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QUOTIENT SETS AND DENSITY RECURRENT SETS

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Dedicated to Dona Strauss on the occasion of her $75^{\rm th}$ *birthday.*

Abstract. Let S be a left amenable semigroup. Say that a subset A of S is large if there is some left invariant mean μ on S with $\mu(\chi_A) > 0$. A subset B of S is *density recurrent* if and only if, whenever A is a large subset of S, there is some $x \in B$ such that $x^{-1}A \cap A$ is large. We show that the set $\mathcal{DR}(S)$ of ultrafilters on S, every member of which is density recurrent, is a compact subsemigroup of the Stone-Čech compactification βS of S containing the idempotents of βS. If S is a group, we show that for every nonprincipal ultrafilter p on S, $p^{-1}p \in \mathcal{DR}(S)$, where $p^{-1} = \{A^{-1} : A \in p\}$. We obtain combinatorial characterizations of sets which are members of a product of k idempotents and of sets which are members of a product of k elements of the form $p^{-1}p$ for each $k \in \mathbb{N}$. We show that $\mathcal{DR}(\mathbb{N},+)$ has substantial multiplicative structure. We show further that if A is a large subset of S, then $\mathcal{DR}(S) \subseteq \overline{AA^{-1}}$, where the quotient set $AA^{-1} = \{x \in S : (\exists y \in A)(xy \in A)\}$. For each positive integer n, we introduce the notion of a *polynomial n-recurrent set* in $\mathbb{N}$. (Such sets provide a generalization of the polynomial Szemerédi Theorem.) We show that the ultrafilters, every member of which is a polynomial n-recurrent set, are a subsemigroup of $(\beta\mathbb{N},+)$ containing the additive idempotents and a left ideal of $(\beta\mathbb{N},\cdot)$.

1. Introduction

Let A be a subset of the set $\mathbb{N}$ of positive integers. The *upper asymptotic density* of A is defined by

$$\overline{d}(A) = \lim\sup_{n\to\infty} |A \cap \{1, 2, \ldots, n\}|/n\,,$$

and the *upper Banach density* of A is defined by

$$d^*(A) = \lim\sup_{n\to\infty} \frac{|A \cap I_n|}{|I_n|}$$

where the supremum is taken over all sequences of intervals $\langle I_n\rangle_{n=1}^\infty$ with length approaching infinity. More formally,

$$d^*(A) = \sup\{\alpha : (\forall n \in \mathbb{N})(\exists m \geq n)(\exists a \in \mathbb{N})(|A\cap\{a+1, a+2, \ldots, a+m\}| \geq \alpha\cdot m)\}\,.$$

It has been known for some time that if either $\overline{d}(A) > 0$ or $d^*(A) > 0$, then the difference set $D(A) = \{x - y : x, y \in A \text{ and } x > y\}$ has substantial algebraic structure. In fact, for such results about $D(A)$ it doesn't matter whether one assumes that $\overline{d}(A) > 0$ or $d^*(A) > 0$. The reason is, if $d^*(A) > 0$, then there exists

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$B \subseteq \mathbb{N}$ such that $\overline{d}(B) > 0$ and $D(B) \subseteq D(A)$. And of course always $d^*(A) \geq \overline{d}(A)$. For example, it is shown in [4, Theorem 2.6], using some results from ergodic theory, that given any function $f : \mathbb{N} \longrightarrow \mathbb{N}$, there must exist a sequence $\langle x_n \rangle_{n=1}^{\infty}$ so that $\big\{ \sum_{n\in F} a_n \cdot x_n : F$ is a finite nonempty subset of $\mathbb{N}$ and for each $n \in F$, $a_n \in \{1, 2, \ldots, f(n)\}\big\} \cup \big\{ \prod_{n\in F} x_n^{a_n} : F$ is a finite nonempty subset of $\mathbb{N}$ and for each $n \in F$, $a_n \in \{1, 2, \ldots, f(n)\}\big\} \subseteq D(A)$.

We shall be concerned in this paper with *quotient sets* of large subsets of left amenable semigroups. Given such a semigroup $(S, \cdot)$ and $A \subseteq S$, we define $AA^{-1} = \{x \in S : (\exists z \in A)(xz \in A)\}$. (If the operation is denoted by $+$ this becomes $A - A = \{x \in S : (\exists z \in A)(x + z \in A)\}$.) The related quotient set

$$A^{-1}A = \{x \in S : (\exists z \in A)(zx \in A)\}$$

would arise if we were dealing with right amenable semigroups. If $A \subseteq \mathbb{N}$, then one has $A - A = D(A)$. We only occasionally assume that our semigroups are commutative or countable.

We present results about quotient sets and the algebraic structure of the Stone-Čech compactification βS of S in Section 3. For example, it is a consequence of Theorem 3.15 that if $A \subseteq \mathbb{N}$, $d^*(A) > 0$, $k \in \mathbb{N}$, $p_1, p_2, \ldots, p_k$ are idempotents in $\beta\mathbb{N}$, and $q_1, q_2, \ldots, q_l$ are any points in $\beta\mathbb{N} \setminus \mathbb{N}$, then $A - A \in p_1 + p_2 + \ldots + p_k$, $A - A \in (-q_1 + q_1) + (-q_2 + q_2) + \ldots + (-q_l + q_l)$, as well as any other sum of the p_i's and $(-q_j + q_j)$'s in any order.

In Section 4 we characterize precisely those subsets of S which are members of a product of a fixed number of idempotents. For example, a subset A of S is a member of the product of two idempotents if and only if there exist sequences $\langle x_{1,t} \rangle_{t=1}^{\infty}$ and $\langle x_{2,t} \rangle_{t=1}^{\infty}$ in S such that all products of the form $\prod_{t\in F} x_{1,t} \prod_{t\in H} x_{2,t}$ are in A where F and H are finite nonempty subsets of $\mathbb{N}$ and $\max F < \min H$. We also obtain combinatorial descriptions of those sets which are members of all products of the form $p_1 p_2 \cdots p_n$ where each p_i is an idempotent. We obtain the unsurprising result that the strength of the assertion that A is a member of a product of n idempotents decreases as n increases.

In Section 5, in the event S is a group or $(\mathbb{N}, +)$, we characterize precisely those subsets which are members of a product of a fixed number of elements of the form $p^{-1}p$.

In Section 6 we restrict our attention to $\mathbb{N}$. We obtain the surprising result that in $(\mathbb{N}, +)$, the assertion that A is a member of a sum of n terms of the form $-p + p$ for $p \in \beta\mathbb{N} \setminus \mathbb{N}$ has no relationship whatever to the corresponding statement about k terms if $k \neq n$. We characterize there sets which are members of certain "polynomials" (such as $2p + qp$) whose terms are additive idempotents.

In this Section 6 we introduce the *polynomial n-recurrent sets*. A set $B \subseteq \mathbb{N}$ is a polynomial n-recurrent set if and only if whenever $A \subseteq \mathbb{N}$ and $d^*(A) > 0$ and $g_1, g_2, \ldots, g_n$ are polynomials with rational coefficients taking integers to integers and 0 to 0, there exists $k \in B$ such that $d^*\big(A \cap \bigcap_{t=1}^{n} (-g_t(k) + A)\big) > 0$. For example if $g_t(x) = tx$ and $d^*(A) > 0$, then the definition tells us that there will exist length $n + 1$ arithmetic progressions in A with increment taken from any polynomial n-recurrent set. We show that the set of all ultrafilters, all of whose members are polynomial n-recurrent sets is a subsemigroup of $(\beta\mathbb{N}, +)$. By [8, Theorem 7.3] it contains the idempotents. We show that it is a left ideal of $(\beta\mathbb{N}, \cdot)$, and is closed under subtraction from the left.

During the course of the paper we introduce several classes of subsets of S as well as several classes of subsets of βS. In a final section we summarize the results about these classes as well as relationships among these classes.

2. Preliminaries

Given a semigroup S, let $l_\infty(S)$ be the Banach space of bounded real valued functions on S with the supremum norm. A *mean* on S is a member μ of the dual space $l_\infty(S)^*$ such that $||\mu|| = 1$ and $\mu(g) \geq 0$ whenever $g \in l_\infty(S)$ and for all $s \in S$, $g(s) \geq 0$. A *left invariant mean* on S is a mean μ such that for all $s \in S$ and all $g \in l_\infty(S)$, $\mu(s \cdot g) = \mu(g)$, where $s \cdot g = g \circ \lambda_s$ and $\lambda_s : S \to S$ is defined by $\lambda_s(t) = st$. A semigroup S is *left amenable* if and only if there exists a left invariant mean on S. In any left amenable semigroup, there is a natural notion of density for subsets of S.

Definition 2.1. Let S be a left amenable semigroup and let $A \subseteq S$. Then

$$d(A) = \sup\{\mu(\chi_A) : \mu \text{ is a left invariant mean on } S\}.$$

For an arbitrary set X, let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. In [10] Følner established that any amenable group satisfies the *Følner Condition*.

$$(FC) \qquad \big(\forall F \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(\forall s \in F\big)\big(|sK \setminus K| < \epsilon \cdot |K|\big)$$

In [11] Frey showed that any left amenable semigroup satisfies the Følner condition. (For a simplified proof see [19, Theorem 3.5].) Later, Argabright and Wilde [1] showed that a left cancellative semigroup is left amenable if and only if it satisfies the *Strong Følner Condition*.

$$(SFC) \qquad \big(\forall F \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(\forall s \in F\big)\big(|K \setminus sK| < \epsilon \cdot |K|\big)$$

Notice that for any finite $K \subseteq S$ and any $s \in S$,

$$|K \setminus sK| + |K \cap sK| = |K| \geq |sK| = |sK \setminus K| + |K \cap sK|$$

so $|K \setminus sK| \geq |sK \setminus K|$ and equality holds if s is left cancelable.

Argabright and Wilde also showed [1] that any semigroup satisfying SFC is left amenable and that any commutative semigroup satisfies SFC. In particular, any commutative semigroup is left amenable. (See [17, Section 7] for a simple elementary proof that any commutative semigroup satisfies SFC.)

If the left amenable semigroup S is left cancellative, the Strong Følner Condition provides a method of calculation of density on S. We will use this theorem in the proof of Theorem 3.22.

Theorem 2.2. *Let S be a left amenable left cancellative semigroup. For $A \subseteq S$,*

$$d(A) = \sup\{\alpha \in [0,1] : \big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big) \\ \big((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|) \text{ and } |A \cap K| \geq \alpha \cdot |K|\big)\}.$$

Proof. [16, Theorems 2.12 and 2.14]. □

Using Theorem 2.2 one easily shows that for the semigroup $(\mathbb{N}, +)$, and any $A \subseteq \mathbb{N}$, $d(A) = d^*(A)$. (See [17, Theorem 1.9].)

Given $A \subseteq S$ and $x \in S$, $x^{-1}A = \{y \in S : xy \in A\}$. (There is no requirement that S have an identity, nor, even if S does have an identity, that x have an inverse.) We shall need the following simple fact.

Theorem 2.3. *Let S be a left amenable semigroup. Let $A \subseteq S$ and let $x \in S$. Then $d(x^{-1}A) = d(A)$. If S is left cancellative, then also $d(xA) = d(A)$.*

Proof. Let μ be a left invariant mean on S. Then $\mu(\chi_{x^{-1}A}) = \mu(\chi_A \circ \lambda_x) = \mu(\chi_A)$. If S is left cancellative, then $x^{-1}xA = A$, so $d(A) = d(x^{-1}xA) = d(xA)$. □

We take the Stone-Čech compactification βS of the discrete semigroup S to be the set of ultrafilters on S. (An ultrafilter is a maximal filter. Alternatively, an ultrafilter p on S may be identified with a $\{0,1\}$-valued finitely additive measure μ on $\mathcal{P}(S)$. The statement "$\mu(A) = 1$" then corresponds to the statement "$A \in p$".)

Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS. We identify the principal ultrafilters with the points of S and thus pretend that $S \subseteq \beta S$. The operation $\cdot$ extends to βS making $(\beta S, \cdot)$ a right topological semigroup (meaning that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for each $x \in S$ the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). As is true of any compact Hausdorff right topological semigroup, βS has idempotents [9, Lemma 1]. If $p, q \in \beta S$ and $A \subseteq S$, one has that $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$. We let $S^* = \beta S \setminus S$.

See [15] for an elementary introduction to the algebraic structure of βS.

3. Quotient sets and density recurrent sets

We begin by introducing the main object of study for this section. See [7] for more information about density intersective sets, sets of density recurrence, and their relation to other sets of recurrence.

Definition 3.1. Let S be a left amenable semigroup.

(a) Let $B \subseteq S$. Then B is *density intersective* if and only if whenever $A \subseteq S$ and $d(A) > 0$, there exists $x \in B$ such that $x^{-1}A \cap A \neq \emptyset$.
(b) Let $B \subseteq S$. Then B is a *density recurrent* set if and only if whenever $A \subseteq S$ and $d(A) > 0$, there exists $x \in B$ such that $d(x^{-1}A \cap A) > 0$.
(c) $\mathcal{DI}(S) = \{p \in \beta S : (\forall B \in p)(B \text{ is density intersective})\}$.
(d) $\mathcal{DR}(S) = \{p \in \beta S : (\forall B \in p)(B \text{ is a density recurrent set})\}$.

We shall show in Theorem 3.14 below that if S is left cancellative, then $\mathcal{DR}(S)$ is a subsemigroup of βS. (And thus, by Corollary 3.4, if S is countable, then $\mathcal{DI}(S)$ is a subsemigroup of βS.) For that, we will need to show that $\mathcal{DR}(S) \neq \emptyset$. The easiest way to do that is to show that $\mathcal{DR}(S)$ contains the idempotents of βS.

We do not know in general whether every density intersective set is a set of density recurrence. However for countable left amenable semigroups the notions coincide, as we shall verify in Theorem 3.3. The proof involves the notion of a *set of measurable recurrence* and is essentially contained in [7]. We present the details for the convenience of the reader.

Definition 3.2. Let S be a semigroup and let $B \subseteq S$. Then B is a *set of measurable recurrence* if and only if for every probability space $(X, \mathcal{B}, \mu)$, every measure preserving action $\langle T_g \rangle_{g \in S}$ of S on X, and every $A \in \mathcal{B}$ such that $\mu(A) > 0$, there exists $g \in B$ such that $\mu(A \cap T_g^{-1}[A]) > 0$. (The family $\langle T_g \rangle_{g \in S}$ is a measure preserving action on $(X, \mathcal{B}, \mu)$ provided that (1) each $T_g : X \to X$, (2) whenever $g \in S$ and $A \in \mathcal{B}$ one has $\mu(T_g^{-1}[A]) = \mu(A)$, and (3) whenever $g, h \in S$, one has $T_g \circ T_h = T_{gh}$.)

Theorem 3.3. *Let S be a countable left amenable semigroup and let $B \subseteq S$. The following statements are equivalent.*

(a) *B is a density recurrent set.*
(b) *B is density intersective.*
(c) *B is a set of measurable recurrence.*

Proof. That (a) implies (b) is trivial.

That (b) implies (c) is an immediate consequence of [7, Theorem 2.2].

To see that (c) implies (a), assume that B is a set of measurable recurrence and let $A \subseteq S$ such that $d(A) > 0$. Pick a left invariant mean μ on S such that $\mu(\chi_A) > 0$. Pick by [7, Theorem 2.1] a probability space $(X, \mathcal{B}, \nu)$, a measure preserving action $\langle T_g \rangle_{g \in S}$ of S on X, and $U \in \mathcal{B}$ such that for all $g, h \in S$, $\nu(T_g^{-1}[U] \cap T_h^{-1}[U]) = \mu(\chi_{g^{-1}A \cap h^{-1}A})$. Taking $g = h$ we have that $\nu(U) = \nu(T_g^{-1}[U]) = \mu(\chi_{g^{-1}A}) = \mu(\chi_A) > 0$ so pick $g \in B$ such that $\nu(U \cap T_g^{-1}[U]) > 0$. Pick any $x \in S$ and let $C = x^{-1}A \cap (gx)^{-1}A$.

Then $T_x^{-1}\big[U \cap T_g^{-1}[U]\big] = T_x^{-1}[U] \cap T_{gx}^{-1}[U]$ so

$$0 < \nu(U \cap T_g^{-1}[U]) = \nu(T_x^{-1}[U] \cap T_{gx}^{-1}[U]) = \mu(\chi_C) \leq d(C)\,.$$

Thus $0 < d(C) = d\big(x^{-1}(A \cap g^{-1}A)\big) = d(A \cap g^{-1}A)$. □

Corollary 3.4. *If S is a countable left amenable semigroup, then $\mathcal{DR}(S) = \mathcal{DI}(S)$.*

Definition 3.5. Let S be a semigroup.

(a) If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in S, then

$$FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\textstyle\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}\,,$$

where the products are taken in increasing order of indices.

(b) If $m \in \mathbb{N}$ and $\langle x_n \rangle_{n=1}^{m}$ is a finite sequence in S, then

$$FP(\langle x_n \rangle_{n=1}^{m}) = \big\{\textstyle\prod_{n \in F} x_n : \emptyset \neq F \subseteq \{1, 2, \ldots, m\}\big\}$$

where the products are taken in increasing order of indices and

$$FP_D(\langle x_n \rangle_{n=1}^{m}) = \big\{\textstyle\prod^{\mathrm{d}}_{n \in F} x_n : \emptyset \neq F \subseteq \{1, 2, \ldots, m\}\big\}$$

where in $\prod^{\mathrm{d}}_{n \in F} x_n$, the products are taken in decreasing order of indices.

(c) $\Gamma(S) = \{p \in \beta S : (\forall A \in p)(\exists \langle x_n \rangle_{n=1}^{\infty})(FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A)\}$.
(d) $\Gamma_{<\omega}(S) = \{p \in \beta S : (\forall A \in p)(\forall m \in \mathbb{N})(\exists \langle x_n \rangle_{n=1}^{m})(FP(\langle x_n \rangle_{n=1}^{m}) \subseteq A)\}$.

If the operation in S is denoted by $+$, we write $FS(\langle x_n \rangle_{n=1}^{\infty})$ instead of writing $FP(\langle x_n \rangle_{n=1}^{\infty})$.

It is trivial that $\Gamma(S) \subseteq \Gamma_{<\omega}(S)$. If S contains a sequence with distinct finite products, then the inclusion is proper. (The sequence $\langle x_n \rangle_{n=1}^{\infty}$ has distinct finite products provided that whenever F and H are distinct members of $\mathcal{P}_f(\mathbb{N})$, one has $\prod_{t \in F} x_t \neq \prod_{t \in H} x_t$. By [15, Lemma 6.31] any cancellative semigroup contains a sequence with distinct finite products.) To verify this assertion, let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence with distinct finite products and let $A = \bigcup_{n=1}^{\infty} FP(\langle x_t \rangle_{t=2^{n-1}}^{2^n - 1})$. Then there is no sequence $\langle y_n \rangle_{n=1}^{\infty}$ with $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A$. See the proof of Theorem 3.9 for the details of why this fact suffices.

