WM GROUPS AND RAMSEY THEORY

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Dedicated to Neil Hindman on the occasion of his 65th birthday

0. Introduction.

Our goal in this paper is to exhibit a connection between two seemingly disparate areas: Ramsey theory and the theory of unitary representations of a class of locally compact groups. The class of groups that we are interested in consists of the so-called "minimally periodic groups" introduced by von Neumann in [N]; namely, groups having the property that they do not admit non-trivial almost periodic functions. This, in turn, is equivalent to the property of not having non-trivial finite dimensional unitary representations. One can show that yet another equivalent form of the above condition is that any ergodic measure preserving action of such a group on a finite measure space is actually weakly mixing. It is this aspect that will interest us and so we call these groups WM groups, and the class of these is denoted WM. As we shall see WM groups that are also amenable have unexpected Ramsey-theoretical properties.

Here are some examples of WM groups:

(i) $\mathrm{SL}(2,\mathbb{R})$, or more generally, any simple non-compact Lie group with finite center.

(ii) The group $\mathrm{Alt}(\mathbb{N})$ of even permutations of $\mathbb{N}$, or more generally, any group which is a direct limit of compact simple groups.

Note that the examples in (i) are non-amenable and those in (ii) are amenable.

Amenability of a locally compact group can be defined in two ways. One of these is in terms of an invariant mean, that is, a functional on either bounded continuous functions or bounded Borel measurable functions having the same value on a function as on its translate: $m(f) = m(f_u)$, where $f_u(g) = f(ug)$. An alternative characterization is in terms of Følner sequences. A sequence of compact sets (F_n) in G is called (left) Følner sequence, if for any $g \in G$ one has

$$\frac{|F_n \cap gF_n|}{|F_n|} \to 1 \text{ as } n \to \infty,$$

where vertical lines refer to Haar measure. In an amenable group there is also a notion of

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"invariant density" for Borel sets, $d(E)$, defined either as $m(1_E)$ or by

$$\lim_{n\to\infty} \frac{|E \cap F_n|}{|F_n|}$$

for an appropriately refined Følner sequence. Note that $d(E) = d(gE)$ for each g in G.

We now shift our discussion to Ramsey theory.

Ramsey theory treats the phenomenon that large subsets of appropriately chosen structures have rich combinatorial properties. In the earlier examples, the notion of largeness was to be one of the several subsets into which a structure is divided in an arbitrary partition to finitely many sets. For example, the classical van der Waerden theorem states that for any finite partition of $\mathbb{Z}$, one of the cells of the partition is AP-rich, i.e. contains arbitrarily long arithmetic progressions. Van der Waerden's theorem is a consequence of the deeper result of Szemerédi which says that any set which has positive density with respect to some sequence of intervals $[a_n, b_n]$ where $b_n - a_n \to \infty$ as $n \to \infty$, is AP-rich. It is customary to say that Szemerédi's theorem is a density version of van der Waerden's. (See [GRS].)

On the other hand, there are partition results which do not necessarily generalize to, for example, sets of positive density (and for which the underlying notion of largeness can often be expressed in the language of ultrafilters). For example, Schur's theorem stipulates that for any finite coloring of $\mathbb{N}$, there is a monochromatic triple of the form $\{x, y, x+y\}$, but clearly there is no such triple in the set of $2\mathbb{N}+1$ which has density $1/2$ with respect to any sequence of intervals. This observation equally applies to Hindman's theorem which forms a far reaching generalization of Schur's result.

Hindman's Theorem ([H]). *For any finite partition of an infinite semigroup S, one of the cells of the partition contains a "finite products" set, namely a set $FP(x_i)$ comprised of an infinite sequence (x_i) together with all finite products of the form*

$$x_{i_1} x_{i_2} \cdots x_{i_k},\ i_1 < i_2 < ... < i_k,\ k \in \mathbb{N}.$$

The finite products sets (or finite sums sets, in the case of additive notation) are also called IP sets and play fundamental role in combinatorial applications of ergodic theory and topological dynamics (see for example [F2], [FK], [FW], [B1], [B2], [BM].)

One of our principal observations is that for WM amenable locally compact groups, a sufficient condition for a set E to contain a Schur triple, and even to contain an entire IP set, is that $d^*(E) > 0$, where $d^*(E) := \sup\{d(E),\ d$ a left-invariant density$\}$. Indeed, provisionally defining a set E to be large if $d^*(E) > 0$, we will prove

Theorem 1. *A locally compact amenable group is a WM group if and only if any large set in G contains and IP set.*

One of the approaches to Hindman's theorem involves the topological algebra in Stone-Čech compactifications. In particular, if the group G is countable, and discrete Hindman's theorem is implied by the fact that given an idempotent ultrafilter p in βG, any member of p contains an IP set. It is also known that any IP set in G is a member of some idempotent in βG. If G is a countable discrete abelian group, then the (two sided) ideal of βG which consists of ultrafilters p with the property that every $E \in p$ satisfies $d^*(E) > 0$, is strictly larger than the closure of all the idempotents in the smallest ideal of βG. (See [HS, Exercise 6.1.4 and Theorem 7.28].) The following corollary of Theorem 1 shows that the structure of βG for any amenable WM group G is quite different.

