alpha',t))d\alpha',$$

then $|Z_\alpha| \equiv s_\alpha = \frac{L}{2\pi}$ independent of α. In this equal arclength formulation, on α-differentiation of (X_t, Y_t), obtain

$$\theta_t(\alpha,t) = \frac{2\pi}{L}U_\alpha(\alpha,t) + \frac{2\pi}{L}T(\alpha,t)\left(1+\theta_\alpha(\alpha,t)\right),$$

$$L_t = -\int_0^{2\pi}\left(1+\theta_\alpha(\alpha,t)\right)U(\alpha,t)d\alpha$$

For given θ, L, $Z(\alpha,t) = \frac{iL}{2\pi}\int_0^{\alpha}\exp\left[i\alpha' + i\theta(\alpha',t)\right]d\alpha' + Z(0,t)$, where $Z(0,t) = X(0,t) + iY(0,t)$ satisfies $(X_t(0,t), Y_t(0,t)) = U(0,t)\mathrm{n}$.

Function Space and Projection

Definition: For $\theta(\alpha,t) = \sum_{k\in\mathbb{Z}} \hat{\theta}(k;t)e^{ik\alpha}$, *define* $\|.\|_r$ *so that*

$$\|\theta(.,t)\|_r^2 = \sum_{k\in\mathbb{Z}} \left(1 + |k|^{2r}\right) |\hat{\theta}(k,t)|^2.$$

We denote this class of function by H_p^r *Definition: For* $f \in H_p^r$, *define projection* $\mathcal{Q}_n$:

$$\mathcal{Q}_n f = f - \sum_{k=-n}^{n} \hat{f}(k)e^{ik\alpha}.$$

Henceforth $\tilde{\theta} = \mathcal{Q}_1\theta$. *We also denote* $\dot{H}^r = \mathcal{Q}_1 H_p^r$. *Note that in this space* $\|\phi\|_r = \|\partial_\alpha^r \phi\|_0$. *Definition: We define*

$$\omega(\alpha,t) = \int_0^\alpha \exp\left[i\alpha' + i\theta(\alpha,t) + i\frac{\pi}{2}\right] d\alpha'$$

Note that $Z(\alpha,t) = Z(0,t) + \frac{L}{2\pi}\omega(\alpha,t)$.

Integral equation to solve for γ

With $\mathcal{H}$ defined as Hilbert-Transform, interfacial condition:

$$\gamma(\alpha, t) = -a_\mu \left[\mathcal{F}[Z(., t)] \gamma(., t) \right](\alpha) + \frac{L}{\pi} \left(1 + \frac{\mu_2 u_0}{\mu_1 + \mu_2} \right) \sin(\alpha + \theta)$$

$$+ \frac{2\pi}{L} \epsilon \theta_{\alpha\alpha} \qquad (1) \quad , \qquad \text{where } a_\mu = \tfrac{\mu_1 - \mu_2}{\mu_1 + \mu_2}, \quad \text{where}$$

$$\mathcal{F}[Z]\gamma = \mathrm{Re}\left[-i \mathcal{G}[Z]\gamma \right] \ , \ \mathcal{G}[Z]\gamma = 2i Z_\alpha \mathcal{K}[Z]\gamma + Z_\alpha \mathcal{H}\left[\frac{\gamma}{Z_\alpha} \right] - \mathcal{H}[\gamma],$$

$$\mathcal{K}[Z]\gamma = \frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} f(\alpha') \left\{ \kappa(\alpha, \alpha') - \frac{1}{2 Z_\alpha(\alpha')} \cot \frac{1}{2}(\alpha - \alpha') \right\} d\alpha',$$

$$\kappa(\alpha, \alpha') = \frac{\beta}{4} \coth \frac{\beta}{4} \left(Z(\alpha) - Z(\alpha') \right) - \frac{\beta}{4} \tanh \frac{\beta}{4} \left(Z(\alpha) - Z^*(\alpha') \right).$$

Equation (1) is a Fredholm type integral equation of second kind for γ. Solution exists for given Z. Further, $\|\gamma\|_{\dot{H}_r} \leq C \|\theta\|_{\dot{H}_{r+2}}$