It is an easy exercise to see that, if S is commutative, then $\Gamma_{<\omega}(S)$ is a subsemigroup of βS. (Let $p, q \in \Gamma_{<\omega}(S)$. To see that $pq \in \Gamma_{<\omega}(S)$, let $A \in pq$ and let $m \in \mathbb{N}$. Since $\{x \in S : x^{-1}A \in q\} \in p$, pick $\langle x_n \rangle_{n=1}^{m}$ such that $FP(\langle x_n \rangle_{n=1}^{m}) \subseteq$

$\{x \in S : x^{-1}A \in q\}$. Let $B = \bigcap\{y^{-1}A : y \in FP(\langle x_n\rangle_{n=1}^m)\}$. Then $B \in q$ so pick $\langle y_n\rangle_{n=1}^m$ such that $FP(\langle y_n\rangle_{n=1}^m) \subseteq B$. Then $FP(\langle x_n y_n\rangle_{n=1}^m) \subseteq A$.)

On the other hand, by [15, Exercise 6.1.4] there exist idempotents p and q in $(\beta\mathbb{N}, +)$ such that $q + p \notin \Gamma(\mathbb{N}, +)$. In particular, neither the set of idempotents in $(\beta\mathbb{N}, +)$ nor $\Gamma(\mathbb{N}, +)$ is a semigroup. (The proof outlined in [15, Exercise 6.1.4] uses the algebraic structure of $(\beta\mathbb{N}, +)$, and establishes a stronger fact. If one wants a more elementary proof that neither the set of idempotents in $(\beta\mathbb{N}, +)$ nor $\Gamma(\mathbb{N}, +)$ is a semigroup, take idempotents $p \in \bigcap_{m=1}^\infty \overline{FS(\langle 2^{2n}\rangle_{n=m}^\infty)}$ and $q \in \bigcap_{m=1}^\infty \overline{FS(\langle 2^{2n+1}\rangle_{n=m}^\infty)}$, which exist by [15, Lemma 5.11]. Show that

$$\{\textstyle\sum_{n\in F} 2^{2n} + \sum_{n\in G} 2^{2n+1} : F, G \in \mathcal{P}_f(\mathbb{N}) \text{ and } \max F < \min G\} \in p + q$$

but this set does not contain $FS(\langle x_n\rangle_{n=1}^\infty)$ for any sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$.)

It is an immediate consequence of [15, Theorem 5.12] that

$$\Gamma(S) = c\ell\{p \in \beta S : pp = p\}\,.$$

Lemma 3.6. *Let S be a left amenable semigroup and let $A \subseteq S$. If $d(A) > \frac{1}{n}$ and $\langle x_t\rangle_{t=1}^n$ is a sequence in S, then there exist $i \neq j$ in $\{1, 2, \ldots, n\}$ such that $d(x_i^{-1}A \cap x_j^{-1}A) > 0$. If S is left cancellative, then there exist $i \neq j$ in $\{1, 2, \ldots, n\}$ such that $d(x_iA \cap x_jA) > 0$.*

Proof. Given $x \in S$, $d(x^{-1}A) = d(A)$ and, if S is left cancellative, then $d(xA) = d(A)$. Also, left invariant means are finitely additive. $\square$

Lemma 3.7. *Let S be a left amenable semigroup, let $A \subseteq S$, let $n \in \mathbb{N}$, assume that $d(A) > \frac{1}{n}$, and let $\langle x_t\rangle_{t=1}^n$ be a sequence in S. There exists $y \in FP_D(\langle x_t\rangle_{t=1}^n)$ such that $d(y^{-1}A \cap A) > 0$. If S is left cancellative, then there exists $y \in FP(\langle x_t\rangle_{t=1}^n)$ such that $d(y^{-1}A \cap A) > 0$.*

Proof. For $i \in \{1, 2, \ldots, n\}$, let $z_i = \prod_{t=1}^i x_t$. By Lemma 3.6 pick $i < j$ in $\{1, 2, \ldots, n\}$ such that $d(z_i^{-1}A \cap z_j^{-1}A) > 0$. Let $y = \prod_{t=i+1}^j$. Then $z_i^{-1}A \cap z_j^{-1}A = z_i^{-1}(A \cap y^{-1}A)$ so $d(A \cap y^{-1}A) > 0$.

Now assume that S is left cancellative. For $i \in \{1, 2, \ldots, n\}$, let $z_i = \prod_{t=1}^i x_t$. By Lemma 3.6 pick $i < j$ in $\{1, 2, \ldots, n\}$ such that $d(z_iA \cap z_jA) > 0$. Let $y = \prod_{t=i+1}^j x_t$. Then $y^{-1}A \cap A = z_j^{-1}(z_iA \cap z_jA)$ and, by Theorem 2.3, $d\big(z_j^{-1}(z_iA \cap z_jA)\big) = d(z_iA \cap z_jA) > 0$. $\square$

As an immediate consequence of Lemma 3.7 we have the following.

Lemma 3.8. *Let S be a left cancellative, left amenable semigroup. Then $\Gamma_{<\omega}(S) \subseteq \mathcal{DR}(S)$.*

Proof. Let $p \in \Gamma_{<\omega}(S)$ and let $B \in p$. To see that B is density recurrent, let $A \subseteq S$ such that $d(A) > 0$. Pick $n \in \mathbb{N}$ such that $d(A) > \frac{1}{n}$. Pick $\langle x_t\rangle_{t=1}^n$ such that $FP(\langle x_t\rangle_{t=1}^n) \subseteq B$ and pick $y \in FP(\langle x_t\rangle_{t=1}^n)$ such that $d(y^{-1}A \cap A) > 0$. $\square$

We pause to observe that the inclusion in Lemma 3.8 can be proper.

Theorem 3.9. $\Gamma_{<\omega}(\mathbb{N}, +) \subsetneq \mathcal{DR}(\mathbb{N}, +)$.

Proof. Let $A = \{n^3 : n \in \mathbb{N}\}$. By [12, Theorem 3.16], A is a set of measurable recurrence. By [7, Theorem 2.7], sets of measurable recurrence are partition regular (meaning that whenever the finite union of sets is a set of measurable recurrence, one of them must be a set of measurable recurrence). Thus by Theorem 3.3 and [15, Theorem 5.7] there exists $p \in \overline{A} \cap \mathcal{DR}(\mathbb{N},+)$. By a special case of Fermat's Last Theorem, which has been known for a long time, A does not contain any $\{x_1, x_2, x_1 + x_2\}$. $\square$

If S is a group and $p \in S^*$, then $p^{-1} = \{A^{-1} : A \in p\}$ is also in S^*, where $A^{-1} = \{a^{-1} : a \in A\}$. Note however, that in this case by [15, Theorem 4.36], S^* is an ideal of βS so $p^{-1}p$ is not the identity of S. If $p \in \mathbb{N}^*$, then since $\mathbb{N} \subseteq \mathbb{Z}$, $-p = \{-A : A \in p\} \in \mathbb{Z}^*$. Also, by [15, Exercise 4.3.5], $\mathbb{N}^*$ is a left ideal of $(\beta\mathbb{Z}, +)$ so if $p \in \mathbb{N}^*$, then $-p + p \in \mathbb{N}^*$. This fact does not carry over to arbitrary semigroups that are embeddable in a group. In fact, if S is a subsemigroup of G, then S^* is a left ideal of βG if and only if for every $x \in G$, $\{y \in S : xy \notin S\}$ is finite. In particular, consider the commutative cancellative countable semigroup $(\mathbb{Q}_d^+, +)$ of positive rationals with the discrete topology. If $p \in \beta\mathbb{Q}_d^+$ and

$$\{\mathbb{Q} \cap (1, 1 + \epsilon) : \epsilon > 0\} \subseteq p\,,$$

then $-p + p \notin (\mathbb{Q}_d^+)^*$ because $\{x \in \mathbb{Q} : x < 0\} \in -p + p$.

Of course, when we say something like "assume that S is a group or $(\mathbb{N}, +)$ and let $p \in S^*$", any reference to $p^{-1}p$ should be interpreted as $-p + p$ if $S = (\mathbb{N}, +)$.

In the following lemma, the computation of $p^{-1}p$ is done in βG. It may or may not be the case that $p^{-1}p \in \beta S$.

Lemma 3.10. *Let S be a subsemigroup of a group G, let $\langle x_n \rangle_{n=1}^\infty$ be an injective sequence in S, let $p \in S^*$ such that $\{x_n : n \in \mathbb{N}\} \in p$, and let $a \in \mathbb{N}$. Then*

$$\{x_k^{-1}x_n : k, n \in \mathbb{N} \text{ and } a < k < n\} \in p^{-1}p\,.$$

Proof. Let $A = \{x_k^{-1}x_n : k, n \in \mathbb{N} \text{ and } a < k < n\}$. Then

$$\{x_k^{-1} : k \in \mathbb{N} \text{ and } a < k\} \subseteq \{y \in S : y^{-1}A \in p\}$$

so $A \in p^{-1}p$. $\square$

All of our results about $p^{-1}p$ deal with S as either a group or $(\mathbb{N}, +)$. We are not concerned with pp^{-1} because, on the one hand, if S is a group, then $pp^{-1} = (p^{-1})^{-1}p^{-1}$, so is already included. If $S = (\mathbb{N}, +)$, then by [15, Exercise 4.3.5], $p + (-p) \notin \mathbb{N}^*$.

Lemma 3.11. *Let S be a subsemigroup of a group G, let $p \in S^*$, let $A \subseteq S$, and let $\langle B_t \rangle_{t=1}^\infty$ be a sequence of members of p. If $A \in p^{-1}p$, then there exists an injective sequence $\langle x_t \rangle_{t=1}^\infty$ in S such that for each t, $x_t \in \bigcap_{j=1}^t B_j$, and*

$$\{x_k^{-1}x_n : k, n \in \mathbb{N} \text{ and } k < n\} \subseteq A\,.$$

Proof. Let $C = \{x \in S : x^{-1}A \in p\}$, and let $D = \{x \in S : xA \in p\}$. Since $A \in p^{-1}p$, we have that $C \in p^{-1}$ and so $D \in p$. Pick $x_1 \in D \cap B_1$ and inductively, given $n > 1$ and having chosen $\langle x_k \rangle_{k=1}^{n-1}$ with each $x_k \in D$, pick

$$x_n \in D \cap \textstyle\bigcap_{k=1}^{n-1} x_k A \cap \bigcap_{k=1}^{n} B_k \setminus \{x_1, x_2, \ldots, x_{n-1}\}\,.$$

(Since $D \cap \bigcap_{k=1}^{n-1} x_k A \cap \bigcap_{k=1}^{n} B_k \in p$, it is infinite.) $\square$

Lemma 3.12. (1) *Let S be a group and let $A \subseteq S$. There exists $p \in S^*$ such that $A \in p^{-1}p$ if and only if there exists an injective sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $\{x_k^{-1}x_n : k, n \in \mathbb{N} \text{ and } k < n\} \subseteq A$.*

(2) *Let $A \subseteq \mathbb{N}$. There exists $p \in \mathbb{N}^*$ such that $A \in (-p+p)$ if and only if there exists an increasing sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $\{x_n - x_k : k, n \in \mathbb{N}$ and $k < n\} \subseteq A$.*

Proof. (1). Necessity. Pick $p \in S^*$ such that $A \in p^{-1}p$ an apply Lemma 3.11.

Sufficiency. Pick $p \in S^*$ such that $\{x_n : n \in \mathbb{N}\} \in p$ and apply Lemma 3.10.

(2). Necessity. We have that $p \in \mathbb{Z}^*$ so by (1) there is an injective sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{Z}$ such that $\{x_n - x_k : k, n \in \mathbb{N}$ and $k < n\} \subseteq A$. Since $A \subseteq \mathbb{N}$, the sequence $\langle x_n \rangle_{n=1}^\infty$ is increasing and so must be eventually in $\mathbb{N}$.

Sufficiency. By (1) pick $p \in \mathbb{Z}^*$ such that $A \in (-p+p)$. Then $p \in \mathbb{N}^*$ or $p \in -\mathbb{N}^*$. Since $A \in (-p+p)$ we have that $(-p+p) \in \mathbb{N}^*$ and therefore by [15, Exercise 4.3.5], $p \in \mathbb{N}^*$. □

Lemma 3.13. *Let S be an amenable group or $(\mathbb{N}, +)$ and let $p \in S^*$. Then $p^{-1}p \in \mathcal{DR}(S)$.*

Proof. Let $B \in p^{-1}p$ and pick by Lemma 3.12 an injective sequence $\langle x_n \rangle_{n=1}^\infty$ such that $\{x_k^{-1}x_n : k, n \in \mathbb{N}$ and $k < n\} \subseteq B$. Let $A \subseteq S$ such that $d(A) > 0$ and pick $n \in \mathbb{N}$ such that $d(A) > \frac{1}{n}$. Pick by Lemma 3.6 $i < j$ in $\{1, 2, \ldots, n\}$ such that $d(x_iA \cap x_jA) > 0$. Now $x_j^{-1}(x_iA \cap x_jA) = A \cap x_j^{-1}x_iA$, $d\big(x_j^{-1}(x_iA \cap x_jA)\big) = d(x_iA \cap x_jA) > 0$, and $x_j^{-1}x_iA = (x_i^{-1}x_j)^{-1}A$. □

Theorem 3.14. *Let S be a left cancellative left amenable semigroup. Then $\mathcal{DR}(S)$ is a subsemigroup of βS containing the idempotents of βS. If S is a group or $(\mathbb{N}, +)$, then $\mathcal{DR}(S)$ contains all elements of the form $p^{-1}p$ for $p \in S^*$ as well as all elements of the form $q^{-1}p$ for $q, p \in \mathcal{DR}(S)$.*

Proof. By Lemma 3.8 we have $\mathcal{DR}(S)$ contains the idempotents of βS, and in particular is nonempty. Let $p, q \in \mathcal{DR}(S)$. To see that $pq \in \mathcal{DR}(S)$, let $B \in pq$. To see that B is density recurrent, let $A \subseteq S$ with $d(A) > 0$. Let

$$C = \{x \in S : x^{-1}B \in q\}\,.$$

Then $C \in p$ so pick $x \in C$ such that $d(x^{-1}A \cap A) > 0$ and let $D = x^{-1}A \cap A$. Since $x^{-1}B \in q$, pick $y \in x^{-1}B$ such that $d(y^{-1}D \cap D) > 0$. Then $xy \in B$ so it suffices to show that $y^{-1}D \cap D \subseteq (xy)^{-1}A \cap A$. To this end, let $z \in y^{-1}D \cap D$. Then $z \in D \subseteq A$ and $z \in y^{-1}D \subseteq y^{-1}(x^{-1}A)$ so $xyz \in A$ and therefore $z \in (xy)^{-1}A \cap A$.

Now assume that S is a group or $(\mathbb{N}, +)$. The first part of the assertion is precisely Lemma 3.13. Now assume that $q, p \in \mathcal{DR}(S)$ and let $B \in q^{-1}p$. Let $A \subseteq S$ with $d(A) > 0$. Let $C = \{x \in S : xB \in p\}$. Then $C \in q$ so pick $x \in C$ such that $d(x^{-1}A \cap A) > 0$. Now $A \cap xA = x(x^{-1}A \cap A)$ so by Theorem 2.3, $d(A \cap xA) > 0$. Let $D = A \cap xA$. Since $xB \in p$, pick $y \in xB$ such that $d(y^{-1}D \cap D) > 0$. Then $x^{-1}y \in B$ and $y^{-1}D \cap D \subseteq (x^{-1}y)^{-1}A \cap A$. □

We would like to have Theorem 3.14 without the assumption that S is left cancellative. Products in decreasing order are produced by Lemma 3.7 without the left cancellative assumption. Such products are associated with βS when it is taken to be left topological, rather than right topological as we have done here. But if we made that choice, then in the proof above we would need $d(Ax^{-1} \cap A) > 0$ and $d(Dy^{-1} \cap D) > 0$.

Theorem 3.15. *Let S be a left amenable semigroup and let $A \subseteq S$ such that $d(A) > 0$. Then $\mathcal{DI}(S) \subseteq \overline{AA^{-1}}$. In particular $\mathcal{DR}(S) \subseteq \overline{AA^{-1}}$.*

Proof. Let $p \in \mathcal{DI}(S)$, suppose that $AA^{-1} \notin p$, and let $B = S \setminus AA^{-1}$. Then $B \in p$ so pick $x \in B$ such that $x^{-1}A \cap A \neq \emptyset$. Pick $z \in x^{-1}A \cap A$. Then $x \in AA^{-1}$, a contradiction. □

We have then immediately the following Ramsey Theoretic corollary. Given $F, H \in \mathcal{P}_f(\mathbb{N})$ we write $F < H$ to mean $\max F < \min H$. Recall that we take products in increasing order of indices.

Corollary 3.16. *Let S be a left amenable and left cancellative semigroup and let $A \subseteq S$ such that $d(A) > 0$. Let $m \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, m\}$, let $\langle x_{i,t}\rangle_{t=1}^{\infty}$ be a sequence in S. Then*

$$\{\textstyle\prod_{i=1}^{m} \prod_{t \in F_i} x_{i,t} : \textit{each } F_i \in \mathcal{P}_f(\mathbb{N}) \textit{ and } F_1 < F_2 < \ldots < F_m\} \cap AA^{-1} \neq \emptyset\,.$$

Proof. For each $i \in \{1, 2, \ldots, m\}$, pick by [15, Lemma 5.11] an idempotent

$$p_i \in \textstyle\bigcap_{n=1}^{\infty} \overline{FS(\langle x_{i,t}\rangle_{t=n}^{\infty})}\,.$$

By Theorems 3.14 and 3.15 we have that $p_1 p_2 \cdots p_m \in \mathcal{DR}(S) \subseteq \overline{AA^{-1}}$. It suffices to show that if

$$B_m = \{\textstyle\prod_{i=1}^{m} \prod_{t \in F_i} x_{i,t} : \text{each } F_i \in \mathcal{P}_f(\mathbb{N}) \text{ and } F_1 < F_2 < \ldots < F_m\}\,,$$

then $B_m \in p_1 p_2 \cdots p_m$. We do this by induction on m.

For $m = 1$, we have that $B_m = FP(\langle x_{1,t}\rangle_{t=1}^{\infty}) \in p_1$. So let $m \in \mathbb{N}$ and assume that $B_m \in p_1 p_2 \cdots p_m$. We claim that $B_m \subseteq \{x \in S : x^{-1}B_{m+1} \in p_{m+1}\}$ so that $B_{m+1} \in p_1 p_2 \cdots p_{m+1}$. So let $x \in B_m$ and pick $F_1 < F_2 < \ldots < F_m$ such that $x = \prod_{i=1}^{m} \prod_{t \in F_i} x_{i,t}$. Let $r = \max F_m$. Then $FP(\langle x_{m+1,t}\rangle_{t=r+1}^{\infty}) \in p_{m+1}$ and $FP(\langle x_{m+1,t}\rangle_{t=r+1}^{\infty}) \subseteq x^{-1}B_{m+1}$. □

We observe that Corollary 3.16 is obtainable directly from Lemma 3.7 without using βS. Notice the similarities with the proof of Theorem 3.14.