Theorem 2. *If G is a discrete, countable amenable WM group, then any large set in G is a member an idempotent ultrafilter.*

The following result demonstrates yet another peculiarity of WM groups.

Theorem 3. *Let G be a locally compact amenable WM group and $d(\cdot)$ an invariant density. If A and B are two Borel subsets satisfying $d(A) > 0$, $d(B) > 0$, the product set AB has density one with respect to any Følner sequence. If G is, in addition, countable and discrete, then AB is a member of any minimal idempotent in βG.*

1. A Weak Correspondence Principle for Amenable Groups.

For a topological group G we speak of a measure preserving action of G if we have a representation of G by measure preserving transformations $\{T_g\}_{g\in G}$ of a measure space $(\Omega, \mathscr{B}, \mu)$ such that the map $(g, A) \to T_g^{-1}A$ is jointly continuous from $G \times \mathscr{B} \to \mathscr{B}$, where the topology on $\mathscr{B}$ is given by the (pseudo-)metric: $\rho(B_1, B_2) = \mu(B_1 \vartriangle B_2)$. We say the action is *ergodic* if for $B_1, B_2 \in \mathscr{B}$ with $\mu(B_1)$, $\mu(B_2) > 0$ $\exists g \in G$ with $\mu(B_1 \vartriangle g^{-1}B_2) > 0$.

The action is *weakly mixing* if the corresponding action on $(\Omega \times \Omega, \mathscr{B} \times \mathscr{B}, \mu \times \mu)$ is ergodic. In this case the action on $(\Omega \times \Omega, \mathscr{B} \times \mathscr{B}, \mu \times \mu)$ is again weakly mixing. WM groups are characterized by the fact that the two notions coincide.

G is *amenable* if there is a left-invariant mean on Borel functions on G, $m(f)$, i.e., for any $\gamma \in G$, $m(f) = m(L_\gamma f)$, where $L_\gamma f(g) \equiv f(\gamma^{-1} g)$. The invariant mean determines a notion of density for Borel sets, $d(B) = m(1_B)$, with $d(B) = d(\gamma^{-1} B)$ for $\gamma \in G$. The invariant mean for an amenable group is (generally) not unique and we can define the *upper density* $d^*(B)$ for a Borel set B as $\sup\{d(B),\ d \text{ a left-invariant density}\}$. We remark for future reference that for weakly mixing actions of an amenable group, if $\mu(B_1) > 0$ and $\mu(B_2) > 0$, then

$$d^*(\{g\colon \mu(g^{-1}B_1 \cap B_2) = 0\}) = 0$$

(cf. [BR, Theorem 4.1]).

For an amenable locally compact group acting ergodically on a space Ω one has the ergodic theorem, which roughly speaking, implies that if a bounded measurable function on Ω is lifted to the group via the group action on almost any point, this correspondence will identify group "averages" with "space averages". One exact version of this, for an ergodic measure preserving action of G on $(\Omega, \mathscr{B}, \mu)$, is that for any measurable sets $B_1, B_2, \ldots, B_k$ there are sets $S_1, S_2, \ldots, S_k$ in G and an invariant density $d(\cdot)$, so that for $\{g_{ij}\colon 1 \le i \le k,\ 1 \le j \le l_i\} \subset G$ the identity

$$d\left(\bigcap_{i,j} g_{ij}^{-1} S_i\right) = \mu\left(\bigcap_{i,j} T_{g_{ij}}^{-1} B_i\right) \tag{1.1}$$

holds.

In this section we shall show how to invert the direction, going rather from subsets in G to ergodic actions and a family of corresponding subsets of the space, again retrieving the identity (1.1). It is not too difficult to do this for discrete groups and here we can formulate the following (strong) correspondence principle in which $d(\cdot)$ is some invariant density:

Let $S_1, S_2, \ldots, S_k$ be arbitrary subset of G. There exists a measure preserving action of G on a space $(\Omega, \mathscr{B}, \mu)$ and sets $B_1, B_2, \ldots, B_k \in \mathscr{B}$, so that for any $\{g_{ij}\colon 1 \le i \le k,\ 1 \le j \le l_i\}$

$$d\left(\bigcap_{i,j} g_{ij}^{-1} S_i\right) = \mu\left(\bigcap_{i,j} T_{g_{ij}}^{-1} B_i\right).$$

When $G = \mathbb{Z}^n$ this principle has been used to deduce Szemerédi type theorems from multiple recurrence results for $\mathbb{Z}^n$-actions. (See, for example, [F1], [FK], [BL], [BM].)