Equivalent System of Equations

Theorem: For symmetric initial conditions or $\beta = 0$*, the system*

$$\theta_t(\alpha,t) = \frac{2\pi}{L} U_\alpha(\alpha,t) + \frac{2\pi}{L} T(\alpha,t)\left(1+\theta_\alpha(\alpha,t)\right), \theta(\alpha,0) = \theta_0,$$

$$L_t = -\int_0^{2\pi} \left(1+\theta_\alpha(\alpha,t)\right) U(\alpha,t) d\alpha,\ L(0) = L_0, \textit{ equivalent to}$$

$$\tilde{\theta}_t = \frac{2\pi}{L} U_\alpha + \frac{2\pi}{L} T(\alpha,t)\left(1+\theta_\alpha(\alpha,t)\right), \tilde{\theta}(\alpha,0) = \mathcal{Q}_1\theta_0,$$

$$\frac{d}{dt}\hat{\theta}(0;t) = \frac{1}{L}\int_0^{2\pi} T(\alpha,t)\left(1+\theta_\alpha(\alpha,t)\right) d\alpha,\ \ \hat{\theta}(0,0) = \hat{\theta}_0(0)$$

$$\int_0^{2\pi} \exp\left[i\tilde{\theta}(\alpha,t) + i\hat{\theta}(1,t)e^{i\alpha} - i\hat{\theta}(-1,t)e^{-i\alpha}\right] d\alpha = 0,$$

$$L(t) = \sqrt{8\pi^2 A}\left[\mathrm{Im}\int_0^{2\pi} \omega_\alpha \omega^* d\alpha\right]^{-1/2},\ \ A:\ \text{bubble area}$$

Comment on the latter form of equations

We did not find the original form of equation suitable for global existence proof since it was difficult to get exponentially decaying estimates on the linearized part of the operator, which was used to control nonlinearity. In the latter form, this is easier. For $\beta = 0$ or for symmetric initial condition for $\beta \neq 0$, $\hat{\theta}(0;t)$ or $Z(0,t)$ does not enter into the equation for $\tilde{\theta}(\alpha,t)$, $\hat{\theta}(1,t)$ and $\hat{\theta}(-1,t)$ in any substantial manner; therefore, exponential decay for linearized equation arrived at easily.

$\beta = 0$ results

When $\beta = 0$, $\theta = 0$, $\gamma = 2\sin\left(\alpha + \hat{\theta}(0;t)\right)$, $u_0 = 0$ corresponds to a a circular translating steadily with speed 2 relative to fluid displacement rate at ∞. The time dependence of γ_s merely corresponds to a convenient shift of change of origin of α and has no effect on the steady flow.

Theorem: For $\beta = 0$ and any surface tension $\epsilon > 0$ and $r \geq 3$, there exists $\delta > 0$ such that if $\|\theta_0\|_r < \delta$ and $|L_0 - 2\pi| < \delta < \frac{1}{2}$, then there exists a unique solution $(\theta, L) \in C\left([0, \infty), H_r \times \mathbb{R}\right)$ to the Hele-Shaw problem. Further, $\theta - \hat{\theta}(0;t)$ approaches 0 exponentially, $\hat{\theta}(0;t)$ remains finite, while L approaches $2\sqrt{\pi A}$ exponentially.

Basic ingredient in the proof for $\beta = 0$

Using properties such as the following for $r \geq 3$:

$$U(\alpha, t) = \frac{\pi}{L} \mathcal{H}\left[\gamma\right] + \frac{\pi}{L} \mathrm{Re}\left[\mathcal{G}[Z]\gamma\right] + (1 + u_0) \cos(\alpha + \theta),$$

$$\|\mathcal{G}[Z^{(1)}]f - \mathcal{G}[Z^{(2)}]f\|_r \leq C\|f\|_1\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r e^{C[\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r]},$$

$$\|\gamma - \frac{2\pi\epsilon}{L}\theta_{\alpha\alpha}\|_{r-2} \leq C \exp\left(C\|\tilde{\theta}\|_{r-2}\right)\|\tilde{\theta}\|_{r-1},$$

one obtains evolution equation of $\tilde{\theta}$ in the form

$$\tilde{\theta}_t = \mathcal{A}[\tilde{\theta}] + \mathcal{N}\left[\tilde{\theta}\right],$$

$$\mathcal{A}[\tilde{\theta}] = \sum_{k=2}^{\infty} e^{ik\alpha}\left(-\epsilon d(k)\hat{\theta}(k, t) + m(k)\hat{\theta}(k + 1, t)\right) + c.c.,$$

$$d(k) = \frac{1}{2}k(k^2 - 1), m(k) = (1 + a_\mu)\frac{(k^2 - 1)(k + 1)}{k(k + 2)}.$$