Alternate Proof. We show by induction on m that there exist $F_1 < F_2 < \ldots < F_m$ in $\mathcal{P}_f(\mathbb{N})$ such that, if $y = \prod_{i=1}^{m} \prod_{t \in F_i} x_{i,t}$, then $d(A \cap y^{-1}A) > 0$ (and in particular $y \in AA^{-1}$). The case $m = 1$ follows immediately from Lemma 3.7. So let $m \in \mathbb{N}$ and assume that we have $F_1 < F_2 < \ldots < F_m$ in $\mathcal{P}_f(\mathbb{N})$ such that $d(A \cap z^{-1}A) > 0$ where $z = \prod_{i=1}^{m} \prod_{t \in F_i} x_{i,t}$. Let $D = A \cap z^{-1}A$, let $r = \max F_m$, and apply Lemma 3.7 to D and the sequence $\langle x_{m+1,t}\rangle_{t=r+1}^{\infty}$. Pick $F_{m+1} \in \mathcal{P}_f(\mathbb{N})$ with $\min F_{m+1} > r$ such that if $w = \prod_{t \in F_{m+1}} x_{m+1.t}$, then $d(D \cap w^{-1}D) > 0$. Then $D \cap w^{-1}D \subseteq A \cap w^{-1}(z^{-1}A) = A \cap (zw)^{-1}A$ and $zw = \prod_{i=1}^{m+1} \prod_{t \in F_i} x_{i,t}$. □

In a similar vein, if G is a group, one has by Theorem 3.14 that $p^{-1}p \in \mathcal{DR}(G)$ for all $p \in G^*$. Consequently, one obtains corollaries such as the following.

Corollary 3.17. *Let G be an amenable group and let $A \subseteq S$ such that $d(A) > 0$. Let $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ be injective sequences in $\mathbb{N}$. Then*

$$\{x_k^{-1}x_n \textstyle\prod_{t \in F} y_t : k, n \in \mathbb{N}\,,\ F \in \mathcal{P}_f(\mathbb{N}), \text{ and } k < n < \min F\} \cap AA^{-1} \neq \emptyset\,.$$

Proof. Pick $q \in G^*$ such that $\{x_t : t \in \mathbb{N}\} \in q$ and pick an idempotent $p \in \bigcap_{n=1}^{\infty} \overline{FS(\langle y_t\rangle_{t=n}^{\infty})}$. By Theorems 3.14 and 3.15 we have that $q^{-1}qp \in \mathcal{DR}(G) \subseteq \overline{AA^{-1}}$. Let $B = \{x_k^{-1}x_n \prod_{t \in F} y_t : k, n \in \mathbb{N}\,,\ F \in \mathcal{P}_f\mathbb{N}, \text{and} k < n < \min F\}$. It suffices to show that $B \in q^{-1}qp$. Let $C = \{x_k^{-1}x_n : k, n \in \mathbb{N} \text{ and } k < n\}$. By Lemma 3.10 we have $C \in q^{-1}q$ so it suffices to show that $C \subseteq \{w \in G : w^{-1}B \in p\}$. So let $k < n$ in $\mathbb{N}$. Then $FP(\langle y_t\rangle_{t=n+1}^{\infty}) \subseteq (x_k^{-1}x_n)^{-1}B$. □

Again, we see that there is an alternative proof not using βG.

Alternate Proof. By Lemma 3.6 pick $k < n$ in $\mathbb{N}$ such that $d(x_kA \cap x_nA) > 0$. Then $d(A \cap x_n^{-1}x_kA) > 0$. Let $D = A \cap x_n^{-1}x_kA$. Pick by Lemma 3.7 some $F \in \mathcal{P}_f(\mathbb{N})$ with $\min F > n$ such that if $w = \prod_{t\in F} y_t$, then $d(D \cap w^{-1}D) > 0$. Then $D \cap w^{-1}D \subseteq A \cap w^{-1}x_n^{-1}x_kA = A \cap (x_k^{-1}x_n \prod_{t\in F} y_t)^{-1}A$. □

We obtained in Corollaries 3.16 and 3.17 certain configurations which must always meet AA^{-1} whenever $d(A) > 0$. We shall give an illustration in Theorem 3.20 of the fact that such results imply the existence of similar configurations contained in AA^{-1}. For this, we need the Milliken-Taylor Theorem, which in turn requires some new notation.

Definition 3.18. (a) Let $\langle F_n\rangle_{n=1}^\infty$ be a sequence in $\mathcal{P}_f(\mathbb{N})$. Then

$$\begin{aligned}\big(FU(\langle F_n\rangle_{n=1}^\infty)\big)^k_< \;=\; & \{(\textstyle\bigcup_{t\in H_1} F_t, \bigcup_{t\in H_2} F_t, \ldots, \bigcup_{t\in H_k} F_t) : \\ & \text{for each } j \in \{1,2,\ldots,k\}\,,\ H_t \in \mathcal{P}_f(\mathbb{N}) \\ & \text{and if } j < k, \text{ then } \max H_j < \min H_{j+1}\}\,.\end{aligned}$$

(b) Let S be a semigroup and let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in S. Then $\langle y_n\rangle_{n=1}^\infty$ is a *product subsystem* of $\langle x_n\rangle_{n=1}^\infty$ if and only if there exists a sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each n, $\max F_n < \min F_{n+1}$ and $y_n = \prod_{t\in F_n} x_t$.

Theorem 3.19 (Milliken-Taylor Theorem). *Let $k, r \in \mathbb{N}$ and let $(\mathcal{P}_f(\mathbb{N}))^k = \bigcup_{i=1}^r A_i$. There exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max F_n < \min F_{n+1}$ for all n and $\big(FU(\langle F_n\rangle_{n=1}^\infty)\big)^k_< \subseteq A_i$.*

Proof. This follows immediately from [20, Lemma 2.2]. (An equivalent version is proved in [18, Theorem 2.2]. See [15, Section 18.1].) □

We can now illustrate the sort of results that follow from theorems such as Corollary 3.17.

Theorem 3.20. *Let G be a group and let $B \subseteq G$. Assume that whenever $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ are injective sequences in G, one has*

$$\{x_k^{-1}x_n \textstyle\prod_{t\in K} y_t : k,n \in \mathbb{N}\,,\ K \in \mathcal{P}_f(\mathbb{N}), \text{ and } k < n < \min K\} \cap B \neq \emptyset\,.$$

Let injective sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in G be given. Then there exist a subsequence $\langle w_n\rangle_{n=1}^\infty$ of $\langle x_n\rangle_{n=1}^\infty$ and a product subsystem $\langle z_n\rangle_{n=1}^\infty$ of $\langle y_n\rangle_{n=1}^\infty$ such that $\{w_k^{-1}w_n \prod_{t\in K} z_t : k,n \in \mathbb{N}\,,\ K \in \mathcal{P}_f(\mathbb{N}), \text{ and } k < n < \min K\} \subseteq B$.

Proof. By [15, Lemma 6.31] there is a subsequence of $\langle y_n\rangle_{n=1}^\infty$ which has distinct finite products, so we may assume that $\langle y_n\rangle_{n=1}^\infty$ has distinct finite products. Let $C_1 = B$ and $C_2 = G \setminus C_1$. For $i \in \{1,2\}$, let

$$A_i = \{(H_1,H_2,H_3) \in \big(\mathcal{P}_f(\mathbb{N})\big)^3 : (x_{\min H_1})^{-1}x_{\min H_2} \textstyle\prod_{t\in H_3} y_t \in C_i\}\,.$$

Pick $i \in \{1,2\}$ and a sequence $\langle F_n\rangle_{n=1}^\infty$ as guaranteed by Theorem 3.19.

For $n \in \mathbb{N}$, let $w_n = x_{\min F_n}$ and let $z_n = \prod_{t\in F_n} y_t$. We shall show that

$$\{w_k^{-1}w_n \textstyle\prod_{t\in K} z_t : k,n \in \mathbb{N}\,,\ K \in \mathcal{P}_f(\mathbb{N}), \text{ and } k < n < \min K\} \subseteq C_i\,.$$

Since $\{w_k^{-1}w_n \prod_{t\in K} z_t : k,n \in \mathbb{N}\,,\ K \in \mathcal{P}_f(\mathbb{N}), \text{ and } k < n < \min K\} \cap B \neq \emptyset$, this will imply that

$$\{w_k^{-1}w_n \textstyle\prod_{t\in K} z_t : k,n \in \mathbb{N}\,,\ K \in \mathcal{P}_f(\mathbb{N}), \text{ and } k < n < \min K\} \subseteq B\,.$$

To this end, let $k, n \in \mathbb{N}$ and let $K \in \mathcal{P}_f(\mathbb{N})$ such that $k < n < \min K$. Let $H_1 = F_k$, let $H_2 = F_n$, and let $H_3 = \bigcup_{m\in K} F_m$. Then $(H_1, H_2, H_3) \in \big(FU(\langle F_n\rangle_{n=1}^{\infty})\big)^3_<$ so $(x_{\min H_1})^{-1} x_{\min H_2} \prod_{t\in H_3} y_t \in C_i$. Since $x_{\min H_1} = x_{\min F_k} = w_k$, $x_{\min H_2} = x_{\min F_n} = w_n$, and $\prod_{t\in H_3} y_t = \prod_{m\in K} \prod_{t\in F_m} y_t = \prod_{m\in K} z_m$ we have that

$$w_k^{-1} w_n \prod_{m\in K} z_m \in C_i$$

as required. □

In [6, Theorem 1.5] it was shown that if, in addition to being left cancellative and left amenable, S is countable, and $A \subseteq S$ with $d(A) > 0$, then all of the idempotents of βS are in $\overline{AA^{-1}}$. As a consequence of Theorems 3.14 and 3.15, we have without the countability assumption that the semigroup generated by the idempotents is contained in $\bigcap\{\overline{AA^{-1}} : A \subseteq S \text{ and } d(A) > 0\}$. We do not know the answer to the following question even in the case that S is $(\mathbb{N}, +)$.

Question 3.21. *Let S be a left cancellative, left amenable semigroup. Let*

$$T = \bigcap\{\overline{AA^{-1}} : A \subseteq S \text{ and } d(A) > 0\}\,.$$

Is T a subsemigroup of βS?

Let $\mathcal{F} = \mathcal{P}_f(\mathbb{N})$. Then the semigroup $(\mathcal{F}, \cup)$ is very non cancellative. We shall see that the conclusion of Theorem 3.14 remains valid for this semigroup. But, unfortunately, this is because most of the issues with which we are dealing are trivial in $\mathcal{F}$, starting with the notion of having positive density.

Theorem 3.22. *Let $\mathcal{A} \subseteq \mathcal{F}$. The following statements are equivalent.*

(a) *$d(\mathcal{A}) > 0$.*
(b) *For all $F \in \mathcal{F}$ there exists $G \in \mathcal{A}$ such that $F \subseteq G$.*
(c) *$d(\mathcal{A}) = 1$.*

Proof. To see that (a) implies (b), assume that $d(\mathcal{A}) > 0$ and let $F \in \mathcal{F}$. Then by Theorem 2.3 $d(F^{-1}\mathcal{A}) > 0$ so $F^{-1}\mathcal{A} \neq \emptyset$. That is, there is some $G \in \mathcal{F}$ such that $F \cup G \in \mathcal{A}$.

To see that (b) implies (c), we use Theorem 2.2. So let $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$ and $\epsilon > 0$ be given. Pick $G \in \mathcal{F}$ such that $\bigcup \mathcal{H} \subseteq G$ and let $\mathcal{K} = \{G\}$. Then given $F \in \mathcal{H}$, we have $F\mathcal{K} = \{F \cup G\} = \{G\} = \mathcal{K}$ so $\mathcal{K} \setminus F\mathcal{K} = \emptyset$ and $\mathcal{A} \cap \mathcal{K} = \mathcal{K}$.

That (c) implies (a) is trivial. □

Corollary 3.23. *A set $\mathcal{B} \subseteq \mathcal{F}$ is density recurrent if and only if $\mathcal{B} \neq \emptyset$. Consequently, $\mathcal{DR}(\mathcal{F}) = \beta\mathcal{F}$.*

Proof. The necessity is trivial. So assume $\mathcal{B} \neq \emptyset$ and let $\mathcal{A} \subseteq \mathcal{F}$ with $d(\mathcal{A}) > 0$. Pick $F \in \mathcal{B}$. Let $\mathcal{C} = \{G \in \mathcal{A} : F \subseteq G\}$. Then $\mathcal{C} \subseteq (F^{-1}\mathcal{A} \cap \mathcal{A})$ and by Theorem 3.22, $d(\mathcal{C}) = 1$. □

We close this section with some remarks about another question which came up in the course of our investigations. Recall the standard statements of the Finite Unions Theorem and the Finite Products Theorem. If $\langle G_n\rangle_{n=1}^{\infty}$ is a sequence in $\mathcal{F}$ we write $FU(\langle G_n\rangle_{n=1}^{\infty}) = \{\bigcup_{n\in H} G_n : H \in \mathcal{P}_f(\mathbb{N})\}$.

Theorem 3.24. (a) *(Finite Unions Theorem). Let $r \in \mathbb{N}$ and let $\mathcal{F} = \bigcup_{i=1}^{r} \mathcal{A}_i$. There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle G_n\rangle_{n=1}^{\infty}$ of pairwise disjoint members of $\mathcal{F}$ such that $FU(\langle G_n\rangle_{n=1}^{\infty}) \subseteq \mathcal{A}_i$.*

(b) *(Finite Products Theorem). Let S be a semigroup, let $r \in \mathbb{N}$, and let $S = \bigcup_{i=1}^r A_i$. There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i$.*

Proof. [15, Corollaries 5.17 and 5.9]. □

The Finite Unions Theorem easily implies the Finite Products Theorem. (If $S = \bigcup_{i=1}^r A_i$, then for $i \in \{1, 2, \ldots, r\}$, let $\mathcal{A}_i = \{F \in \mathcal{F} : \prod_{t \in F} x_t \in A_i\}$.)

And of course, the Finite Products Theorem applies to the semigroup $(\mathcal{F}, \cup)$. However, the Finite Products Theorem in $\mathcal{F}$ is trivial, even if one demands that the sequence be injective. If $\mathcal{F} = \bigcup_{i=1}^r \mathcal{A}_i$, then necessarily some $\mathcal{A}_i$ satisfies $d(\mathcal{A}_i) > 0$. (And one need not resort to a left invariant mean to show this. If for each i there were some F_i with no superset in $\mathcal{A}_i$, then $\bigcup_{i=1}^r F_i$ could not be in any cell.) Thus there is a sequence $\langle G_n \rangle_{n=1}^\infty$ in $\mathcal{A}_i$ with $G_n \subsetneq G_{n+1}$. And $FU(\langle G_n \rangle_{n=1}^\infty) = \{G_n : n \in \mathbb{N}\}$.

Now suppose we modify the statement of the Finite Unions Theorem by requiring that for $n \neq m$, neither of G_n or G_m contains the other. Can one prove that version without proving the full Finite Unions Theorem?

4. IP^n SETS

A subset A of a semigroup S is an *IP set* if and only if A contains $FP(\langle x_n \rangle_{n=1}^\infty)$ for some sequence $\langle x_n \rangle_{n=1}^\infty$ in S and A is an *IP* set* if and only if it has nonempty intersection with each IP set. By [15, Theorem 5.12] A is an IP set if and only if there is an idempotent $p \in \beta S$ such that $A \in p$. Consequently, A is an IP* set if and only if for every idempotent $p \in \beta S$ one has $A \in p$. By Theorems 3.14 and 3.15, if S is a left cancellative and left amenable semigroup, $A \subseteq S$, and $d(A) > 0$, then AA^{-1} is an IP* set. But in fact much more is true as a consequence of those same theorems. That is, AA^{-1} is a member of any finite product of idempotents in βS.

In this section we introduce IP^n sets and characterize them as precisely those sets which are members of a product of a fixed number of idempotents.

Definition 4.1. Let $n \in \mathbb{N}$, let S be a semigroup, and let $A \subseteq S$. Then A is an IP^n set if and only if there exist for each $i \in \{1, 2, \ldots, n\}$ a sequence $\langle x_{i,t} \rangle_{t=1}^\infty$ such that $\{\prod_{i=1}^n \prod_{t \in H_i} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N})$ and $H_1 < H_2 < \ldots < H_n\} \subseteq A$. Also, A is an IP^{n*} set if and only if A has nonempty intersection with every IP^n set in S.

The notion of an IP^n set should not be confused with the notion of an IP$_n$ set defined in [5] (which in turn is different from the notion of an IP$_n$ set defined in [14]). There we said that A is an IP$_n$ set if and only if whenever S was finitely partitioned, one cell contained $FP(\langle x_t \rangle_{t=1}^n)$ for some sequence $\langle x_t \rangle_{t=1}^n$ in S. Thus by definition, the notion of IP$_n$ set is partition regular. We shall show in Corollary 4.6 that the notion of IP^n is also partition regular.

Lemma 4.2. *Let $n \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, n\}$ let $\langle x_{i,t} \rangle_{t=1}^\infty$ be a sequence in S and let p_i be an idempotent in βS such that $p_i \in \bigcap_{m=1}^\infty \overline{FP(\langle x_{i,t} \rangle_{t=m}^\infty)}$. Let $A \in p_1 p_2 \cdots p_n$. Then there exist $H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N})$ such that such that $H_1 < H_2 < \ldots < H_n$ and $\prod_{i=1}^n \prod_{t \in H_i} x_{i,t} \in A$.*

Proof. We proceed by induction. For $n = 1$ we have that

$$A \in p_1 \text{ and } FP(\langle x_{1,t}\rangle_{t=1}^{\infty}) \in p_1$$

so $A \cap FP(\langle x_{1,t}\rangle_{t=1}^{\infty}) \neq \emptyset$.

Now let $n > 1$ and assume the statement is true for $n-1$. Let

$$B = \{y \in S : y^{-1}A \in p_n\}.$$

Then $B \in p_1p_2\cdots p_{n-1}$ so pick $H_1, H_2, \ldots, H_{n-1} \in \mathcal{P}_f(\mathbb{N})$ such that such that $H_1 < H_2 < \ldots < H_{n-1}$ and $\prod_{i=1}^{n-1}\prod_{t\in H_i} x_{i,t} \in B$. Let $y = \prod_{i=1}^{n-1}\prod_{t\in H_i} x_{i,t}$ and let $m = \max H_{n-1} + 1$. Then $y^{-1}A \in p_n$ and $FP(\langle x_{n,t}\rangle_{t=m}^{\infty}) \in p_n$ so pick $H_n \in \mathcal{P}_f(\mathbb{N})$ with $\min H_n \geq m$ such that $\prod_{t\in H_n} x_{n,t} \in y^{-1}A$. □

Theorem 4.3. *Let S be a semigroup, let $n \in \mathbb{N}$, and let $A \subseteq S$. Then A is an IP^n set if and only if there exist idempotents $p_1, p_2, \ldots, p_n$ in βS such that $A \in p_1p_2\cdots p_n$.*

Proof. Necessity. Pick for each $i \in \{1, 2, \ldots, n\}$ a sequence $\langle x_{i,t}\rangle_{t=1}^{\infty}$ such that $\{\prod_{i=1}^{n}\prod_{t\in H_i} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq A$. For each $i \in \{1, 2, \ldots, n\}$ pick by [15, Lemma 5.11] an idempotent

$$p_i \in \bigcap_{m=1}^{\infty} \overline{FP(\langle x_{i,t}\rangle_{t=m}^{\infty})}.$$

To see that $A \in p_1p_2\cdots p_n$ suppose instead that $S \setminus A \in p_1p_2\cdots p_n$. By Lemma 4.2 pick $H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N})$ such that such that $H_1 < H_2 < \ldots < H_n$ and $\prod_{i=1}^{n}\prod_{t\in H_i} x_{i,t} \in S \setminus A$. This is a contradiction.