We shall be interested in an analogous result for non-discrete groups. We shall also be interested in obtaining ergodic actions which for WM groups are automatically weakly mixing. This will give us great flexibility in using the correspondence. The price that will be paid is that we won't achieve equality as in (1.1), but we can still conclude that when the right hand side in (1.1) is positive, the left hand side will not vanish.

We begin with a preliminary result in which we don't achieve ergodicity, but it will clarify the steps we take in going from discrete to non-discrete groups. For this part of the discussion G is a topological amenable group and $d(\cdot)$ is a fixed left invariant density. The following definition is related to this density.

Definition 1.1. A Borel subset S is *substantial* if $S \supset UW$ for some non-empty open set U and a Borel set W with $d(W) > 0$.

Remark. This notion appears (albeit not under this name) in [FKW].

For a discrete amenable group, the condition for substantiality reduces to $d(S) > 0$. Note that the open set in the definition can be chosen to be a neighborhood of the identity.

We can now formulate a weak correspondence principle. We remark that from the proof of this weakened form one can easily deduce the stronger version for the case of discrete countable group. See also a counterexample in [BBB, Section 4], which shows that the "strong" correspondence principle does not, in general, hold for locally compact groups.

Theorem 1.1. *Let S_1, S_2,…,S_k be substantial sets in an amenable, locally compact group G. There exists a measure preserving action of G on a space $(\Omega, \mathscr{B}, \mu)$, sets B_1, B_2,…, B_k of positive measure in $\mathscr{B}$ and positive constants c_1, c_2, …, c_k so that for any $\{g_{ij} \colon 1 \le i \le k,\ 1 \le j \le l_i\}$ in G*

$$d\left(\bigcap_{i,j} g_{ij}^{-1} S_i\right) \ge \left(\prod_i c_i^{l_i}\right) \mu\left(\bigcap_{i,j} T_{g_{ij}}^{-1} B_i\right). \tag{1.2}$$

Moreover, when the expression on the right is positive, the set $\bigcap g_{ij}^{-1} S_i$ appearing on the left is again substantial.

In the proof of this theorem the space Ω will be the compact space turning up in the Gelfand representation of a particular algebra of functions on G. The group G will act on this algebra by automorphisms and these induce homeomorphisms of Ω which will give us the representation $g \mapsto T_g$. We use the following notion of "left-uniformly continuous" functions:

Definition 1.2. A function $f(g)$ on G is *LUC* if for any $\varepsilon > 0$ there exists a neighborhood V of the identity so that

$$|f(vg) - f(g)| < \varepsilon$$

for all $v \in V$ and $g \in G$.

We denote by $\mathcal{LUC}$ the algebra of all bounded complex-valued functions on G. With the sup norm $\mathcal{LUC}$ is a commutative C^*-algebra with involution. It is not hard to check that L_γ, defined by $L_\gamma(f)(g) = f(\gamma^{-1}g)$, takes $\mathcal{LUC}$ to itself defining an automorphism.

For G locally compact we can use Haar measure to define convolution of functions in $L^1(G)$ and $L^\infty(G)$ respectively:

$$\psi * f(g) = \int \psi(h) f(h^{-1}g) dh.$$

The convolution of two such functions is always LUC. It is natural to interpret $\psi * f$ as a limit of finite linear combinations of translates of f. Taking into account that m is left invariant, we arrive at the following convenient formula.

$$m(\psi * f) = \int \psi(h) dh \cdot m(f). \tag{1.3}$$

We now consider the Gelfand representation $\mathcal{LUC} \xrightarrow{\sim} C(\Omega)$. Denote by $\tilde{f}$ the function on Ω associated to a function $f \in \mathcal{LUC}$. The automorphisms $L_{\gamma^{-1}}$ induce automorphisms of $C(\Omega)$; these take maximal ideals to maximal ideals, thereby defining maps $T_\gamma : \Omega \to \Omega$ satisfying $\tilde{f}(T_\gamma \omega) = (L_{\gamma^{-1}} f)^{\sim}(\omega)$.

Finally, we can transfer the invariant mean $m(f)$ from $\mathcal{LUC}$ to $C(\Omega)$, and by the usual properties we find that there exists a measure μ on Ω such that

$$m(f) = \int_\Omega \tilde{f}(\omega) d\mu(\omega).$$

Since we have

$$\int \tilde{f}(T_\gamma \omega) d\mu(\omega) = \int (L_{\gamma^{-1}} f)^\sim(\omega) d\mu(\omega) = m(L_{\gamma^{-1}} f) = m(f) = \int \tilde{f}(\omega) d\mu(\omega)$$

and since every function in $C(\Omega)$ is the image of a function in $\mathcal{LUC}$, the foregoing identity implies that μ is invariant under each T_γ.

To complete the construction of a measure preserving action of G we still need to check the continuity of the map $(g, B) \to T_g^{-1} B$. More generally, we shall have that $(g, f) \mapsto f \circ g^{-1}$ is jointly continuous where $f \in L^1(\Omega, \mathscr{B}, \mu)$. It is convenient to denote the operator $f \mapsto f \circ g^{-1}$ by $T_g f$.