Weighted Sobolev Norm and $e^{t\mathcal{A}}$ semi-group

Definition: Define $K(\epsilon) = 2$ *if* $\epsilon \geq 1$ *and* $K(\epsilon) = \left[\sqrt{1 + \frac{6}{\epsilon}}\right] + 1$ *otherwise.*

For $r \geq 0$*, on* $\dot{H}^r$ *define inner-product and corresponding norm* $\|.\|_{w,r}$*:*

$$(v, u)_{w,r} = \sum_{k \neq 0, \pm 1} w^2(\epsilon, k) |k|^{2r} \hat{v}^*(k) \hat{u}(k)$$

Define for $r \geq 3$*,* $H^r_\epsilon \equiv C\left([0, \infty), \dot{H}^r\right) \cap L^2\left([0, \infty), \dot{H}^{r+3/2}\right)$ *with*

$$\|u\|^2_{H^r_\epsilon} = \sup_{t>0} e^{t\epsilon} \|u(., t)\|^2_{w,r} + \frac{\epsilon}{4} \int_0^\infty e^{\epsilon t} \|u(., t)\|^2_{w,r+3/2} dt$$

$$Lemma : \|e^{t\mathcal{A}} v_0\|_{H^r_\epsilon} \leq \|v_0\|_{w,r} \;,\; \| \int_0^t e^{(t-\tau)\mathcal{A}} f(., \tau) \|_{H^r_\epsilon} \leq C \|f\|_{H^{r-3}_\epsilon}$$

We apply contraction mapping in the space H^r_ϵ *for sufficiently small* $\|\theta_0\|_{w,r}$ *on*

$$\tilde{\theta}(., t) = e^{t\mathcal{A}} \mathcal{Q}_1 \theta_0 + \int_0^t e^{(t-\tau)\mathcal{A}} \mathcal{N}[\tilde{\theta}](\tau) d\tau$$

Results for $0 < \beta << 1$–steady state existence

Theorem (Ye & T., 11): For any $\epsilon > 0$*, and* $r \geq 3$*, there exist for* $\epsilon > 0$ *two balls* $O_1 = \{\beta \in \mathbb{R} : |\beta| < \Upsilon\}$*, and* $O_2 = \left\{(u, v) \in H_p^r \times \mathbb{R} | \|u\|_r < \delta, |v| < \delta\right\}$ *so that for sufficiently small* δ *and* Υ*, there exists* $\left(\theta^{(s)}, u_0\right)^T : O_1 \rightarrow O_2$ *is a unique real valued map determining the shape and velocity of a steady translating bubble for* $\beta \in O_1$*. Furthermore, there exists* C *independent of* δ *and* Υ *such that*

$$\|\theta^{(s)}\|_r + |u_0| + \|\gamma^{(s)} - 2\sin()\|_{r-2} \leq C\beta^2$$

These results were obtained formally in '86 (T.). Small ϵ selection for arbitrary bubble size given through formal asymptotics (Combescot & Dombre, '88 and T.'88), justified by Xie '07

The idea of the proof is to control the linearized part of the operator through a suitable Lax-Milgram formulation.

Global existence and stability results for $0 < \beta << 1$

Define $\theta = \theta^{(s)} + \Theta$, $\gamma = \gamma^{(s)} + \Gamma$. We have the following:

Theorem: For any $\epsilon > 0$ and $r \geq 3$, there exists δ, Υ so that if $\|\Theta(.,0)\|_r < \delta$, $|L_0 - 2\pi| < \delta$ and $\beta < \Upsilon$, with $\Theta(-\alpha, 0) = -\Theta(\alpha, 0)$, then there exists unique global solution $(\theta, L) \in C\left([0,\infty), H_p^r \times \mathbb{R}\right)$. For this solution, $\theta(-\alpha, t) = -\theta(\alpha, t)$. Furthermore, $\|\Theta(., t)\|_r$ decays exponentially as $t \to \infty$, while L approaches $2\sqrt{\pi A}$ exponentially. Thus the translating steady bubble is asymptotically stable for sufficiently small initially symmetric disturbances in the H_p^r space.

The proof is similar to $\beta = 0$. The smallness of β is crucial to the proof since it is based on a contraction argument exploiting the smallness of both initial conditions and β. Without symmetry, $Z(0, t)$ couples with evolution of θ and the problem is more difficult.