Sufficiency. We proceed by induction. If p_1 is an idempotent in βS and $A \in p_1$, then by [15, Theorem 5.8] A is an IP set which is the same as an IP^1 set. So let $n \in \mathbb{N}$ and assume that the implication is valid for n. Let $p_1, p_2, \ldots, p_{n+1}$ be idempotents in βS and assume that $A \in p_1p_2\cdots p_{n+1}$. Let $B = \{y \in S : y^{-1}A \in p_{n+1}\}$. Then $B \in p_1p_2\cdots p_n$ so B is an IP^n set. Pick sequences $\langle x_{i,t}\rangle_{t=1}^{\infty}$ for $i \in \{1, 2, \ldots, n\}$ such that $\{\prod_{i=1}^{n}\prod_{t\in H_i} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq B$. For $t \in \{1, 2, \ldots, n\}$, pick $x_{n+1,t}$ arbitrarily. For $m > n$, let

$$C_m = \{\textstyle\prod_{i=1}^{n}\prod_{t\in H_i} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\{1, 2, \ldots, m-1\}) \\ \text{and } H_1 < H_2 < \ldots < H_n\}$$

and let $D_m = \bigcap_{y\in C_m} y^{-1}A$. Let $D_m^{\star} = \{z \in D_m : z^{-1}D_m \in p_{n+1}\}$. Since p_{n+1} is an idempotent, each $D_m^{\star} \in p_{n+1}$. Further by [15, Lemma 4.14], if $z \in D_m^{\star}$, then $z^{-1}D_m^{\star} \in p_{n+1}$.

Pick $x_{n+1,n+1} \in D_{n+1}^{\star}$. Let $m > n+1$ and assume we have chosen $x_{n+1,k}$ for all $k \in \{n+1, n+2, \ldots, m-1\}$ so that

$$(*)\qquad \text{if } \emptyset \neq G \subseteq \{n+1, n+2, \ldots, m-1\} \text{ and } n+1 \leq k \leq \min G, \text{ then} \\ \textstyle\prod_{t\in G} x_{n+1,t} \in D_k^{\star}.$$

Pick

$$x_{n+1,m} \in D_m^{\star} \cap \bigcap_{k=n+1}^{m-1} \bigcap\left\{\left(\textstyle\prod_{t\in G} x_{n+1,t}\right)^{-1} D_k^{\star} : \emptyset \neq G \subseteq \{k, k+1, \ldots, m-1\}\right\}.$$

(The listed intersection is an element of p_{n+1} and so is nonempty.)

To verify $(*)$, let $\emptyset \neq G \subseteq \{n+1, n+2, \ldots, m\}$ and let $n+1 \leq k \leq \min G$. If $m \notin G$, then $(*)$ holds by assumption, so assume that $m \in G$. If $G = \{m\}$, then $\prod_{t\in G} x_{n+1,t} = x_{n+1,m} \in D_m^{\star} \subseteq D_k^{\star}$. So assume $|G| > 1$, and let $F = G \setminus \{m\}$.

Then $\prod_{t\in F} x_{n+1,t} \in D_k^\star$ and $x_{n+1,m} \in \left(\prod_{t\in F} x_{n+1,t}\right)^{-1} D_k^\star$ so $\prod_{t\in G} x_{n+1,t} \in D_k^\star$ as required.

The construction being complete, assume that $H_1, H_2, \ldots, H_{n+1} \in \mathcal{P}_f(\mathbb{N})$ and $H_1 < H_2 < \ldots < H_{n+1}$. Let $k = \min H_{n+1}$ and let $y = \prod_{i=1}^n \prod_{t\in H_i} x_{i,t}$. Then $y \in C_k$ and $\prod_{t\in H_{n+1}} x_{n+1,t} \in D_k^\star \subseteq y^{-1}A$ so $\prod_{i=1}^{n+1} \prod_{t\in H_i} x_{i,t} \in A$ as required. □

Corollary 4.4. *Let S be a semigroup, let $n \in \mathbb{N}$, and let $A \subseteq S$. Then A is an IP^{n*} set if and only if for all idempotents $p_1, p_2, \ldots, p_n$ in βS one has $A \in p_1 p_2 \cdots p_n$.*

Proof. The set A is an IP^{n*} set if and only if $S \setminus A$ is not an IP^n set. □

Corollary 4.5. *Let S be a left cancellative left amenable semigroup and let $A \subseteq S$ with $d(A) > 0$. Then AA^{-1} is IP^{n*} for every $n \in \mathbb{N}$.*

Proof. By Theorems 3.15 and 3.14, $\overline{AA^{-1}}$ contains a subsemigroup of βS containing the idempotents so Theorem 4.3 applies. □

Corollary 4.6. *Let S be a semigroup, let $n \in \mathbb{N}$, let A be an IP^n set in S, and let $\mathcal{F}$ be a finite partition of A. Then there exists $B \in \mathcal{F}$ such that B is an IP^n set.*

Proof. Pick idempotents $p_1, p_2, \ldots, p_n$ in βS such that $A \in p_1 p_2 \cdots p_n$ by Theorem 4.3. Since $p_1 p_2 \cdots p_n$ is an ultrafilter, there exists $B \in \mathcal{F}$ such that $B \in p_1 p_2 \cdots p_n$. Applying Theorem 4.3 again, we have that B is an IP^n set. □

We now set out to verify that the relationship among these notions is what we would expect.

Theorem 4.7. *Let S be a semigroup, let $n \in \mathbb{N}$, and let A be an IP^n set in S. Then A is an IP^{n+1} set in S.*

Proof. Pick sequences $\langle x_{i,t}\rangle_{t=1}^\infty$ for $i \in \{1, 2, \ldots, n\}$ such that

$$\{\prod_{i=1}^n \prod_{t\in H_i} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq A\,.$$

For each $t \in \mathbb{N}$, let $x_{n+1,t} = x_{n,t}$ and let $H_1, H_2, \ldots, H_{n+1} \in \mathcal{P}_f(\mathbb{N})$ such that $H_1 < H_2 < \ldots < H_{n+1}$. For $i \in \{1, 2, \ldots, n-1\}$, if any, let $G_i = H_i$ and let $G_n = H_n \cup H_{n+1}$. Then $\prod_{i=1}^{n+1} \prod_{t\in H_i} x_{i,t} = \prod_{i=1}^n \prod_{t\in G_i} x_{i,t} \in A$. □

A somewhat shorter, though less elementary, proof of Theorem 4.7 is to pick idempotents $p_1, p_2, \ldots, p_n$ such that $A \in p_1 p_2 \cdots p_n$ and let $p_{n+1} = p_n$ so that $A \in p_1 p_2 \cdots p_n p_n = p_1 p_2 \cdots p_{n+1}$.

Now we see that the strength of the assertion that A is an IP^n in $(\mathbb{N}, +)$ is strictly decreasing as n increases. For $x \in \mathbb{N}$ we define $\text{supp}(x)$ as the subset of $\omega = \mathbb{N} \cup \{0\}$ such that $x = \sum_{t\in \text{supp}(x)} 2^t$. Given a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$, $\langle y_n\rangle_{n=1}^\infty$ is a *sum subsystem* of $\langle x_n\rangle_{n=1}^\infty$ if and only if there exists a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $H_n < H_{n+1}$ for each $n \in \mathbb{N}$ and $y_n = \sum_{t\in H_n} x_t$. Notice that if $\langle y_n\rangle_{n=1}^\infty$ is a sum subsystem of $\langle x_n\rangle_{n=1}^\infty$, then $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq FS(\langle x_n\rangle_{n=1}^\infty)$.

Theorem 4.8. *For each $n \in \mathbb{N}$ there is an IP^{n+1} set in the semigroup $(\mathbb{N}, +)$ which is not an IP^n set.*

Proof. Let $A = \{\sum_{i=1}^{n+1} \sum_{t\in H_i} 2^{t(n+1)+i} : H_1, H_2, \ldots, H_{n+1} \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_{n+1}\}$. Then immediately we have that A is an IP^{n+1} set.

Suppose that A is an IP^n set and pick for each $i \in \{1, 2, \ldots, n\}$ a sequence $\langle x_{i,t}\rangle_{t=1}^\infty$ such that

$$\{\sum_{i=1}^n \sum_{t\in H_i} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq A\,.$$

For $x \in \mathbb{N}$, let $\varphi(x) = \{i \in \{1, 2, \ldots, n+1\} : \mathrm{supp}(x) \cap \big((n+1)\mathbb{N} + i\big) \neq \emptyset\}$. Notice that if $x \in A$, then $\varphi(x) = \{1, 2, \ldots, n+1\}$ and for each $i \in \{1, 2, \ldots, n\}$, $\max\big(\mathrm{supp}(x) \cap \big((n+1)\mathbb{N} + i\big)\big) < \min\big(\mathrm{supp}(x) \cap \big((n+1)\mathbb{N} + i + 1\big)\big)$.

By [15, Corollary 5.15] we may choose for each $i \in \{1, 2, \ldots, n\}$ a sum subsystem $\langle y_{i,t}\rangle_{t=1}^\infty$ of $\langle x_{i,t}\rangle_{t=1}^\infty$ such that φ is constant on $FS(\langle y_{i,t}\rangle_{t=1}^\infty)$. Let $\psi(i)$ be that constant value. By passing to sum subsystems again, we may presume that for each $i \in \{1, 2, \ldots, n\}$ and each $t \in \mathbb{N}$, $\max\mathrm{supp}(y_{i,t}) < \min\mathrm{supp}(y_{i,t+1})$. (See [15, Exercise 5.2.2].) Finally, by successively thinning the sequences, we may presume that if $i \in \{1, 2, \ldots, n-1\}$ and $t \in \mathbb{N}$, then $\max\mathrm{supp}(y_{i,t}) < \min\mathrm{supp}(y_{i+1,t+1})$ and that $\{\sum_{i=1}^n \sum_{t\in H_i} y_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq A$.

Now $y_{1,1} + y_{2,2} + \ldots + y_{n,n} \in A$ so $\bigcup_{i=1}^n \varphi(y_{i,i}) = \varphi(y_{1,1} + y_{2,2} + \ldots + y_{n,n}) = \{1, 2, \ldots, n+1\}$ and therefore for some $j \in \{1, 2, \ldots, n\}$, $\psi(j)$ is not a singleton, and so we have some $k < l$ such that $\{k, l\} \subseteq \psi(j)$. Now consider

$$z = \sum_{i=1}^{j-1} y_{i,i} + y_{j,j} + y_{j,j+1} + \sum_{i=j+1}^n y_{i,i+1}$$

where $\sum_{i=1}^{j-1} y_{i,i} = 0$ if $j = 1$ and $\sum_{i=j+1}^n y_{i,i+1} = 0$ if $j = n$. The support of z has an element congruent to $l \pmod{n+1}$ (as part of the support of $y_{j,j}$) followed by an element congruent to $k \pmod{n+1}$ (as part of the support of $y_{j,j+1}$) and so $z \notin A$, a contradiction. □

We now obtain combinatorial descriptions of IP^{n*} sets.

Theorem 4.9. *Let S be a semigroup, let $n \in \mathbb{N}$, and let $A \subseteq S$. The following statements are equivalent.*

(a) *A is an IP^{n*} set.*
(b) *Whenever $\langle x_{1,t}\rangle_{t=1}^\infty, \langle x_{2,t}\rangle_{t=1}^\infty, \ldots, \langle x_{n,t}\rangle_{t=1}^\infty$ are sequences in S, there exists a sequence $\langle F_k\rangle_{k=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $F_k < F_{k+1}$ for all $k \in \mathbb{N}$ and $\{\prod_{i=1}^n \prod_{k\in H_i} \prod_{t\in F_k} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < \ldots < H_n\} \subseteq A$*
(c) *Whenever $\langle x_{1,t}\rangle_{t=1}^\infty, \langle x_{2,t}\rangle_{t=1}^\infty, \ldots, \langle x_{n,t}\rangle_{t=1}^\infty$ are sequences in S, there exist for each $i \in \{1, 2, \ldots, n\}$ a product subsystem $\langle y_{i,k}\rangle_{k=1}^\infty$ of $\langle x_{i,t}\rangle_{t=1}^\infty$ such that $\{\prod_{i=1}^n \prod_{k\in H_i} y_{i,k} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < \ldots < H_n\} \subseteq A$.*

Proof. (a) implies (b). Let $\langle x_{1,t}\rangle_{t=1}^\infty, \langle x_{2,t}\rangle_{t=1}^\infty, \ldots, \langle x_{n,t}\rangle_{t=1}^\infty$ be sequences in S. Let $B_0 = \big\{(F_1, F_2, \ldots, F_n) \in \big(\mathcal{P}_f(\mathbb{N})\big)^n : \prod_{i=1}^n \prod_{t\in F_i} x_{i,t} \in A\}$ and let $B_1 = \big(\mathcal{P}_f(\mathbb{N})\big)^n \setminus B_0$. Pick by Theorem 3.19, $j \in \{0, 1\}$ and a sequence $\langle F_k\rangle_{k=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max F_k < \min F_{k+1}$ for all k and $\big(FU(\langle F_k\rangle_{k=1}^\infty)\big)^n_< \subseteq B_j$. For each $k \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, n\}$, let $y_{i,k} = \prod_{t\in F_k} x_{i,t}$. For each $i \in \{1, 2, \ldots, n\}$ pick by [15, Lemma 5.11] an idempotent $p_i \in \bigcap_{m=1}^\infty \overline{FP(\langle y_{i,k}\rangle_{k=m}^\infty)}$. By Corollary 4.4, we have that $A \in p_1 p_2 \cdots p_n$. By Lemma 4.2 pick $H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N})$ such that such that $H_1 < H_2 < \ldots < H_n$ and $\prod_{i=1}^n \prod_{k\in H_i} y_{i,k} \in A$. For $i \in \{1, 2, \ldots, n\}$, let $G_i = \bigcup_{k\in H_i} F_k$. Then $(G_1, G_2, \ldots, G_n) \in \big(FU(\langle F_k\rangle_{k=1}^\infty)\big)^n_<$ and $\prod_{i=1}^n \prod_{t\in G_i} x_{i,t} = \prod_{i=1}^n \prod_{k\in H_i} y_{i,k} \in A$ so $j = 0$. Consequently,

$$\{\prod_{i=1}^n \prod_{k\in H_i} y_{i,k} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq A\,.$$

Trivially (b) implies (c).

(c) implies (a). Suppose that A is not an IP^{n*} set, so that $S\setminus A$ is an IP^n set and pick sequences $\langle x_{1,t}\rangle_{t=1}^\infty, \langle x_{2,t}\rangle_{t=1}^\infty, \ldots, \langle x_{n,t}\rangle_{t=1}^\infty$ such that

$$\{\textstyle\prod_{i=1}^n \prod_{t\in H_i} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq S\setminus A\,.$$

If for each $i \in \{1,2,\ldots,n\}$, $\langle y_{i,t}\rangle_{t=1}^\infty$ is a product subsystem of $\langle x_{i,t}\rangle_{t=1}^\infty$, then also $\{\prod_{i=1}^n \prod_{t\in H_i} y_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq S\setminus A$. $\square$

We introduce a stronger notion, whose definition drops the requirement that $H_1 < H_2 < \ldots < H_n$. (The "E" in the name stands for "enhanced".)

Definition 4.10. Let S be a semigroup, let $n\in\mathbb{N}$, and let $A\subseteq S$. Then A is an EIP^{n*} set if and only if whenever $\langle x_{1,t}\rangle_{t=1}^\infty, \langle x_{2,t}\rangle_{t=1}^\infty, \ldots, \langle x_{n,t}\rangle_{t=1}^\infty$ are sequences in S, there exists a sequence $\langle F_k\rangle_{k=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $F_k < F_{k+1}$ for all $k\in\mathbb{N}$ and $\{\prod_{i=1}^n \prod_{k\in H_i}\prod_{t\in F_k} x_{i,t} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N})\cup\{\emptyset\} \text{ and some } H_i\neq\emptyset\} \subseteq A$.

In [8, Definition 6.1], the notion of E-IP* set is defined for subsets of $\mathbb{Z}^k$ for some k. A subset of $\mathbb{Z}^k$ is an E-IP* set if and only if it is an EIP^{n*} set for each $n\in\mathbb{N}$ as defined here.

Note that the notions IP^{1*} and EIP^{1*} are synonymous. However, for $n>1$, in the semigroup $(\mathbb{N},+)$, EIP^{n*} is strictly stronger than IP^{n*}. In fact we have the following.

Theorem 4.11. *There is a set $A\subseteq\mathbb{N}$ such that A is an IP^{n*} set for every $n\in\mathbb{N}$, but A is not an EIP^{2*} set.*

Proof. Let $B = \{\sum_{t\in F_1} 2^{2t} + \sum_{t\in F_2} 2^{2t-1} + \ldots + \sum_{t\in F_{2k}} 2^{2t-1} + \sum_{t\in F_{2k+1}} 2^{2t} : k\in\mathbb{N}\,, F_1, F_2, \ldots, F_{2k+1} \in \mathcal{P}_f(\mathbb{N})\,, k = \min F_1, \text{ and } F_1 < F_2 < \ldots < F_{2k+1}\}$ and let $A = \mathbb{N}\setminus B$. Thus, if $x\in B$, then $\min\operatorname{supp}(x) = 2k$ for some $k\in\mathbb{N}$ and, if the elements of $\operatorname{supp}(x)$ are listed in order, there are precisely $2k$ alterations between even and odd.

Suppose first that A is an EIP^{2*} set. For each $t\in\mathbb{N}$, let $x_{1,t} = 2^{2t}$ and let $x_{2,t} = 2^{2t-1}$. Pick a sequence $\langle F_k\rangle_{k=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $F_k < F_{k+1}$ for all $k\in\mathbb{N}$ and

$$\{\textstyle\sum_{k\in H_1}\sum_{t\in F_k} x_{1,t} + \sum_{k\in H_2}\sum_{t\in F_k} x_{2,t} : H_1, H_2 \in \mathcal{P}_f(\mathbb{N})\cup\{\emptyset\} \text{ and some } H_i\neq\emptyset\}\subseteq A\,.$$

Let $H_1 = \{1,3,\ldots,2k+1\}$ and let $H_2 = \{2,4,\ldots,2k\}$. Then

$$\textstyle\sum_{k\in H_1}\sum_{t\in F_k} x_{1,t} + \sum_{k\in H_2}\sum_{t\in F_k} x_{2,t} \in B\,,$$

a contradiction.

Now let $n\in\mathbb{N}$. We shall show that A is an IP^{n*} set. To this end, let sequences $\langle x_{1,t}\rangle_{t=1}^\infty, \langle x_{2,t}\rangle_{t=1}^\infty, \ldots, \langle x_{n,t}\rangle_{t=1}^\infty$ in $\mathbb{N}$ be given. Let

$$\begin{aligned} C_0 &= \{x\in\mathbb{N} : \operatorname{supp}(x)\subseteq 2\omega\}\,,\\ C_1 &= \{x\in\mathbb{N} : \operatorname{supp}(x)\subseteq 2\omega+1\}, \text{ and}\\ C_2 &= \mathbb{N}\setminus(C_0\cup C_1)\,. \end{aligned}$$

For each $i\in\{1,2,\ldots,n\}$, pick by [15, Corollary 5.15] $j(i)\in\{0,1,2\}$ and a sum subsystem $\langle y_{i,t}\rangle_{t=1}^\infty$ of $\langle x_{i,t}\rangle_{t=1}^\infty$ such that $FS(\langle y_{i,t}\rangle_{t=1}^\infty)\subseteq C_{j(i)}$.