T_g is an isometry on L^1 and it can be seen that the continuity in question will follow from continuity restricted to the dense subset $\widetilde{\mathcal{LUC}} \subset L^1$, and here it follows from the definition of $\mathcal{LUC}$.

Turn now to the substantial sets $S_1, S_2, \ldots, S_k$ in the statement of theorem 1.1. We have for each i, $S_i \supset U_i W_i$ with $d(W_i) > 0$ and U_i a non-empty open set. Let $\psi_i \geq 0$ be continuous with support in U_i and

$$0 < \int \psi_i(g) dg \leq 1.$$

By definition of the convolution we see that $\psi_i * 1_{W_i}(g) \neq 0$ only for $g \in U_i W_i \subset S_i$ so that

$$1_{S_i} \geq \psi_i * 1_{W_i} \tag{1.4}$$

We will use (1.4) to prove our theorem. We begin by defining for each i the set B_i and the constant $c_i > 0$. Namely, since $\psi_i * 1_{W_i} \in \mathcal{LUC}$, the function $\varphi_i = (\psi_i * 1_{W_i})^\sim$ is defined on Ω with $\varphi_i \geq 0$ and

$$\int \varphi_i d\mu = m(\psi_i * 1_{W_i}) = \left(\int \psi_i(h) dh \right) d(W_i) > 0.$$

Thus $\varphi_i(\omega) > 0$ for some set of positive μ-measure and we can write

$$\varphi_i(\omega) > c_i 1_{B_i}(\omega)$$

for all ω and for appropriate c_i, B_i, where $\mu(B_i) > 0$.

We turn to (1.2) replacing sets by functions:

$$d\left(\bigcap_{i,j} g_{ij}^{-1}S_i\right) = m\left(\prod_{i,j} L_{g_{ij}^{-1}}1_{S_i}\right) \geq m\left(\prod_{i,j} L_{g_{ij}^{-1}}\psi_i * 1_{W_i}\right).$$

The latter can be evaluated as an integral over Ω:

$$\int \prod_{i,j}(L_{g_{ij}^{-1}}\psi_i * 1_{W_i})^{\sim}(\omega)d\mu(\omega) = \int \prod_{i,j}(\psi_i * 1_{W_i})^{\sim}(T_{g_{ij}}(\omega))d\mu(\omega)$$
$$\geq \prod_i c_i^{l_i} \int \prod_{i,j} 1_{B_i}(T_{g_{ij}}(\omega))d\mu(\omega) = \prod_i c_i^{l_i}\mu\left(\bigcap T_{g_{ij}^{-1}}B_i\right).$$

This proves the first part of the theorem. To prove the second part we make the following observation. If $S \supset UW$, where U is non-empty open neighborhood of identity, we can find non-empty open neighborhoods of identity U',U'' with $U'U'' \subset U$ so that $S \supset U'(U''W)$, therefore a substantial set always contains a "thickening" of a smaller substantial set.

Turning to S_1, S_2, ..., S_k in our theorem we can suppose $S_i \supset U_i'S_i'$. We can also suppose without loss of generality that the sets U_i' are neighborhoods of identity. The first part of the theorem is valid for the sets S_i' for an appropriate measure preserving action, sets B_i, and constants c_i. We now use the fact that

$$\bigcap_{i,j} g_{ij}^{-1}U_i'S_i' \supset \left(\bigcap_{i,j} g_{ij}^{-1}U_i'g_{ij}\right)\cdot\left(\bigcap_{ij} g_{ij}^{-1}S_i'\right) = U''W''$$

we conclude that when $\bigcap_{i,j} g_{ij}^{-1}S_i'$ has positive density, then $\bigcap_{i,j} g_{ij}^{-1}S_i$ is substantial. □

In the foregoing theorem, there is no reason that the action on $(\Omega, \mathscr{B}, \mu)$ be ergodic. To achieve ergodicity we need another condition which need not hold for arbitrary substantial sets. For, suppose we have a correspondence $S_i \leftrightarrow B_i$ with $\mu(B_i) > 0$. If the action is ergodic then there exist group elements γ_i with $\mu\left(\bigcap_i T_{\gamma_i}^{-1}B_i\right) > 0$. This should imply $d\left(\bigcap_i \gamma_i^{-1}S_i\right) > 0$. So we need to assume that a condition of this type is given. We make this precise in the following:

Definition 1.3. A family of sets $\{S_i\}_{1\leq i\leq k}$ in G is *coalescent* if $\exists$ $\{\gamma_i\}_{1\leq i\leq k}$ with $d^*\left(\bigcap_i \gamma_i^{-1}S_i\right) > 0$.

Note that we are not fixing a particular invariant mean here.

Definition 1.4. A family of Borel sets $\{S_i\}_{1\le i\le k}$ in G is *coherent* if for each i, $S_i \supset U_iW_i$, where U_i is a non-empty open set, and the family $\{W_i\}_{1\le i\le k}$ is coalescent.