Now let $D_0 = A$ and $D_1 = B$. For $v \in \{0,1\}$ let

$$E_v = \left\{(H_1, H_2, \ldots, H_n) \in \left(\mathcal{P}_f(\mathbb{N})\right)^n : \textstyle\sum_{i=1}^n \sum_{t\in H_i} y_{i,t} \in D_v\right\}.$$

Pick by Theorem 3.19 an increasing sequence $\langle F_m\rangle_{m=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ and $v \in \{0,1\}$ such that $\left(FU(\langle F_m\rangle_{m=1}^\infty)\right)_<^n \subseteq E_v$. For each $i \in \{1,2,\ldots,n\}$ and each $m \in \mathbb{N}$, let $z_{i,m} = \sum_{t\in F_m} y_{i,t}$. Then $\langle z_{i,m}\rangle_{m=1}^\infty$ is a sum subsystem of $\langle y_{i,t}\rangle_{t=1}^\infty$ and so $FS(\langle z_{i,m}\rangle_{m=1}^\infty) \subseteq C_{j(i)}$. Also, $\langle z_{i,m}\rangle_{m=1}^\infty$ is a sum subsystem of $\langle x_{i,t}\rangle_{t=1}^\infty$. Now we claim that

$$\{\textstyle\sum_{i=1}^n \sum_{m\in H_i} z_{i,m} : H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \text{ and } H_1 < H_2 < \ldots < H_n\} \subseteq D_v\,.$$

To see this let $H_1 < H_2 < \ldots < H_n$ be given and for $i \in \{1,2,\ldots,n\}$ let $L_i = \bigcup_{m\in H_i} F_m$. Then $(L_1, L_2, \ldots, L_n) \in \left(FU(\langle F_m\rangle_{m=1}^\infty)\right)_<^n \subseteq E_v$ so

$$\textstyle\sum_{i=1}^n \sum_{m\in H_i} z_{i,m} = \sum_{i=1}^n \sum_{t\in L_i} y_{i,t} \in D_v$$

as required.

To complete the proof, we show that $v = 0$. Suppose instead that $v = 1$. Pick $r \in \mathbb{N}$ such that $\min\operatorname{supp}(z_{1,r}) \geq n$ Now $\sum_{i=1}^n z_{i,r+i-1} \in D_1 = B$ so $\min\operatorname{supp}(z_{1,r})$ is even. Let $2k = \min\operatorname{supp}(z_{1,r})$. If for some $u \in \{1,2,\ldots,n\}$, $j(u) = 2$, then pick $H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N})$ with $r = \min H_1$, $H_1 < H_2 < \ldots < H_n$, and $|H_u| = 2k+1$. Then when the elements of the support of $\sum_{i=1}^n \sum_{m\in H_i} z_{i,m}$ are written in order, there are at least $2k+1$ alterations between even and odd so $\sum_{i=1}^n \sum_{m\in H_i} z_{i,m} \notin B$. Thus, for each $i \in \{1,2,\ldots,n\}$, we have that $j(i) \in \{0,1\}$. But now, if the elements of the support of $\sum_{i=1}^n z_{i,r+i-1}$ are written in order, there are at most $n-1$ alterations between even and odd, and $n-1 < 2k = \min\operatorname{supp}(z_{1,r})$ so $\sum_{i=1}^n z_{i,r+i-1} \notin B$, a contradiction. □

We have by Corollary 4.5 that if $(G,+)$ is an abelian group, $A \subseteq G$ and $d(A) > 0$, then $A - A$ is IP^{n*} for every $n \in \mathbb{N}$. And $A - A = \{x \in G : A \cap (A - x) \neq \emptyset\}$. We shall see, using some powerful results of Furstenberg and Katznelson, that much stronger results are true. While Theorem 4.13 is not stated in [13], it is implicitly contained there. Also, Theorem 4.13 is a corollary of Theorem 4.16, but its proof is much simpler, so we present that proof separately.

Lemma 4.12. *Let $(G,+)$ be a countable abelian group, let $A \subseteq G$ with $d(A) > 0$, and let K be a finite set of commuting endomorphisms of G. Then*

$$\left\{x \in G : d\left(\textstyle\bigcap_{g\in K} (A - g(x))\right) > 0\right\}$$

is an IP^{1*} set.

Proof. Using Theorem 2.2 pick a sequence $\langle K_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(G)$ such that for each $x \in G$, $\displaystyle\lim_{n\to\infty} \frac{|K_n \setminus (x + K_n)|}{|K_n|} = 0$ and $d(A) = \displaystyle\lim_{n\to\infty} \frac{|A \cap K_n|}{|K_n|}$. (A sequence satisfying the first of these requirements is called a *Følner sequence*.) By [3, Theorem 4.17] pick a probability space $(X, \mathcal{B}, \mu)$, a measure preserving action $\langle T_x\rangle_{x\in G}$ of G on X, and a set $B \in \mathcal{B}$ such that $\mu(B) = d(A)$ and for every $F \in \mathcal{P}_f(G)$,

$$d\left(\textstyle\bigcap_{z\in F} (A - z)\right) \geq \mu\left(\textstyle\bigcap_{z\in F} T_z^{-1}[A]\right).$$

Now let a sequence $\langle x_n\rangle_{n=1}^\infty$ in G be given. For $n \in \mathbb{N}$ and $g \in K$, let $R_n^{(g)} = T_{g(x_n)}$. Given $F \in \mathcal{P}_f(\mathbb{N})$ let $i_1, i_2, \ldots, i_l$ list the elements of F in increasing order and let for each $g \in K$, $S_F^{(g)} = R_{i_1}^{(g)} \circ R_{i_2}^{(g)} \circ \ldots \circ R_{i_l}^{(g)}$. For example, if $F = \{1,3,4\}$, then $S_F^{(g)} = R_1^{(g)} \circ R_3^{(g)} \circ R_4^{(g)} = T_{g(x_1)} \circ T_{g(x_3)} \circ T_{g(x_4)} = T_{g(x_1+x_3+x_4)}$. In general, if $z = \sum_{i\in F} x_i$, then $S_F^{(t)} = T_{g(z)}$. Pick by [13, Theorem A] some $F \in \mathcal{P}_f(\mathbb{N})$ such that $\mu(\bigcap_{g\in K} (S_F^{(g)})^{-1}[B]) > 0$. Let $z = \sum_{i\in F} x_i$. Then $d\big(\bigcap_{g\in K} (A - g(z))\big) \geq \mu(\bigcap_{g\in K} T_{g(z)}^{-1}[B]) > 0$. □

Theorem 4.13. *Let $(G,+)$ be a countable abelian group, let $A \subseteq G$ with $d(A) > 0$, let K be a finite set of commuting endomorphisms of G, and let $n \in \mathbb{N}$. Then*

$$\Big\{x \in G : d\Big(\bigcap_{g\in K} \big(A - g(x)\big)\Big) > 0\Big\}$$

is an IP^{n*} set.

Proof. We proceed by induction on n, the case $n = 1$ being Lemma 4.12.

Let $n \in \mathbb{N}$ and assume the result is true for n. Let

$$B = \{x \in G : d\big(\bigcap_{g\in K} (A - g(x))\big) > 0\}\,.$$

By Corollary 4.4 it suffices to let $p_1, p_2, \ldots, p_{n+1}$ be idempotents in βG and show that $B \in p_1 + p_2 + \ldots + p_{n+1}$. To this end, since (again by Corollary 4.4) $B \in p_1 + p_2 + \ldots + p_n$ it suffices to show that $B \subseteq \{x \in G : -x + B \in p_{n+1}\}$, so let $x \in B$. Let $C = \bigcap_{g\in K} \big(A - g(x)\big)$ and let $D = \{y \in G : d\Big(\bigcap_{g\in K} \big(C - g(y)\big)\Big) > 0\}$. Then $d(C) > 0$ so by Lemma 4.12 D is an IP^{1*} set and thus $D \in p_{n+1}$. Given $y \in D$, one has $\bigcap_{g\in K} \big(C - g(y)\big) \subseteq \bigcap_{g\in K} \big(A - g(x+y)\big)$ and so $x + y \in B$. Thus $-x + B \in p_{n+1}$ as required. □

The next lemma is a version of Furstenberg's Correspondence Principle.

Lemma 4.14. *Let $(G,+)$ be a countable abelian group, let λ be a left invariant mean on G, and let $A \subseteq G$ such that $\lambda(\chi_A) > 0$. There exist a compact metric space X, a countably generated σ-algebra $\mathcal{B}$ of subsets of X, a clopen set $U \in \mathcal{B}$, a countably additive measure μ on $\mathcal{B}$, and a measure preserving action $\langle S_x\rangle_{x\in G}$ of G on $(X, \mathcal{B}, \mu)$ such that for all $F \in \mathcal{P}_f(G)$, if $B = \bigcap_{x\in F}(A - x)$, then $\mu(\bigcap_{x\in F} S_x^{-1}[U]) = \lambda(\chi_B)$.*

Proof. This is what was shown in the proof of [7, Theorem 2.1]. □

Lemma 4.15. *Let $(G,+)$ be a countable abelian group, let K be a finite set of commuting endomorphisms of G, let $n \in \mathbb{N}$, and for each $i \in \{1, 2, \ldots, n\}$, let $\langle x_{i,t}\rangle_{t=1}^{\infty}$ be a sequence in G, let $A \subseteq G$ with $d(A) > 0$, and let $l \in \mathbb{N}$. Then there exists $M \in \mathcal{P}_f(\mathbb{N})$ such that $\min M > l$ and*

$$d\big(A \cap \bigcap \big\{A - g(\textstyle\sum_{i\in F} \sum_{t\in M} x_{i,t}) : g \in K \text{ and } \emptyset \neq F \subseteq \{1, 2, \ldots, n\}\big\}\big) > 0\,.$$

Proof. Pick an invariant mean λ on G such that $\lambda(\chi_A) > 0$. Pick $(X, \mathcal{B}, \mu)$, U, and $\langle S_x\rangle_{x\in G}$ as guaranteed by Lemma 4.14 for λ and A. For $g \in K$, $i \in \{1, 2, \ldots, n\}$, and $H \in \mathcal{P}_f(\mathbb{N})$, let $T_H^{g,i} = S_{g(\Sigma_{t\in H} x_{i,t})}$.

Pick by the Main Theorem of [13], an increasing sequence $\langle L_k\rangle_{k=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ and $M \in FU(\langle L_k\rangle_{k=l}^{\infty})$ such that

$$\mu\big(U \cap \bigcap \big\{(\textstyle\prod_{i\in F} T_M^{g,i})^{-1}[U] : g \in K \text{ and } \emptyset \neq F \subseteq \{1, 2, \ldots, n\}\big\}\big) > 0\,.$$

(In the notation of the Main Theorem of [13], Σ is the group generated by

$$\big\{\langle T_H^{g,i}\rangle_{H\in\mathcal{P}_f(\mathbb{N})} : g \in K \text{ and } i \in \{1, 2, \ldots, n\}\big\}\,,$$

$\mathcal{F}^{(1)}$ is $FU(\langle L_k\rangle_{k=1}^{\infty})$, r is $|K \times \{1, 2, \ldots, n\}|$, $\{T^{(1)}, T^{(2)}, \ldots, T^{(r)}\}$ is

$$\big\{\langle T_H^{g,i}\rangle_{H\in\mathcal{P}_f(\mathbb{N})} : g \in K \text{ and } i \in \{1, 2, \ldots, n\}\big\}\,,$$

and $\Lambda = \{\emptyset, M\}$. Then $\prod_{i\in F} T_M^{g,i} = \prod_{t=1}^r T_{\lambda_t}^{(t)}$, where

$$\lambda_t = M \text{ if } T^{(t)} = \langle T_H^{g,i}\rangle_{H\in\mathcal{P}_f(\mathbb{N})}$$

and $\lambda_t = \emptyset$ otherwise.)

Now, given $g \in K$ and $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, we have that $U = (S_0)^{-1}[U]$ and

$$\begin{aligned}\textstyle\prod_{i\in F} T_M^{g,i} &= \textstyle\prod_{i\in F} S_{g(\Sigma_{t\in M} x_{i,t})}\\ &= S_{\Pi_{i\in F} g(\Sigma_{t\in M} x_{i,t})}\\ &= S_{g(\Sigma_{i\in F}\Sigma_{t\in M} x_{i,t})}\,,\end{aligned}$$

so by Lemma 4.14, if

$$B = A \cap \bigcap \big\{A - g(\textstyle\sum_{i\in F}\sum_{t\in M} x_{i,t}) : g \in K \text{ and } \emptyset \neq F \subseteq \{1,2,\ldots,n\}\big\}\,,$$

then $\lambda(\chi_B) > 0$ and consequently $d(B) > 0$. □

Theorem 4.16. *Let $(G,+)$ be a countable abelian group, let $A \subseteq G$ with $d(A) > 0$, let K be a finite set of commuting endomorphisms of G, and let $n \in \mathbb{N}$. Then*

$$\Big\{x \in G : d\Big(\textstyle\bigcap_{g\in K} \big(A - g(x)\big)\Big) > 0\Big\}$$

is an EIP^{n*} set.

Proof. Let $A_1 = A$, and by Lemma 4.15 pick $M_1 \in \mathcal{P}_f(\mathbb{N})$ such that, letting $A_2 = A_1 \cap \bigcap\big\{A_1 - g(\sum_{i\in F}\sum_{t\in M_1} x_{i,t} : g \in K \text{ and } \emptyset \neq F \subseteq \{1,2,\ldots,n\}\big\}$, we have that $d(A_2) > 0$.

Inductively, given $k > 1$, A_k, and M_{k-1}, let $l = \max M_{k-1}$ and pick by Lemma 4.15, $M_k \in \mathcal{P}_f(\mathbb{N})$ such that $\min M_k > l$ and, letting

$$A_{k+1} = A_k \cap \bigcap\big\{A_k - g(\textstyle\sum_{i\in F}\sum_{t\in M_k} x_{i,t} : g \in K \text{ and } \emptyset \neq F \subseteq \{1,2,\ldots,n\}\big\}\,,$$

we have that $d(A_{k+1}) > 0$.

The induction being complete, for each $i \in \{1,2,\ldots,n\}$ and each $k \in \mathbb{N}$, let $y_{i,k} = \sum_{t\in M_k} x_{i,t}$. We show by induction on $|\bigcup_{i=1}^n H_i|$ that if $H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N}) \cup \{\emptyset\}$, some $H_i \neq \emptyset$, and $m = \max\bigcup_{i=1}^n H_i$, then

$$A_{m+1} \subseteq \textstyle\bigcap_{g\in K}\big(A - g(\sum_{i=1}^n\sum_{k\in H_i} y_{i,k})\big)$$

so that $\sum_{i=1}^n\sum_{k\in H_i} y_{i,k} \in \Big\{x \in G : d\Big(\bigcap_{g\in K}\big(A - g(x)\big)\Big) > 0\Big\}$ as required.

Assume first that $\bigcup_{i=1}^n H_i = \{m\}$ and let $F = \{i \in \{1,2,\ldots,n\} : m \in H_i\}$. Then

$$\begin{aligned}A_{m+1} &\subseteq \textstyle\bigcap_{g\in K}\big(A_m - g(\sum_{i\in F}\sum_{t\in M_m} x_{i,t})\big)\\ &= \textstyle\bigcap_{g\in K}\big(A_m - g(\sum_{i\in F} y_{i,m})\big)\\ &\subseteq \textstyle\bigcap_{g\in K}\big(A - g(\sum_{i\in F} y_{i,m})\big)\\ &= \textstyle\bigcap_{g\in K}\big(A - g(\sum_{i=1}^n\sum_{k\in H_i} y_{i,k})\big)\,.\end{aligned}$$

Now assume that $|\bigcup_{i=1}^n H_i| > 1$, let $m = \max\bigcup_{i=1}^n H_i$, and let

$$F = \{i \in \{1,2,\ldots,n\} : m \in H_i\}\,.$$

For $i \in \{1,2,\ldots,n\}$, let $D_i = H_i \setminus \{m\}$ (so if $i \notin F$, then $D_i = H_i$). Then some $D_i \neq \emptyset$. Let $l = \max\bigcup_{i=1}^n D_i$. Then by the induction hypothesis we have

$A_m \subseteq A_{l+1} \subseteq \bigcap_{g\in K} \big(A - g(\sum_{i=1}^n \sum_{k\in D_i} y_{i,k})\big)$. Thus

$$\begin{aligned} A_{m+1} &\subseteq \textstyle\bigcap_{g\in K} \big(A_m - g(\sum_{i\in F}\sum_{t\in M_m} x_{i,t})\big) \\ &= \textstyle\bigcap_{g\in K} \big(A_m - g(\sum_{i\in F} y_{i,m})\big) \\ &\subseteq \textstyle\bigcap_{g\in K} \Big(\big(A - g(\sum_{i=1}^n \sum_{k\in D_i} y_{i,k})\big) - g(\sum_{i\in F} y_{i,m})\Big)\,. \end{aligned}$$

If $i \in F$, then $H_i = D_i \cup \{m\}$ while if $i \notin F$, then $H_i = D_i$ so given $g \in K$,

$$\begin{gathered} \big(A - g(\textstyle\sum_{i=1}^n \sum_{k\in D_i} y_{i,k})\big) - g(\sum_{i\in F} y_{i,m}) = \\ A - g(\textstyle\sum_{i=1}^n \sum_{k\in D_i} y_{i,k} + \sum_{i\in F} y_{i,m}) = \\ A - g(\textstyle\sum_{i=1}^n \sum_{k\in H_i} y_{i,k})\,. \end{gathered}$$

□

5. Δ^n SETS

We now turn our attention to Δ^n sets. A set $A \subseteq \mathbb{N}$, is a Δ set if and only if there is an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that

$$\{x_m - x_n : n, m \in \mathbb{N} \text{ and } n < m\} \subseteq A$$

and we can extend that notion to a subset A of an arbitrary group S by requiring that there exists an injective sequence $\langle x_n\rangle_{n=1}^\infty$ in S with

$$\{x_n^{-1}x_m : n, m \in \mathbb{N} \text{ and } n < m\} \subseteq A\,.$$

(In [5] we did not require the sequence to be injective. This has the drawback that $\{e\}$ is then a Δ set, where e is the identity of S.)

Definition 5.1. Let S be a group or $(\mathbb{N}, +)$, let $A \subseteq S$, and let $n \in \mathbb{N}$. Then A is a Δ^n set in S if and only if there exist for each $i \in \{1, 2, \ldots, n\}$ an injective sequence $\langle x_{i,t}\rangle_{t=1}^\infty$ in S such that

$$\begin{aligned} \{\textstyle\prod_{i=1}^n (x_{i,k(i)}^{-1} x_{i,m(i)}) : k(1), m(1), k(2), \ldots, k(n), m(n) \in \mathbb{N} \\ \text{and } k(1) < m(1) < k(2) < \ldots < k(n) < m(n)\} \subseteq A\,. \end{aligned}$$

Also, A is a Δ^{n*} set if and only if A has nonempty intersection with every Δ^n set in S.

As with the IP^n sets, we set out to characterize the Δ^n sets in terms of products of members of βS.

Lemma 5.2. *Let S be a group or $(\mathbb{N}, +)$, let $n \in \mathbb{N}$, and for $i \in \{1, 2, \ldots, n\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ be an injective sequence in S. Assume that for each $i \in \{1, 2, \ldots, n\}$, $p_i \in S^* \cap \overline{\{x_{i,t} : t \in \mathbb{N}\}}$. Then*

$$\begin{aligned} &\{\textstyle\prod_{i=1}^n (x_{i,k(i)}^{-1} x_{i,m(i)}) : \ k(1), m(1), k(2), \ldots, k(n), m(n) \in \mathbb{N} \\ &\textit{and } k(1) < m(1) < k(2) < \ldots < k(n) < m(n)\} \in \textstyle\prod_{i=1}^n (p_i^{-1} p_i)\,. \end{aligned}$$

Proof. We proceed by induction, the case $n = 1$ following from Lemma 3.10. So let $n \in \mathbb{N}$ and assume that the statement is true for n. Let

$$\begin{aligned} A = \{\textstyle\prod_{i=1}^{n+1} (x_{i,k(i)}^{-1} x_{i,m(i)}) : k(1), m(1), k(2), \ldots, k(n+1), m(n+1) \in \mathbb{N} \\ \text{and } k(1) < m(1) < k(2) < \ldots < k(n+1) < m(n+1)\} \end{aligned}$$

and let

$$B = \{\prod_{i=1}^{n}(x_{i,k(i)}^{-1}x_{i,m(i)}) : k(1), m(1), k(2), \ldots, k(n), m(n) \in \mathbb{N} \text{ and } k(1) < m(1) < k(2) < \ldots < k(n) < m(n)\}\,.$$

Then $B \in \prod_{i=1}^{n}(p_i^{-1}p_i)$ We claim that $B \subseteq \{y \in S : y^{-1}A \in p_{n+1}^{-1}p_{n+1}\}$, so let $y = \prod_{i=1}^{n}(x_{i,k(i)}^{-1}x_{i,m(i)}) \in B$. Let

$$C = \{x_{n+1,k(n+1)}^{-1}x_{n+1,m(n+1)} : k(n+1), m(n+1) \in \mathbb{N} \text{ and } m(n) < k(n+1) < m(n+1)\}\,.$$

By Lemma 3.10, $C \in p_{n+1}^{-1}p_{n+1}$. Since $C \subseteq y^{-1}A$, we have that $A \in \prod_{i=1}^{n+1}(p_i^{-1}p_i)$ as required. □

Lemma 5.3. *Let S be a group or $(\mathbb{N}, +)$, let $n \in \mathbb{N}$, and let $A \subseteq S$. For each $i \in \{1, 2, \ldots, n\}$ let $p_i \in S^*$, let $\langle B_{i,t}\rangle_{t=1}^{\infty}$ be a sequence of members of p_i, and assume that $A \in \prod_{i=1}^{n}(p_i^{-1}p_i)$. There exist for each $i \in \{1, 2, \ldots, n\}$ an injective sequence $\langle x_{i,t}\rangle_{t=1}^{\infty}$ such that for each $t \in \mathbb{N}$, $x_{i,t} \in \bigcap_{j=1}^{t} B_{i,j}$ and*

$$\{\{\prod_{i=1}^{n}(x_{i,k(i)}^{-1}x_{i,m(i)}) : k(1), m(1), k(2), \ldots, k(n), m(n) \in \mathbb{N} \text{ and } k(1) < m(1) < k(2) < \ldots < k(n) < m(n)\} \subseteq A\,.$$

In particular, A is a Δ^n set.

Proof. We proceed by induction, the $n = 1$ case following from Lemma 3.11. So let $n \in \mathbb{N}$ and assume the implication holds for n. Pick $p_1, p_2, \ldots, p_{n+1} \in S^*$ such that $A \in \prod_{i=1}^{n+1}(p_i^{-1}p_i)$ and let $B = \{y \in S : y^{-1}A \in p_{n+1}^{-1}p_{n+1}\}$. Then $B \in \prod_{i=1}^{n}(p_i^{-1}p_i)$ so choose for each $i \in \{1, 2, \ldots, n\}$ an injective sequence $\langle x_{i,t}\rangle_{t=1}^{\infty}$ in S such that for each t, $x_{i,t} \in \bigcap_{j=1}^{t} B_{i,j}$ and

$$\{\prod_{i=1}^{n}(x_{i,k(i)}^{-1}x_{i,m(i)}) : k(1), m(1), k(2), \ldots, k(n), m(n) \in \mathbb{N} \text{ and } k(1) < m(1) < k(2) < \ldots < k(n) < m(n)\} \subseteq B\,.$$

For $t \leq 2n$, choose $x_{n+1,t} \in \bigcap_{j=1}^{t} B_{n+1,j}$ arbitrarily (preserving injectivity). For $l > 2n$, let

$$C_l = \{\prod_{i=1}^{n}(x_{i,k(i)}^{-1}x_{i,m(i)}) : k(1), m(1), k(2), \ldots, k(n), m(n) \in \{1, 2, \ldots, l-1\} \text{ and } k(1) < m(1) < k(2) < \ldots < k(n) < m(n)\}\,,$$

let $D_l = \bigcap_{y \in C_l} y^{-1}A$, and let $E_l = \{w \in S : wD_l \in p_{n+1}\}$. Since $C_l \subseteq B$, $D_l \in p_{n+1}^{-1}p_{n+1}$ and so $E_l \in p_{n+1}$.

Choose $x_{n+1,2n+1} \in E_{2n+1} \cap \bigcap_{j=1}^{2n+1} B_{n+1,j}$ and for $l > 2n+1$, choose

$$x_{n+1,l} \in E_l \cap \bigcap_{t=2n+1}^{l-1} x_{n+1,t}D_t \cap \bigcap_{j=1}^{l} B_{n+1,j}\,.$$

Now let $k(1), m(1), k(2), \ldots, k(n), m(n) \in \mathbb{N}$ such that $k(1) < m(1) < k(2) < \ldots < k(n+1) < m(n+1)$. Then

$$x_{n+1,k(n+1)}^{-1}x_{n+1,m(n+1)} \in D_{k(n+1)} \text{ and } \prod_{i=1}^{n}(x_{i,k(i)}^{-1}x_{i,m(i)}) \in C_{k(n+1)}$$

so $\prod_{i=1}^{n+1}(x_{i,k(i)}^{-1}x_{i,m(i)}) \in A$ as required. □

Theorem 5.4. *Let S be a group or $(\mathbb{N}, +)$, let $n \in \mathbb{N}$, and let $A \subseteq S$. Then A is a Δ^n set if and only if there exist $p_1, p_2, \ldots, p_n \in S^*$ such that $A \in \prod_{i=1}^{n}(p_i^{-1}p_i)$.*

Proof. Necessity. Choose for each $i \in \{1, 2, \ldots, n\}$ an injective sequence $\langle x_{i,t}\rangle_{t=1}^{\infty}$ in S such that

$$\{\textstyle\prod_{i=1}^{n}(x_{i,k(i)}^{-1}x_{i,m(i)}) : k(1), m(1), k(2), \ldots, k(n), m(n) \in \mathbb{N} \text{ and } k(1) < m(1) < k(2) < \ldots < k(n) < m(n)\} \subseteq A\,.$$

For each $i \in \{1, 2, \ldots, n\}$ pick $p_i \in S^*$ such that $\{x_{i,t} : t \in \mathbb{N}\} \in p_i$. By Lemma 5.2, $A \in \prod_{i=1}^{n}(p_i^{-1}p_i)$.

The sufficiency is an immediate consequence of Lemma 5.3. □

We immediately get corollaries corresponding to Corollaries 4.4, 4.5, and 4.6.

Corollary 5.5. *Let S be a group or $(\mathbb{N}, +)$, let $n \in \mathbb{N}$, and let $A \subseteq S$. Then A is a Δ^{n*} set if and only if if for all $p_1, p_2, \ldots, p_n$ in S^* one has $A \in \prod_{i=1}^{n}(p_i^{-1}p_i)$.*

Proof. The set A is a Δ^{n*} set if and only if $S \setminus A$ is not a Δ^n set. □

Corollary 5.6. *Let S be an amenable group or $(\mathbb{N}, +)$ and let $A \subseteq S$ with $d(A) > 0$. Then AA^{-1} is Δ^{n*} for every $n \in \mathbb{N}$.*

Proof. By Theorems 3.15 and 3.14, $\overline{AA^{-1}}$ contains a subsemigroup of βS containing $p^{-1}p$ for all $p \in S^*$ so Theorem 5.4 applies. □

Corollary 5.7. *Let S be a group or $(\mathbb{N}, +)$, let $n \in \mathbb{N}$, let A be a Δ^n set in S, and let $\mathcal{F}$ be a finite partition of A. Then there exists $B \in \mathcal{F}$ such that B is a Δ^n set.*

Proof. Pick by Theorem 5.4 $p_1, p_2, \ldots, p_n$ in S^* such that $A \in \prod_{i=1}^{n}(p_i^{-1}p_i)$. Since $\prod_{i=1}^{n}(p_i^{-1}p_i)$ is an ultrafilter, there exists $B \in \mathcal{F}$ such that $B \in \prod_{i=1}^{n}(p_i^{-1}p_i)$. Applying Theorem 5.4 again, we have that B is a Δ^n set. □

By contrast with the situation regarding the IP^n property, we shall show in Theorem 6.25 that in $(\mathbb{N}, +)$ there is no relationship whatsoever between the properties Δ^n and Δ^k when $n \neq k$.

6. Density recurrent, polynomial recurrent, and Δ^n sets in $\mathbb{N}$

We begin this section by showing that $\mathcal{DR}(\mathbb{N}, +)$ has substantial multiplicative structure. (And consequently, by Theorem 3.15, so does $A - A$ whenever $A \subseteq \mathbb{N}$ with $d(A) > 0$.)

Theorem 6.1. *$\mathcal{DR}(\mathbb{N}, +)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.*

Proof. Let $p \in \beta\mathbb{N}$ and $q \in \mathcal{DR}(\mathbb{N}, +)$. To see that $p \cdot q \in \mathcal{DR}(\mathbb{N}, +)$, let $B \in p \cdot q$. To see that B is a density recurrent set let $A \subseteq \mathbb{N}$ such that $d(A) > 0$. Since $B \in p \cdot q$, pick $m \in \mathbb{N}$ such that $m^{-1}B \in q$. Pick $t \in \{0, 1, \ldots, m-1\}$ such that $d\big(A \cap (m\mathbb{N} + t)\big) > 0$ and let $C = \{n \in \mathbb{N} : mn + t \in A\}$. Then $d(C) > 0$ so pick $n \in m^{-1}B$ such that $d\big(C \cap (-n + C)\big) > 0$. Then $d\big(mC \cap (-mn + mC)\big) > 0$ and $mC \cap (-mn + mC) \subseteq (-t + A) \cap (-mn - t + A)$ so $d\big((-t + A) \cap (-mn - t + A)\big) > 0$ and by Theorem 2.3 $d\big(A \cap (-mn + A)\big) = d\big((-t + A) \cap (-mn - t + A)\big)$. □

We now turn our attention to sets of multiple recurrence, establishing that much, but not all, of the structure of $\mathcal{DR}(\mathbb{N})$ carries over to the set of ultrafilters all of whose members satisfy a strong multiple recurrence property.

Definition 6.2. $\mathcal{R} = \{g : g$ is a polynomial with rational coefficients, $g[\mathbb{Z}] \subseteq \mathbb{Z}$, and $g(0) = 0\}$.

Theorem 6.3. *Let $F \in \mathcal{P}_f(\mathcal{R})$ and let $A \subseteq \mathbb{N}$ with $d(A) > 0$. Then*

$$\{n \in \mathbb{N} : d\big(\bigcap_{g\in F} \big((A - g(n))\big) > 0\}$$

is an IP* set.

Proof. [8, Theorem 7.3]. $\square$

Notice that, since the function $\overline{0} \in \mathcal{R}$, the assertion that for each $F \in \mathcal{P}_f(\mathcal{R})$, $\{n \in \mathbb{N} : d\big(\bigcap_{g\in F} (A - g(n)\big) > 0\}$ is an IP* set is the same as the assertion that for each $F \in \mathcal{P}_f(\mathcal{R})$, $\{n \in \mathbb{N} : d\big(A \cap \bigcap_{g\in F} (A - g(n)\big) > 0\}$ is an IP* set.

Given a set X and $n \in \mathbb{N}$, $[X]^n = \{A \subseteq X : |A| = n\}$.

Definition 6.4. (a) Let $n \in \mathbb{N}$ and let $B \subseteq \mathbb{N}$. Then B is a *polynomial n-recurrent set* if and only if whenever $A \subseteq \mathbb{N}$ with $d(A) > 0$, and $F \in [\mathcal{R}]^n$, there exists $k \in B$ such that

$$d\big(A \cap \bigcap_{g\in F} \big(A - g(k)\big)\big) > 0\,.$$

(b) Let $n \in \mathbb{N}$. Then $\mathcal{PR}_n = \{p \in \beta\mathbb{N} : (\forall B \in p)(B$ is a polynomial n-recurrent set$)\}$.

(c) $\mathcal{PR} = \bigcap_{n=1}^{\infty} \mathcal{PR}_n$.

Theorem 6.5. *Let $n \in \mathbb{N}$. Then $\mathcal{PR}_n$ is a subsemigroup of $(\beta\mathbb{N}, +)$ containing the idempotents, and consequently so is $\mathcal{PR}$.*

Proof. By Theorem 6.3, $\mathcal{PR}_n$ contains the idempotents and, in particular, $\mathcal{PR}_n \neq \emptyset$.

Now let $p, q \in \mathcal{PR}_n$ and let $B \in p + q$. To see that B is polynomial n-recurrent, let $A \subseteq \mathbb{N}$ and let $F \in [\mathcal{R}]^n$. Let $C = \{m \in \mathbb{N} : -m + B \in q\}$. Then $C \in p$ so pick $m \in C$ such that $d\big(A \cap \bigcap_{g\in F} \big(A - g(m)\big)\big) > 0$. Let

$$D = A \cap \bigcap_{g\in F} \big(A - g(m)\big)\,.$$

For $g \in F$, define $h_g(x) = g(m + x) - g(m)$ and let $H = \{h_g : g \in F\}$. Then $H \in [\mathcal{R}]^n$. Pick $k \in -m + B$ such that $d\big(D \cap \bigcap_{g\in F} \big(D - h_g(k)\big)\big) > 0$. Then $m + k \in B$ and $D \cap \bigcap_{g\in F} \big(D - h_g(k)\big) \subseteq A \cap \bigcap_{g\in F} \big(A - g(m + k)\big)$, so $d\big(A \cap \bigcap_{g\in F} \big(A - g(m + k)\big)\big) > 0$. $\square$

Theorem 6.6. *Let $n \in \mathbb{N}$ and let $p, q \in \mathcal{PR}_n$. Then $-p + q \in \mathcal{PR}_n$. Therefore, if $p, q \in \mathcal{PR}$, so is $-p + q$.*

Proof. Let $B \in -p + q$. To see that B is polynomial n-recurrent, let $A \subseteq \mathbb{N}$ and let $F \in [\mathcal{R}]^n$. For $g \in F$, let $f_g(x) = g(-x)$. Let $C = \{m \in \mathbb{N} : m + B \in q\}$. Then $C \in p$ so pick $m \in C$ such that $d\big(A \cap \bigcap_{g\in F} \big(A - f_g(m)\big)\big) > 0$. Let $D = A \cap \bigcap_{g\in F} \big(A - f_g(m)\big)$. For $g \in F$, define $h_g(x) = g(x - m) - f_g(m)$ and let $H = \{h_g : g \in F\}$. Then $H \in [\mathcal{R}]^n$. Pick $k \in m + B$ such that $d\big(D \cap \bigcap_{g\in F} \big(D - h_g(k)\big)\big) > 0$. Then $k - m \in B$ and $D \cap \bigcap_{g\in F} \big(D - h_g(k)\big) \subseteq A \cap \bigcap_{g\in F} \big(A - g(k - m)\big)$, so $d\big(A \cap \bigcap_{g\in F} \big(A - g(k - m)\big)\big) > 0$. $\square$

Recall from Theorem 3.14 that whenever $p \in \mathbb{N}^*$, $-p + p \in \mathcal{DR}(\mathbb{N})$. We shall see in Corollary 6.20 that there exists $p \in \mathbb{N}^*$ such that $-p + p \notin \mathcal{PR}$. We shall see now that $\mathcal{PR}$ does share with $\mathcal{DR}$ the property of being a left ideal of $(\beta\mathbb{N}, \cdot)$.

Theorem 6.7. *Let $n \in \mathbb{N}$. Then $\mathcal{PR}_n$ is a left ideal of $(\beta\mathbb{N}, \cdot)$, and consequently so is $\mathcal{PR}$.*

Proof. Let $p \in \beta\mathbb{N}$ and let $q \in \mathcal{PR}_n$. Let $B \in p \cdot q$. To see that B is polynomial n-recurrent, let $A \subseteq \mathbb{N}$ and let $F \in [\mathcal{R}]^n$. Pick $m \in \mathbb{N}$ such that $m^{-1}B \in q$. For $g \in F$, define $f_g \in \mathcal{R}$ by $f_g(x) = g(mx)$. Pick $k \in m^{-1}B$ such that

$$d\big(A \cap \textstyle\bigcap_{g\in F}\left(A - f_g(k)\right)\big) > 0\,.$$

Then $mk \in B$ and $d\big(A \cap \bigcap_{g\in F}\left(A - g(mk)\right)\big) > 0$. □

As a consequence of Theorem 3.14, we have that $\mathcal{DR}(\mathbb{N},+)$ is a subsemigroup of $(\beta\mathbb{N},+)$ containing the idempotents, containing all elements of the form $-p+p$ for $p \in \mathbb{N}^*$, and closed under subtraction with the negative term on the left. By Theorem 6.1 we have that $\mathcal{DR}(\mathbb{N},+)$ is also a left ideal of $(\beta\mathbb{N},\cdot)$. And we have just seen that $\mathcal{PR}$ shares all of these properties except that $-p+p$ need not be in $\mathcal{PR}$ for all $p \in \mathbb{N}^*$. Therefore, $\mathcal{PR}$ contains all polynomials formed from additive idempotents as long as the rightmost coefficient is positive. For example, if p, q, and r are aditive idempotents, then $3pq - 2qr + rqp \in \mathcal{PR}$. It will also contain things which one does not usually refer to as polynomials, such as $p(q+r)$. (This is not the same as $pq + pr$. See [15, Corollary 13.27].) In particular, if $A \subseteq \mathbb{N}$ and $d(A) > 0$, then $A - A$ is a member of all such expressions.

Given a sequence corresponding to each variable in the polynomial, sums of a certain form must lie in any member of the polynomial. We make this statement precise in Theorem 6.10 below. This result is due to Kendall Williams and forms part of his dissertation at Howard University. We are grateful for his permission to present the theorem and its proof here.

In the following lemmas, the closure is taken in $\beta\mathbb{Q}_d$, where $\mathbb{Q}_d$ is the set of rationals with the discrete topology. If the given sequences are sequences of integers, of course one will have each $p_j \in \beta\mathbb{Z}$.