Remark. The sets of a coherent family are necessarily substantial for some invariant density $d(\cdot)$.

For discrete groups the two notions of the foregoing definitions coincide. Note also that the open sets $\{U_i\}$ in Definition 1.4 can be assumed to be neighborhoods of the identity.

The main result in this section is the following:

Theorem 1.2. *Let $\{S_1, S_2, \ldots, S_k\}$ be a coherent family of Borel sets in G. There is an ergodic action of G on a space $(\Omega, \mathscr{B}, \mu)$ and sets $B_1, B_2, \ldots, B_k \in \mathscr{B}$ with $\mu(B_i) > 0$, so that for any $\{g_{ij} \colon 1 \le i \le k,\ 1 \le j \le l_i\}$ in G, if*

$$\mu\left(\bigcap_{i,j} T_{g_{ij}}^{-1} B_i\right) > 0 \tag{1.5}$$

then

$$d^*\left(\bigcap_{i,j} g_{ij}^{-1} S_i\right) > 0. \tag{1.6}$$

Proof. We write $S_i \supset U_iW_i$, $1 \le i \le k$, and having assumed that the U_i are neighborhoods of the identity and setting $U = \bigcap_i U_i$ we have $S_i \supset UW_i$, where furthermore $\{W_i\}$ forms a coalescent family. We can find an invariant density and elements $\gamma_i \in G$ with $d\left(\bigcap_i \gamma_i^{-1} W_i\right) > 0$. We follow the construction in the proof of theorem 1.1, using the Gelfand representation of $\mathcal{LUC}$ to $C(\Omega)$ to obtain an action of G on $(\Omega, \mathscr{B}, \mu)$ with $m(f) = \int \tilde{f} d\mu$ for all functions $f \in \mathcal{LUC}$. If the resulting action on $(\Omega, \mathscr{B}, \mu)$ is ergodic we are through. Otherwise consider the ergodic decomposition

$$\mu = \int_\Theta \mu_\theta \, d\sigma(\theta)$$

where the measures μ_θ are ergodic (i.e., they are invariant under $\{T_g\}_{g\in G}$, and the actions are ergodic). We shall show that for some θ the ergodic action on $(\Omega, \mathscr{B}, \mu_\theta)$ provides the desired correspondence.

As in the proof of theorem 1.1, let $\psi \in L^1(G)$ with $\psi(g) \ge 0$ and having support in $\bigcap_i \gamma_i U \gamma_i^{-1}$. Moreover, we suppose that $\int \psi(g) dg = 1$.

Convolution with ψ is an averaging process and so we can write

$$\prod_i \psi * \left(L_{\gamma_i^{-1}} 1_{W_i}\right) = \prod_i \psi * \left(L_{\gamma_i^{-1}} 1_{W_i}\right)^k \geq \psi * \prod_i \left(L_{\gamma_i^{-1}} 1_{W_i}\right) = \psi * 1_{\bigcap \gamma_i^{-1} W_i} \tag{1.7}$$

Convolution with ψ preserves mean and so

$$m\left(\prod_i \psi * \left(L_{\gamma_i^{-1}} 1_{W_i}\right)\right) > 0$$

and by the correspondence $\mathcal{LUC} \cong C(\Omega)$ we have

$$\int \prod_i (\psi * (L_{\gamma_i^{-1}} 1_{W_i}))^{\sim} d\mu > 0. \tag{1.8}$$

Since $\mu = \int \mu_\theta \, d\sigma$ we can find θ with

$$\int \prod_i (\psi * (L_{\gamma_i^{-1}} 1_{W_i}))^{\sim} d\mu_\theta > 0. \tag{1.9}$$

Write $\mu' = \mu_\theta$. We can find a set $B \in \mathscr{B}$ with $\mu'(B) > 0$, and a positive constant c, so that

$$\prod_i (\psi * (L_{\gamma_i^{-1}} 1_{W_i}))^{\sim} \geq c 1_B. \tag{1.10}$$

on Ω. We set $B_i = T_{\gamma_i} B$.

Note that (1.10) implies that for each i

$$(\psi * (L_{\gamma_i^{-1}} 1_{W_i}))^{\sim} \geq c 1_B$$

and so

$$T_{\gamma_i} \left(\psi * (L_{\gamma_i^{-1}} 1_{W_i})\right)^{\sim} \geq c 1_{B_i}$$

or

$$\left(L_{\gamma_i} \left(\psi * L_{\gamma_i^{-1}} 1_{W_i}\right)\right)^{\sim} \geq c 1_{B_i}$$