Because of the generality of Theorem 6.10, it can be a bit difficult to understand what it says. The reader may wish to bear in mind the following special case. Let $g(z_1,z_2,z_3) = -\frac{2}{3}z_1z_3 + z_3z_2 + 3z_1z_1z_3 + z_2z_1$. Assume that for $j \in \{1,2,3\}$, $\langle x_{j,t}\rangle_{t=1}^\infty$ is a sequence in $\mathbb{N}$ and $p_j \in \bigcap_{l=1}^\infty \overline{FS(\langle x_{j,t}\rangle_{t=l}^\infty)}$. Given $F,G \in \mathcal{P}_f(\mathbb{N})$, write $F < G$ to mean $\max F < \min G$. Then Theorem 6.10 asserts that

$$\begin{aligned}&\{-\tfrac{2}{3}(\textstyle\sum_{t\in F_1} x_{1,t})(\sum_{t\in F_2} x_{3,t}) + (\sum_{t\in F_3} x_{3,t})(\sum_{t\in F_4} x_{2,t})\\ &+3(\textstyle\sum_{t\in F_5} x_{1,t})(\sum_{t\in F_6} x_{1,t})(\sum_{t\in F_7} x_{3,t}) + (\sum_{t\in F_8} x_{2,t})(\sum_{t\in F_9} x_{1,t}) :\\ &\text{each } F_i \in \mathcal{P}_f(\mathbb{N}) \text{ and } F_1 < F_2 < \ldots < F_9\} \in g(p_1,p_2,p_3)\,.\end{aligned}$$

In particular, if p_1, p_2, and p_3 are idempotents, then the listed set will be a polynomial n-recurrent set for each n.

Lemma 6.8. *Let $m,k,s \in \mathbb{N}$ and for $j \in \{1,2,\ldots,k\}$, let $\langle x_{j,t}\rangle_{t=1}^\infty$ be a sequence in $\mathbb{Q}$ and let $p_j \in \bigcap_{l=1}^\infty \overline{FS(\langle x_{j,t}\rangle_{t=l}^\infty)}$. Let $a \in \mathbb{Q}\setminus\{0\}$, let $f : \{1,2,\ldots,m\} \to \{1,2,\ldots,k\}$, and let $s \in \mathbb{N}$. Then*

$$\begin{aligned}&\{a(\textstyle\sum_{t\in F_1} x_{f(1),t})\cdots(\sum_{t\in F_m} x_{f(m),t}) : \text{each } F_i \in \mathcal{P}_f(\mathbb{N}) \text{ and}\\ &\{s\} < F_1 < \ldots < F_m\} \in ap_{f(1)}\cdots p_{f(m)}\,.\end{aligned}$$

Proof. We proceed by induction on m. If $m = 1$, we have that $FS(\langle x_{f(1),t}\rangle_{t=s+1}^\infty) \in p_{f(1)}$ so $\{a(\sum_{t\in F} x_{f(1),t}) : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } s < \min F\} \in ap_{f(1)}$.

Now assume that $m > 1$ and the result holds for $m-1$. let

$$\begin{aligned} B = \{&a(\textstyle\sum_{t\in F_1} x_{f(1),t})\cdots(\sum_{t\in F_m} x_{f(m),t}) : \\ &\text{each } F_i \in \mathcal{P}_f(\mathbb{N}) \text{ and } \{s\} < F_1 < \ldots < F_m\} \text{ and let} \\ C = \{&a(\textstyle\sum_{t\in F_1} x_{f(1),t})\cdots(\sum_{t\in F_{m-1}} x_{f(m-1),t}) : \\ &\text{each } F_i \in \mathcal{P}_f(\mathbb{N}) \text{ and } \{s\} < F_1 < \ldots < F_{m-1}\}\,. \end{aligned}$$

Then by assumption $C \in ap_{f(1)}\cdots p_{f(m-1)}$. We claim that

$$C \subseteq \{y \in \mathbb{Q} : y^{-1}B \in p_{f(m)}\}\,,$$

so that $B \in ap_{f(1)}\cdots p_{f(m)}$ as required. To this end let $y \in C$ and pick $F_1, F_2, \ldots, F_{m-1} \in \mathcal{P}_f(\mathbb{N})$ such that $\{s\} < F_1 < \ldots < F_{m-1}$ and

$$y = a(\textstyle\sum_{t\in F_1} x_{f(1),t})\cdots(\sum_{t\in F_{m-1}} x_{f(m-1),t})\,.$$

Let $r = \max F_{m-1}$. Then $FS(\langle x_{f(m),t}\rangle_{t=r+1}^{\infty}) \in p_{f(m)}$ and $FS(\langle x_{f(m),t}\rangle_{t=r+1}^{\infty}) \subseteq y^{-1}B$. □

Lemma 6.9. *Let $k, m \in \mathbb{N}$, let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$, and for $j \in \{1, 2, \ldots, k\}$, let $\langle x_{j,t}\rangle_{t=1}^{\infty}$ be a sequence in $\mathbb{Q}$ and let $p_j \in \bigcap_{l=1}^{\infty} \overline{FS(\langle x_{j,t}\rangle_{t=l}^{\infty})}$. Let $a \in \mathbb{Q}\setminus\{0\}$, let $q \in \beta\mathbb{Q}_d$, let $D \in q$, and let $\varphi : D \to \mathbb{N}$. Then*

$$\begin{aligned} &\{y + a(\textstyle\sum_{t\in F_1} x_{f(1),t})\cdots(\sum_{t\in F_m} x_{f(m),t}) : y \in D, \text{ each } F_i \in \mathcal{P}_f(\mathbb{N}), \text{ and} \\ &\{\varphi(y)\} < F_1 < \ldots < F_m\} \in q + ap_{f(1)}\cdots p_{f(m)}\,. \end{aligned}$$

Proof. Let

$$\begin{aligned} B = \{&y + a(\textstyle\sum_{t\in F_1} x_{f(1),t})\cdots(\sum_{t\in F_m} x_{f(m),t}) : \\ &y \in D, \text{ each } F_i \in \mathcal{P}_f(\mathbb{N}), \text{ and } \{\varphi(y)\} < F_1 < \ldots < F_m\} \end{aligned}$$

We claim that $D \subseteq \{y \in \mathbb{Q} : -y + B \in ap_{f(1)}\cdots p_{f(m)}\}$ so that

$$B \in q + ap_{f(1)}\cdots p_{f(m)}\,.$$

So let $y \in D$ and let

$$\begin{aligned} C = \{&a(\textstyle\sum_{t\in F_1} x_{f(1),t})\cdots(\sum_{t\in F_{m-1}} x_{f(m),t}) : \\ &\text{each } F_i \in \mathcal{P}_f(\mathbb{N}) \text{ and } \{\varphi(y)\} < F_1 < \ldots < F_m\}\,. \end{aligned}$$

By Lemma 6.8, $C \in ap_{f(1)}\cdots p_{f(m)}$ and $C \subseteq -y + B$. □

In the statement of the following theorem, if

$$g(z_1, z_2, z_3) = -\tfrac{2}{3}z_1z_3 + z_3z_2 + 3z_1z_1z_3 + z_2z_1$$

as in the paragraph before Lemma 6.8, then

$$h_g(y_1, y_2, \ldots, y_m) = -\tfrac{2}{3}y_1y_2 + y_3y_4 + 3y_5y_6y_7 + y_8y_9$$

and the function $f = \{(1,1), (2,3), (3,3), (4,2), (5,1), (6,1), (7,3), (8,2), (9,1)\}$.

We do not demand that each of the listed variables occur in g.

Theorem 6.10 (Kendall Williams). *Let $k \in \mathbb{N}$. For $j \in \{1, 2, \ldots, k\}$, let $\langle x_{j,t}\rangle_{t=1}^{\infty}$ be a sequence in $\mathbb{Q}$ and let $p_j \in \bigcap_{l=1}^{\infty} \overline{FS(\langle x_{j,t}\rangle_{t=l}^{\infty})}$. Let $g(z_1, z_2, \ldots, z_k)$ be a polynomial with rational coefficients. Let m be the number of occurrences of a variable in g, and let $h_g(y_1, y_2, \ldots, y_m)$ be the polynomial obtained by replacing the i^{th} occurrence of a variable by y_i. Define $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$*

by $f(i) = j$ if the i^{th} occurrence of a variable is z_j. (Then $g(z_1, z_2, \ldots, z_k) = h_g(z_{f(1)}, z_{f(2)}, \ldots, z_{f(m)})$. Let

$$B = \{h_g(\textstyle\sum_{t\in F_1} x_{f(1),t}, \ldots, \sum_{t\in F_m} x_{f(m),t}) : \\ \text{each } F_i \in \mathcal{P}_f(\mathbb{N}), \text{ and } F_1 < \ldots < F_m\}\,.$$

Then $B \in g(p_1, p_2, \ldots, p_k)$.

Proof. We proceed by induction on the number of terms in g. If g has one term, the result follows from Lemma 6.8. So assume that g has $n > 1$ terms and the result is valid for polynomials with $n-1$ terms.

Let r be the number of occurrences of variables in the n^{th} term of g, so that this term is $a_n z_{f(m-r+1)} z_{f(m-r+2)} \cdots z_{f(m)}$. Let $\widehat{g}$ consist of the first $n-1$ terms of g, so that $\widehat{g}(z_1, z_2, \ldots, z_k) = h_{\hat{g}}(y_1, y_2, \ldots, y_{m-r})$. Let

$$D = \{h_{\hat{g}}(\textstyle\sum_{t\in F_1} x_{f(1),t}, \ldots, \sum_{t\in F_{m-r}} x_{f(m-r),t}) : \\ \text{each } F_i \in \mathcal{P}_f(\mathbb{N}), \text{ and } F_1 < \ldots < F_{m-r}\}\,.$$

Then by assumption $D \in \widehat{g}(p_1, p_2, \ldots, p_k)$. Also,

$$g(p_1, p_2, \ldots, p_k) = \widehat{g}(p_1, p_2, \ldots, p_k) + a_n p_{f(m-r+1)} p_{f(m-r+2)} \cdots p_{f(m)}\,.$$

Given $y \in D$, pick $F_1, F_2, \ldots, F_{m-r} \in \mathcal{P}_f(\mathbb{N})$ with $F_1 < F_2 < \ldots < F_{m-r}$ and define $\varphi(y) = \max F_{m-r}$.

Let

$$C = \{y + a_n(\textstyle\sum_{t\in F_{m-r+1}} x_{f(m-r+1),t}) \cdots (\sum_{t\in F_m} x_{f(m),t}) : \\ y \in D, \text{ each } F_i \in \mathcal{P}_f(\mathbb{N}), \text{ and } \{\varphi(y)\} < F_{m-r+1} < \ldots < F_m\}\,.$$

By Lemma 6.9, $C \in g(p_1, p_2, \ldots, p_k)$ and $C \subseteq B$. □

The following example of the sort of combinatorial consequences of Theorems 6.5, 6.7, and 6.10 is a very special case of a general phenomenon.

Corollary 6.11. *Let $\langle x_t\rangle_{t=1}^\infty$, $\langle y_t\rangle_{t=1}^\infty$, and $\langle w_t\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$. Let*

$$B = \{2(\textstyle\sum_{t\in F_1} x_t)(\sum_{t\in F_2} y_t) + 3(\sum_{t\in F_3} w_t)(\sum_{t\in F_4} w_t)(\sum_{t\in F_5} x_t) : \\ \text{each } F_i \in \mathcal{P}_f(\mathbb{N}) \text{ and } F_1 < F_2 < F_3 < F_4 < F_5\}\,.$$

Then B is a polynomial n-recurrent set for every $n \in \mathbb{N}$.

Proof. Let $g(z_1, z_2, z_3) = 2z_1z_2 + 3z_3z_3z_1$. Pick by [15, Lemma 5.11] idempotents $p \in \bigcap_{m=1}^\infty \overline{FS(\langle x_t\rangle_{t=m}^\infty)}$, $q \in \bigcap_{m=1}^\infty \overline{FS(\langle y_t\rangle_{t=m}^\infty)}$, and $r \in \bigcap_{m=1}^\infty \overline{FS(\langle w_t\rangle_{t=m}^\infty)}$. By Theorem 6.10, $B \in g(p, q, r)$ and by Theorems 6.5 and 6.7, $g(p, q, r) \in \mathcal{PR}$. □

The assertion that a set B "is a polynomial n-recurrent set for every $n \in \mathbb{N}$" is the same as saying that for each $H \in \mathcal{P}_f(\mathcal{R})$ and each $A \subseteq S$ with $d(A) > 0$, $B \cap \{n \in \mathbb{N} : d\big(\bigcap_{g\in H} (A - g(n))\big) > 0\} \neq \emptyset$.

Corollary 6.12. *Let $\langle x_t\rangle_{t=1}^\infty$, $\langle y_t\rangle_{t=1}^\infty$, and $\langle w_t\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$, let $A \subseteq \mathbb{N}$ with $d(A) > 0$, and let $H \in \mathcal{P}_f(\mathcal{R})$. There exist sum subsytems $\langle u_t\rangle_{t=1}^\infty$ of $\langle x_t\rangle_{t=1}^\infty$, $\langle v_t\rangle_{t=1}^\infty$ of $\langle y_t\rangle_{t=1}^\infty$, and $\langle z_t\rangle_{t=1}^\infty$ of $\langle w_t\rangle_{t=1}^\infty$ such that*

$$B = \{2(\textstyle\sum_{t\in F_1} u_t)(\sum_{t\in F_2} v_t) + 3(\sum_{t\in F_3} z_t)(\sum_{t\in F_4} z_t)(\sum_{t\in F_5} u_t) : \\ \text{each } F_i \in \mathcal{P}_f(\mathbb{N}) \text{ and } F_1 < F_2 < F_3 < F_4 < F_5\} \\ \subseteq \{n \in \mathbb{N} : d\big(\textstyle\bigcap_{g\in H} (A - g(n))\big) > 0\}\,.$$

Proof Sketch. Use Theorem 3.19 as in the proof of Theorem 3.20. □

A stronger result than that of Corollary 6.12 is available. According to [8, Theorem 7.3] one can demand that $F_1 = F_2 = F_3 = F_4 = F_5$, or that just some of these sets are equal. We note that such a conclusion cannot be derived from the fact that $B \cap \{n \in \mathbb{N} : d\big(\bigcap_{g\in H} (A - g(n)\big) > 0\} \neq \emptyset$ for each choice of $\langle x_t\rangle_{t=1}^\infty$, $\langle y_t\rangle_{t=1}^\infty$, and $\langle w_t\rangle_{t=1}^\infty$. For example, let $C = \mathbb{N} \setminus \{x^2 : x \in \mathbb{N}\}$. Then given any sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$, one has $\{(\sum_{t\in F_1} x_t)(\sum_{t\in F_2} x_t) : F_1, F_2 \in \mathcal{P}_f(\mathbb{N})$ and $F_1 < F_2\} \cap C \neq \emptyset$ and so given any sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$, there will exist a sum subsystem $\langle y_n\rangle_{n=1}^\infty$ of $\langle x_n\rangle_{n=1}^\infty$ such that $\{(\sum_{t\in F_1} y_t)(\sum_{t\in F_2} y_t) : F_1, F_2 \in \mathcal{P}_f(\mathbb{N})$ and $F_1 < F_2\} \subseteq C$. One clearly cannot require $F_1 = F_2$.

Many other results can be proved in a similar manner. For example, if $p, q \in \mathbb{N}^*$ and r is an idempotent in $\mathbb{N}^*$, then $p(-q + q) + 3pr \in \mathcal{DR}(\mathbb{N})$. As a consequence, we get the following theorem, whose proof we leave to the reader.

Theorem 6.13. *Let $\langle x_n\rangle_{n=1}^\infty$, $\langle y_n\rangle_{n=1}^\infty$, and $\langle w_n\rangle_{n=1}^\infty$ be injective sequences in $\mathbb{N}$. Then for each $n \in \mathbb{N}$, $\{x_j(y_m - y_k) + 3x_l(\sum_{t\in F} w_t) : j, k, m, l \in \mathbb{N}, F \in \mathcal{P}_f(\mathbb{N})$, and $j < k < m < l < \min F\}$ is a polynomial n-recurrent set.*

There is an intricate relationship between members of polynomials on $\beta\mathbb{N}$ and the ability to find expressions using sum subsystems and subsequences of specified sequences in certain subsets of $\mathbb{N}$. It is our intention to explore this relationship in quite some detail in a forthcoming paper which we expect to write with Kendall Williams. We shall illustrate aspects of this relationship with a few results involving a specific polynomial, namely $f(p, q) = 2p + qp$.

Theorem 6.14. *Let p and q be idempotents in $\beta\mathbb{N}$ and let $A \in 2p + qp$. There exist sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that*

$$\{2\textstyle\sum_{t\in F_1} x_t + (\sum_{t\in F_2} y_t)(\sum_{t\in F_3} x_t) : F_1, F_2, F_3 \in \mathcal{P}_f(\mathbb{N}) \text{ and } F_1 < F_2 < F_3\} \subseteq A\,.$$

Proof. Let $B = 2^{-1}\{x \in \mathbb{N} : -x + A \in qp\}$. Then $B \in p$. Let

$$B^\star(p) = \{x \in B : -x + B \in p\}\,.$$

By [15, Lemma 4.14], $B^\star(p) \in p$ and if $x \in B^\star(p)$, then $-x + B^\star(p) \in p$. Pick $x_1 \in B^\star(p)$ and let $C_1 = -2x_1 + A$. Then $C_1 \in qp$. Let $D_1 = \{y \in \mathbb{N} : y^{-1}C_1 \in p\}$. Then $D_1 \in q$. Pick $y_1 \in D_1^\star(q)$ and let $E_1 = y_1^{-1}C_1$. Then $E_1 \in p$.

Pick $x_2 \in B^\star(p) \cap \big(-x_1 + B^\star(p)\big) \cap E_1^\star(p)$. Let

$$C_2 = (-2x_1 + A) \cap (-2x_2 + A) \cap (-2(x_1 + x_2) + A)\,.$$

Then $C_2 \in pq$. Let $D_2 = \{y \in \mathbb{N} : y^{-1}C_2 \in p\}$. Then $D_2 \in q$. Pick

$$y_2 \in D_2^\star(q) \cap \big(-y_1 + D_1^\star(q)\big)\,.$$

Let $E_2 = y_1^{-1}C_1 \cap (y_1 + y_2)^{-1}C_1 \cap y_2^{-1}C_2$. Then $E_2 \in p$.

Inductively, let $k \geq 2$ and assume that we have chosen $\langle x_t\rangle_{t=1}^k$, $\langle y_t\rangle_{t=1}^k$, $\langle C_t\rangle_{t=1}^k$, $\langle D_t\rangle_{t=1}^k$, and $\langle E_t\rangle_{t=1}^k$. For $l, m \in \{1, 2, \ldots, k\}$ with $l \leq m$, let

$$M_{l,m} = \{\textstyle\sum_{t\in F} x_t : \emptyset \neq F \subseteq \{l, l+1, \ldots, m\} \text{ and } l \in F\}$$

and let

$$N_{l,m} = \{\textstyle\sum_{t\in F} y_t : \emptyset \neq F \subseteq \{l, l+1, \ldots, m\} \text{ and } l \in F\}\,.$$

Assume that for each $m \in \{1, 2, \ldots, k\}$ the following induction hypotheses hold.

(1) If $l \in \{1, 2, \ldots, m\}$ and $z \in M_{l,m}$, then $z \in B^\star(p)$.