We can write

$$(\psi_i * 1_{W_i})^{\sim} \geq c 1_{B_i}, \tag{1.11}$$

where $\psi_i(g) = \psi(\gamma_i^{-1} g \gamma_i)$. By our assumption on ψ, the function $\psi_i * 1_{W_i} \leq 1_{S_i}$ and we shall use this to relate the two expressions in our theorem. To do this we define a new invariant mean on the bounded Borel functions on G. Namely, for $f \in \mathcal{LUC}$ we define

$$m'(f) = \int \tilde{f} d\mu'$$

and we extend m' from $\mathcal{LUC}$ to all bounded Borel functions using the Hahn-Banach extension theorem. Corresponding to this mean m' there is an invariant density d', and by the foregoing inequality , for any $\{g_{ij}\}$

$$d'\left(\bigcap_{i,j} g_{ij}^{-1}S_i\right) = m'\left(\prod_{i,j} L_{g_{ij}}^{-1}1_{S_i}\right) \geq m'\left(\prod_{i,j} L_{g_{ij}}^{-1}\psi_i * 1_{W_i}\right) = \int \prod_{i,j} T_{g_{ij}}^{-1}\left(\psi_i * 1_{W_i}\right)^{\sim} d\mu'$$
$$\geq c^{\sum l_i}\mu'\left(\bigcap T_{g_{ij}}^{-1}B_i\right).$$

Since d^* for a set in G is the sup of d for all invariant densities the foregoing inequality is valid with d^* instead of d'. Thus the action of G on $(\Omega, \mathscr{B}, \mu')$ is the ergodic action that we seek. □

We take as a final definition of largeness for arbitrary amenable locally compact groups: $S \supset UW$, where U is non-empty open set, and $d^*(W) > 0$. Then it is easily seen, imitating the argument in the proof of Theorem 1.1, that in Theorem 1.2, when (1.5) is positive, the set appearing in (1.6) is large.

2. Variations on a theme of Hindman.

Given a sequence $(x_i) \subset G$ we use the notation $x_\alpha := \prod_{i\in\alpha} x_i$, where $\alpha = \{i_1, \ldots, i_k\}$ is a finite non-empty subset in $\mathbb{N}$ and the product is taken in the order of the increasing indices (that is, we assume that $i_1 < i_2 < \cdots < i_k$ and $\prod_{i\in\alpha} x_i = x_{i_1}x_{i_2}\cdots x_{i_k}$).

We call x_α *even*, if $|\alpha|$ is even and *odd* if $|\alpha|$ is odd.

Theorem 2.1. *Let G be a locally compact WM group and let S_0, $S_1 \subset G$ be a coherent pair of sets. Then there exists a sequence $(g_i)_{i\in\mathbb{N}}$ such that $g_\alpha \in S_0$ for all even g_α and $g_\alpha \in S_1$ for all odd g_α.*

Before giving the proof, we need the following lemma.

Lemma 2.2. *Let S_1, S_2, S_3 be large sets in a locally compact amenable WM group G and assume that S_1 and S_2 are coherent. There exists $g \in S_3$ such that, simultaneously, $S_1 \cap g^{-1}S_2$ and $S_2 \cap g^{-1}S_1$ are large.*

Proof. Let $(X, \mathscr{B}, \mu, (T_g)_{g\in G})$ be the weakly mixing measure preserving system "generated" by S_1, S_2 and let C_1, C_2 be the corresponding images of S_1, S_2 in $\mathscr{B}$. Utilizing the fact that the product system $(X \times X, \mathscr{B} \otimes \mathscr{B}, \mu \times \mu, (T_g \times T_g)_{g \in G})$ is also weakly mixing, we can find $g \in S_3$ such that

$$(\mu \times \mu)\left((C_1 \times C_2) \bigcap (T_g \times T_g)(C_2 \times C_1)\right) > 0.$$

This implies $\mu(C_1 \cap T_g C_2) > 0$ and $\mu(C_2 \cap T_g C_1) > 0$, which, by the correspondence principle, implies that the sets $S_1 \cap g^{-1} S_2$ and $S_2 \cap g^{-1} S_1$ are large. □

Proof of Theorem 2.1. The proof is based on repeated application of the above lemma. Start with picking $g_1 \in S_0$ so that the sets $S_0 \cap g_1^{-1} S_1$ and $S_1 \cap g_1^{-1} S_0$ are large.

Now pick $g_2 \in S_0 \cap g_1^{-1} S_1$ (note that $g_2 \in S_0$ and $g_1 g_2 \in S_1$) so that the sets

$$(S_0 \cap g_1^{-1} S_1) \bigcap g_2^{-1}(S_1 \cap g_1^{-1} S_0)$$

and

$$(S_1 \cap g_1^{-1} S_0) \bigcap g_2^{-1}(S_0 \cap g_1^{-1} S_1)$$

are large.

At the next stage we pick $g_3 \in (S_0 \cap g_1^{-1} S_1) \bigcap g_2^{-1}(S_1 \cap g_1^{-1} S_0)$ so that the sets

$$\left[(S_0 \cap g_1^{-1} S_1) \bigcap g_2^{-1}(S_1 \cap g_1^{-1} S_0)\right] \bigcap g_3^{-1} \left[(S_1 \cap g_1^{-1} S_0) \bigcap g_2^{-1}(S_0 \cap g_1^{-1} S_1)\right]$$

and

$$\left[(S_1 \cap g_1^{-1} S_0) \bigcap g_2^{-1}(S_0 \cap g_1^{-1} S_1)\right] \bigcap g_3^{-1} \left[(S_0 \cap g_1^{-1} S_1) \bigcap g_2^{-1}(S_1 \cap g_1^{-1} S_0)\right]$$

are large. Note that our choice of g_3 implies that all g_α with $|\alpha| \leq 3$ satisfy the assertion of the theorem. Continuing this process, we arrive at the sequence $(g_i)_{i \in \mathbb{N}}$ with the desired properties. □

Remarks. (i) Taking $S_0 = S_1$ we obtain Theorem 1 formulated in the Introduction.

(ii) For a discrete countable amenable group G, existence of coherent families of sets $\{S_1, \ldots, S_k\}$ in G follows from the correspondence, implicit in equation (1.1), between sets $B_1, \ldots, B_k$ in Ω for an ergodic measure preserving action of G on $(\Omega, \mathscr{B}, \mu)$ and subsets $S_1, \ldots, S_k$ in G. In particular, a natural example of a pair of coherent sets in a discrete countable amenable group is given by $\{S, S^c\}$, where both S and S^c are sets of positive density.

(iii) Theorem 2.1 is easily extended to a coherent family $\{S_0, S_1, \ldots, S_{k-1}\}$, where in the conclusion of the theorem the parity of $|\alpha|$ is replaced by $|\alpha| \mod k$.

Theorem 2.3. *Let G be a locally compact amenable WM group. Let $k \in \mathbb{N}$ and assume that for each $\alpha \subset \{1, 2, \ldots, k\}$ we are given coherent large sets $S_\alpha \subset G$. Then there exist $g_1, \ldots, g_k \in G$ such that $g_\alpha \in S_\alpha$ for any $\alpha \subset \{1, 2, \ldots, k\}$.*

Proof. We shall show how to prove the result for $k = 3$, the general case is entirely similar.

Pick $g_1 \in S_1$ so that the sets $S_2 \cap g_1^{-1}S_{12}$, $S_3 \cap g_1^{-1}S_{13}$ and $S_{23} \cap g_1^{-1}S_{123}$ are large (the fact that we can choose such g_1 follows from an obvious modification of the above Lemma 2.2. This remark also applies to the choices of g_2 and g_3 made below.)

Now pick $g_2 \in S_2 \cap g_1^{-1}S_{12}$ so that $(S_3 \cap g_1^{-1}S_{13}) \cap g_2^{-1}(S_{23} \cap g_1^{-1}S_{123})$ is large. Note that $g_2 \in S_2$, $g_1g_2 \in S_{12}$.

Finally, pick $g_3 \in (S_3 \cap g_1^{-1}S_{13}) \cap g_2^{-1}(S_{23} \cap g_1^{-1}S_{123})$. Then $g_3 \in S_3$, $g_1g_3 \in S_{13}$, $g_2g_3 \in S_{23}$, $g_1g_2g_3 \in S_{123}$ and we are done. □

Remark. For a similar phenomenon in the framework of the so called quasirandom finite groups see [G, Theorem 5.2].

We conclude this section with an observation that the fact that large sets always contain infinite IP sets actually characterizes the locally compact amenable groups with WM property.

Theorem 2.4. *A locally compact amenable group G is a WM group if and only if any large set in G contains an IP set.*

Proof. In light of Theorem 1 from the Introduction (which, in turn, is a corollary of Theorem 2.1 in this section) we need only to prove one direction. We shall presently see that already a weaker result, namely the fact that any large set contains a triple $\{x, y, xy\}$ implies the WM property.

Indeed, assume that G is not a WM group. Then there exists a nontrivial unitary representation $(U_g)_{g\in G}$ on a finite-dimensional space V.

Pick a non-zero element $f \in V$ and consider the orbit closure $K = \overline{\{U_g f,\ g \in G\}}$. Clearly, K is a compact subset of V. Let $\varepsilon > 0$ and let $\{f_1, f_2, \ldots, f_k\}$ be an ε-separating set in K. (This means that $\|f_i - f_j\| \geq \varepsilon$ for $i \neq j$ and that for any $\varphi \in K$ there exists $i \in \{1, \ldots, k\}$ such that $\|\varphi - f_i\| \leq \varepsilon$). Note now that for any $\varepsilon_1 > 0$ and any $i, j \in \{1, \ldots, k\}$ the set $S = \{g \in G\colon \|U_g f_i - f_j\| < \varepsilon_1\}$ is large.

But if $i \neq j$ and ε_1 is small enough, the set S cannot contain a triple $\{x, y, xy\}$. □

3. Some unexpected properties of the Stone-Čech compactification of a countable WM group.

In this section we restrict to countable WM groups and make a connection between large sets in G (in the sense defined in Section 1) and properties of idempotents in βG, the Stone-Čech compactification of G. We start with a very brief review of some basic definitions. For more information see [B2] and [HS].

The elements of βG are *ultrafilters*, namely families of sets in G which are maximal with respect to the finite intersection property. It is convenient to think of an ultrafilter p on G as a $\{0,1\}$-valued, finitely additive probability measure on the power set of G. If $A \subset G$ has p-measure 1, we write $A \in p$ and say that A is p-large.

The group operation on G extends naturally to βG by the rule

$$A \in p \cdot q \iff \{g \in G \colon Ag^{-1} \in p\} \in q.$$

For $A \subset G$, let $\bar{A} = \{p \in \beta G \colon A \in p\}$. One can check that the family $\mathcal{A} = \{\bar{A} \colon A \subset G\}$ is a basis for a topology on βG and that, under this topology and under the operation introduced above, βG becomes a compact Hausdorff left topological semigroup. (The last condition means that for any fixed $q \in \beta G$, the map $p \to q \cdot p$ is continuous.)

By a theorem of Ellis ([E]), any compact left topological semigroup has an idempotent. One can show that βG has 2^c idempotents and that an ultrafilter p belongs to the closure of the set of idempotents if and only if every p-large set contains an IP set. Moreover, one can show that any IP set is p-large for some idempotent $p \in \beta G$. See [BH, Lemma 5.11].

Let now $A \subset G$ be a large set. Since, as we have seen in the previous section, A contains an IP set, we have the following fact.

Theorem 3.1. *If $A \subset G$ is large then A is p-large for some idempotent $p \in \beta G$.*

Remark. As was mentioned in the Introduction, this is special for WM groups. One can, for example, show that if G is a countable discrete abelian group then for any $\varepsilon > 0$ there exists a set $A \subset G$ which has density larger than $1 - \varepsilon$ and yet contains no shift of an IP set. (This fact, in the framework of $(\mathbb{N},+)$, was first established by E. Strauss, see [BBHS, Theorem 2.20].)

To formulate our next result we need to introduce a few more notions. A *right ideal* (respectively, *left ideal*) in βG is a set $J \subset \beta G$ such that for every $q \in \beta G$ and every $p \in J$, $p \cdot q \in J$ (respectively, $q \cdot p \in J$.) An *ideal* is a set $I \subset \beta G$ which is both a left and right ideal. A *minimal right ideal* is a nonempty right ideal J, containing no proper nonempty set which is itself a right ideal.

Let $\mathcal{K}$ be the union of minimal right ideals in βG. Then one can show that $\mathcal{K}$ is a two-sided ideal and, in fact, the smallest two-sided ideal. It contains (plenty of) idempotents and any idempotent $p \in \mathcal{K}$ is called minimal.

The significance of minimal idempotents in Ramsey theory stems from the fact that sets which are members of minimal idempotents (these sets are called *central* sets) have very rich combinatorial properties. For example, central sets in $(\mathbb{Z},+)$ not only contain IP sets but also contain arbitrarily long arithmetic progressions. See [F2]. (The notion of central sets in $\mathbb{Z}$ is defined in [F2] in terms of topological dynamics; the fact that a set in $\mathbb{Z}$ is central if and only if it is a member of a minimal idempotent in $\beta\mathbb{Z}$ is established in [BH].)

It is not hard to show that if a set $A \subset G$ has density 1 with respect to some invariant mean, then A is p-large for some minimal idempotent $p \in \beta G$. Moreover, if a set $B \subset G$ has the property that it has density 1 with respect to any invariant mean (for example, for any large set $A \subset G$, the set $B = \{g \colon A \cap g^{-1}A \text{ is large}\}$ has this "universal" property), then B is p-large for any minimal idempotent p. Note now that if A is large then

$$A^{-1}A = \{g \in G \colon A \cap g^{-1}A \neq \varnothing\} \supset \{g \in G \colon A \cap g^{-1}A \text{ is large}\}.$$

The above remarks can now be summarized in the following statement.

Theorem 3.2. *Let G be a countable WM group. If $A \subset G$ is large, then $A^{-1}A$ is p-large for any minimal idempotent $p \in \beta G$.*

A set $T \subset G$ is called *thick* if for any finite set F there exists $x \in G$ such that $Fx \subset T$. It is not hard to show that any thick set is central, i.e. is a member of a minimal idempotent. [reference] We can apply this fact to WM groups as follows. Let $A, B \subset G$ be two large, not necessarily coherent sets. One can show (see [J], [BFW], [BBF]) that the product set AB is not just large, but is piecewise syndetic, i.e. is an intersection of a syndetic set with a thick set. Moreover, the set AB is actually a piecewise Bohr set (see [BFW] and [BBF] for the details). The relevant corollary of this fact for the situation at hand is that if G is a WM group then AB has actually to be thick. This implies the following statement.

Theorem 3.3. *If G is a countable WM group and A, B are (not necessarily coherent) large sets in G, then $AB = \{xy\colon x \in A,\ y \in B\}$ is a central set.*

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