(2) If $m>1$, $l\in\{2,3,\ldots,m\}$, and $z\in M_{l,m}$, then $z\in E^\star_{l-1}(p)$.
(3) $C_m=\bigcap_{l=1}^m\bigcap_{z\in M_{l,m}}(-2z+A)$.
(4) $D_m=\{y\in\mathbb{N}:y^{-1}C_m\in p\}$ and $D_m\in q$.
(5) If $l\in\{1,2,\ldots,m\}$ and $z\in N_{l,m}$, then $z\in D^\star_l(q)$.
(6) $E_m=\bigcap_{l=1}^m\bigcap_{z\in N_{l,m}}z^{-1}C_l$ and $E_m\in p$.

All hypotheses are satisfied for $m=1$ and $m=2$. By hypothesis (1) we have $\bigcap_{l=1}^k\bigcap_{z\in M_{l,k}}\big(-z+B^\star(p)\big)\in p$. By hypothesis (2), $\bigcap_{l=2}^k\bigcap_{z\in M_{l,k}}\big(-z+E^\star_{l-1}(p)\big)\in p$. Pick $x_{k+1}\in B^\star(p)\cap\bigcap_{l=1}^k\bigcap_{z\in M_{l,k}}\big(-z+B^\star(p)\big)\cap\bigcap_{l=2}^k\bigcap_{z\in M_{l,k}}\big(-z+E^\star_{l-1}(p)\big)$. Then hypotheses (1) and (2) hold for $m=k+1$. In particular, if $l\in\{1,2,\ldots,k+1\}$ and $z\in M_{l,k+1}$, then $-2z+A\in qp$. Let $C_{k+1}=\bigcap_{l=1}^{k+1}\bigcap_{z\in M_{l,k+1}}(-2z+A)$ and let $D_{k+1}=\{y\in\mathbb{N}:y^{-1}C_{k+1}\in p\}$. Then $C_{k+1}\in qp$ so $D_{k+1}\in q$. By hypothesis (5) we have that $\bigcap_{l=1}^k\bigcap_{z\in N_{l,k}}\big(-z+D^\star_l(q)\big)\in q$. Pick $y_{k+1}\in D^\star_{k+1}\cap\bigcap_{l=1}^k\bigcap_{z\in N_{l,k}}\big(-z+D^\star_l(q)\big)$. Then hypotheses (3), (4), and (5) hold for $m=k+1$. Let $E_{k+1}=\bigcap_{l=1}^{k+1}\bigcap_{z\in N_{l,k+1}}z^{-1}C_l$ Given $l\in\{1,2,\ldots,k+1\}$ and $z\in N_{l,k+1}$ we have that $z\in D^\star_l(q)$ so $z^{-1}C_l\in p$. Thus $E_{k+1}\in p$.

The construction being complete, let $F_1,F_2,F_3\in\mathcal{P}_f(\mathbb{N})$ and assume that

$$\max F_1\le\min F_2\text{ and }\max F_2<\min F_3\,.$$

Let $l=\min F_3$. By hypothesis (2), $\sum_{t\in F_3}x_t\in E_{l-1}$. Let $u=\min F_2$. Then $\sum_{t\in F_2}y_t\in N_{u,l-1}$ so by hypothesis (6), $E_{l-1}\subseteq(\sum_{t\in F_2}y_t)^{-1}C_u$ so

$$(\textstyle\sum_{t\in F_2}y_t)(\sum_{t\in F_3}x_t)\in C_u\,.$$

Let $v=\min F_1$. Then $\sum_{t\in F_1}x_t\in M_{v,u}$ so by hypothesis (3), $C_u\subseteq-2(\sum_{t\in F_1}x_t)+A$ so $2(\sum_{t\in F_1}x_t)+(\sum_{t\in F_2}y_t)(\sum_{t\in F_3}x_t)\in A$ as required. □

Note that a set A satisfying any (and hence all) of the statements in the following theorem must be quite large. By way of contrast, any finite partition of $\mathbb{N}$ will yield some set which is a member of $2p+qp$ for any p and qp.

Theorem 6.15. *Let $A\subseteq\mathbb{N}$. The following statements are equivalent.*

(a) *Whenever p and q are idempotents in $(\beta\mathbb{N},+)$, $A\in 2p+qp$.*
(b) *Whenever $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ are sequences in $\mathbb{N}$, there exist $F_1,F_2,F_3\in\mathcal{P}_f(\mathbb{N})$ such that $F_1<F_2<F_3$ and $2\sum_{t\in F_1}x_t+(\sum_{t\in F_2}y_t)(\sum_{t\in F_3}x_t)\in A$.*
(c) *Whenever $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ are sequences in $\mathbb{N}$, there exist a sum subsystem $\langle u_n\rangle_{n=1}^\infty$ of $\langle x_n\rangle_{n=1}^\infty$ and a sum subsystem $\langle v_n\rangle_{n=1}^\infty$ of $\langle y_n\rangle_{n=1}^\infty$ such that*

$$\{2\textstyle\sum_{t\in F_1}u_t+(\sum_{t\in F_2}v_t)(\sum_{t\in F_3}u_t):F_1,F_2,F_3\in\mathcal{P}_f(\mathbb{N})\text{ and }F_1<F_2<F_3\}\subseteq A\,.$$

Proof. (a) ⇒ (b). Pick by [15, Lemma 5.11] idempotents $p\in\bigcap_{m=1}^\infty\overline{FS(\langle x_t\rangle_{t=m}^\infty)}$ and $q\in\bigcap_{m=1}^\infty\overline{FS(\langle y_t\rangle_{t=m}^\infty)}$. Let $g(z_1,z_2)=2z_1+z_2z_1$. By Theorem 6.10 we have that

$$\begin{aligned}&\{2\textstyle\sum_{t\in F_1}x_t+(\sum_{t\in F_2}y_t)(\sum_{t\in F_3}x_t):\\&F_1,F_2,F_3\in\mathcal{P}_f(\mathbb{N})\text{ and }F_1<F_2<F_3\}\in 2p+qp\,.\end{aligned}$$

Thus this set has a nonempty intersection with A.

(b) $\Rightarrow$ (a). Let p and q be idempotents in $(\beta\mathbb{N},+)$ and suppose that $A \notin 2p+pq$. By Theorem 6.14 there exist sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that

$$\{2\textstyle\sum_{t\in F_1} x_t + (\sum_{t\in F_2} y_t)(\sum_{t\in F_3} x_t) :\\ F_1,F_2,F_3 \in \mathcal{P}_f(\mathbb{N}) \text{ and } F_1 < F_2 < F_3\} \subseteq \mathbb{N}\setminus A\,.$$

But then $A \cap (\mathbb{N}\setminus A) \neq \emptyset$, a contradiction.

(b) $\Rightarrow$ (c). Use Theorem 3.19 as in the proof of Theorem 3.20.

(c) $\Rightarrow$ (b). This is trivial. $\square$

We now turn our attention to Δ^n sets in $\mathbb{N}$, developing some strong contrasts with IP^n sets. Recall that by Corollary 5.6, if $A \subseteq \mathbb{N}$ and $d(A) > 0$, then $A - A$ is a Δ^{n*} set for each $n \in \mathbb{N}$. Further $A - A = \{x \in \mathbb{N} : A \cap (A - x) \neq \emptyset\}$. We have by Lemma 4.12 that if $B \subseteq \mathbb{N}$ and $d(B) > 0$, then $\{x \in \mathbb{N} : d\big(B \cap (B-x) \cap (B-2x)\big) > 0\}$ is an IP* set. We shall see in Corollary 6.19 that for each $n \in \mathbb{N}$, there is a subset $B \subseteq \mathbb{N}$ such that $d(B) > 0$ and $\{x \in \mathbb{N} : B \cap (B-x) \cap (B-2x) \neq \emptyset\}$ is not a Δ^{n*} set.

The construction used in Theorem 6.18 is a minor modification of a construction in [12, pp. 177-178]. Therein we let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, representing the points of $\mathbb{T}$ by elements of $[0,1)$. Given $\theta \in [0,1)$, we let $||\theta|| = \min\{\theta, 1-\theta\}$. Further, given $\theta, \phi \in [0,1)$, $\theta + \phi$ denotes the addition in $\mathbb{T}$, that is, the element of $[0,1)$ congruent to the ordinary sum mod 1.

Lemma 6.16. *Let α be an irrational element of $[0,1)$, let $\beta \in (0,1)$, and let $0 < \delta < \epsilon$. For each $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with $n > m$ such that $||n^2\alpha - \beta|| < \epsilon$ and $||n\alpha|| < \delta$.*

Proof. Define a transformation T of $\mathbb{T}\times\mathbb{T}$ by $T(\theta,\phi) = (\theta+\alpha, \theta+\phi)$. Let $\mu = \min\{\delta, \frac{\epsilon-\delta}{2}\}$. By [12, Lemma 1.25] $\{T^n(0,0) : n \in \mathbb{N}\}$ is dense in $\mathbb{T}\times\mathbb{T}$ and for $n \in \mathbb{N}$, $T^n(0,0) = (n\alpha, \binom{n}{2}\alpha)$. Pick $n > m$ such that $||n\alpha|| < \mu$ and $||\binom{n}{2}\alpha - \frac{\beta}{2}|| < \mu$. Then $||n^2\alpha - \beta|| \leq ||(n^2-n)\alpha - \beta|| + ||n\alpha|| < \epsilon - \delta + \delta$. $\square$

Lemma 6.17. *Let $(X, \mathcal{B}, \mu)$ be a probability measure space, let $a > 0$, and assume that for each $n \in \mathbb{N}$, $A_n \in \mathcal{B}$ and $d(A_n) = a$. Then there exists $C \subseteq \mathbb{N}$ such that $d(C) > 0$ and for any $F \in \mathcal{P}_f(C)$, $\mu(\bigcap_{n\in F} A_n) > 0$.*

Proof. [2, Theorem 1.1]. $\square$

Theorem 6.18. *Let $k \in \mathbb{N}$. There exist a set $B \subseteq \mathbb{N}$ such that $d(B) > 0$ and an increasing sequence $\langle s_t\rangle_{t=1}^\infty$ in $\mathbb{N}$ such that*

$$\big\{\textstyle\sum_{t\in F} j_t s_t : F \in \mathcal{P}_f(\mathbb{N}),\, |F| = k, \text{ and each } j_t \in \{1,-1\}\big\} \cap\\ \{n \in \mathbb{N} : d\big(B \cap (B-n) \cap (B-2n)\big) \neq \emptyset\} = \emptyset\,.$$

Proof. Let $\epsilon = \frac{1}{12+2k^2}$. Choose $s_1 \in \mathbb{N}$ such that $||s_1^2\alpha - \frac{1}{2k}|| < \epsilon$. Inductively, having chosen increasing $s_1, s_2, \ldots, s_t$, choose by Lemma 6.16 $s_{t+1} > s_t$ such that

$$||s_{t+1}^2\alpha - \frac{1}{2k}|| < \epsilon \text{ and } ||s_{t+1}\alpha|| < \frac{\epsilon}{s_t}\,.$$

(So for $m \in \{1, 2, \ldots, t\}$, $||s_m s_{t+1}\alpha|| < \epsilon$.) Then, for any $F \in \mathcal{P}_f(\mathbb{N})$ with $|F| = k$, and any choice of $j_t \in \{1, -1\}$ for $t \in F$ we have

$$\begin{aligned} ||(\textstyle\sum_{t \in F} j_t s_t)^2 \alpha - \frac{1}{2}|| &\leq \textstyle\sum_{t \in F} ||s_t^2 \alpha - \frac{1}{2k}|| + \sum\{||2 s_m s_t \alpha|| : m, t \in F \text{ and } m \neq t\} \\ &< k\epsilon + (k^2 - k)\epsilon \\ &= k^2 \epsilon \,. \end{aligned}$$

Now let $\mathbb{T} \times \mathbb{T}$ have normalized Lebesgue measure μ, so that $\mu(\mathbb{T} \times \mathbb{T}) = 1$, and let T be the transformation of $\mathbb{T} \times \mathbb{T}$ defined in the proof of Lemma 6.16. Let $A = \{(\theta, \phi) \in \mathbb{T} \times \mathbb{T} : ||\theta|| < \epsilon \text{ and } ||\phi|| < \epsilon\}$. Then $\mu(A) = 4\epsilon^2 > 0$.

Let $D = \{n \in \mathbb{N} : A \cap T^{-n}[A] \cap T^{-2n}[A] \neq \emptyset\}$. Then as shown in [12, page 178], if $n^2 \in D$, then $||n^2\alpha|| < 6\epsilon$. We claim that

$$D \cap \big\{ \textstyle\sum_{t \in F} j_t s_t : F \in \mathcal{P}_f(\mathbb{N}) \,, |F| = k, \text{ and each } j_t \in \{1, -1\} \big\} = \emptyset \,.$$

Indeed, suppose that n is in this intersection. Then as we saw above, $||n^2\alpha - \frac{1}{2}|| < k^2\epsilon$ while $||n^2\alpha|| < 6\epsilon$. So $\frac{1}{2} < (k^2 + 6)\epsilon = \frac{1}{2}$, a contradiction.

Now pick by Lemma 6.17 a set $B \subseteq \mathbb{N}$ such that $d(B) > 0$ and for any $F \in \mathcal{P}_f(B)$, $\mu(\bigcap_{n \in F} T^{-n}[A]) > 0$. We claim that B is as required. So suppose instead that we have $n \in \mathbb{N}$, $F \in \mathcal{P}_f(\mathbb{N})$ such that $|F| = k$ and for each $t \in F$, $j_t \in \{-1, 1\}$, $n = \sum_{t \in F} j_t s_t$, and $B \cap (B - n) \cap (B - 2n) \neq \emptyset$. Pick $x \in B \cap (B - n) \cap (B - 2n)$ and pick $y \in T^{-x}[A] \cap T^{-n-x}[A] \cap T^{-2n-x}[A]$. Then $T^x(y) \in A \cap T^{-n}[A] \cap T^{-2n}[A]$, so $n \in D$, a contradiction. □

Corollary 6.19. *Let $n \in \mathbb{N}$. There is a set $B \subseteq \mathbb{N}$ such that $d(B) > 0$ and $\{x \in \mathbb{N} : B \cap (B - x) \cap (B - 2x) \neq \emptyset\}$ is not a Δ^{n*} set.*

Proof. In Theorem 6.18, let $k = 2n$. Given $F \in \mathcal{P}_f(\mathbb{N})$ with $|F| = k$, let $t_1, t_2, \ldots, t_k$ be the elements of F in increasing order and for $i \in \{1, 2, \ldots, k\}$, let $j_{t_i} = (-1)^i$. □

Recall that by Lemma 3.13, if $p \in \mathbb{N}^*$, then $-p + p \in \mathcal{DR}(\mathbb{N})$. So the next corollary provides a contrast between $\mathcal{DR}(\mathbb{N})$ and $\mathcal{PR}$.

Corollary 6.20. *There exists $p \in \mathbb{N}^*$ such that $-p + p \notin \mathcal{PR}_2$.*

Proof. In Theorem 6.18 let $k = 2$ and pick $B \subseteq \mathbb{N}$ and an increasing sequence $\langle s_t \rangle_{t=1}^{\infty}$ in $\mathbb{N}$ such that $d(B) > 0$ and whenever $r < t$,

$$s_t - s_r \notin \{n \in \mathbb{N} : d\big(B \cap (B - n) \cap (B - 2n)\big) \neq \emptyset\} \,.$$

Pick $p \in \mathbb{N}^*$ such that $\{s_t : t \in \mathbb{N}\} \in p$. Then by Lemma 3.10,

$$C = \{s_t - s_r : r < t\} \in -p + p \,.$$

Thus $-p + p \notin \mathcal{PR}_2$. □

As our final contrast between IP^n sets and Δ^n sets, we show, as promised, that there is no relationship at all between Δ^n sets and Δ^k sets when $n \neq k$. We fix the following notation for the rest of this section.

Definition 6.21. Let $g < k \leq r$ in ω. Let

$$\begin{aligned} A_{r,k,g} = \{\textstyle\sum_{i=g+1}^{k} (2^{rm(i)+i} - 2^{rn(i)+i}) &: n(g+1), m(g+1), \ldots, n(k), m(k) \in \mathbb{N} \\ &\text{and } n(g+1) < m(g+1) < n(g+2) < \ldots < n(k) < m(k)\} \,. \end{aligned}$$

We have immediately that $A_{r,k,g}$ is a Δ^{k-g} set in $(\mathbb{N}, +)$. Notice also that any member of $A_{r,k,g}$ has a binary expansion with exactly $k-g$ blocks of 1's, and each of these blocks has length divisible by r.

Lemma 6.22. *Let $g<k\le r$ in ω and let $v>k-g$. Then $A_{r,k,g}$ is not a Δ^v set.*

Proof. It suffices to show that there do not exist $a\in\mathbb{N}$ and an increasing sequence $\langle x_{i,t}\rangle_{t=1}^{\infty}$ for each $i\in\{1,2,\ldots,k-g\}$ such that

$$\begin{aligned}&\{a+\textstyle\sum_{i=1}^{k-g}(x_{i,m(i)}-x_{i,n(i)}):\ n(1),m(1),n(2),\ldots,n(k-g),m(k-g)\in\mathbb{N}\\ &\text{and } n(1)<m(1)<n(2)<\ldots<n(k-g)<m(k-g)\}\subseteq A_{r,k,g}\end{aligned}$$

so suppose we have such a and such sequences. Pick $l_1\in\mathbb{N}$ such that $2^{l_1}>a$. Pick $n(1)<m(1)$ such that $x_{1,n(1)}\equiv x_{1,m(1)} \pmod{2^{l_1}+1}$. Given $i\in 1,2,\ldots,k-g-1$ and $m(i)$, pick l_{i+1} such that $2^{l_{i+1}}>x_{i,m(i)}$ and pick $n(i+1)$ and $m(i+1)$ such that $m(i)<n(i+1)<m(i+1)$ and

$$x_{i+1,n(i+1)}\equiv x_{i+1,m(i+1)} \pmod{2^{l_{i+1}}+1}\,.$$

Then the binary expansion of $a+\sum_{i=1}^{k-g}(x_{i,m(i)}-x_{i,n(i)})$ has at least $k-g+1$ blocks of 1's so $a+\sum_{i=1}^{k-g}(x_{i,m(i)}-x_{i,n(i)})\notin A_{r,k,g}$. □

Lemma 6.23. *Let $g<k\le r$ in ω with $k-g>1$. Then $A_{r,k,g}$ is not a Δ^1 set.*

Proof. Suppose that we have an increasing sequence $\langle y_t\rangle_{t=1}^{\infty}$ in $\mathbb{N}$ such that

$$\{y_s-y_t: s,t\in\mathbb{N} \text{ and } t<s\}\subseteq A_{r,k,g}\,.$$

For each $t\in\mathbb{N}\setminus\{1\}$, pick $n(t,g+1),m(t,g+1),n(t,g+2),\ldots,n(t,k),m(t,k)\in\mathbb{N}$ such that $n(t,g+1)<m(t,g+1)<n(t,g+2)<\ldots<n(t,k)<m(t,k)$ and $y_t-y_1=\sum_{i=g+1}^{k}(2^{rm(t,i)+i}-2^{rn(t,i)+i})$. We may presume by thinning the sequences that for each $i\in\{g+1,g+2,\ldots,k\}$, the sequence $\langle n(t,i)\rangle_{t=2}^{\infty}$ is either constant or strictly increasing and the sequence $\langle m(t,i)\rangle_{t=2}^{\infty}$ is either constant or strictly increasing. Further, if $\langle n(t,i)\rangle_{t=2}^{\infty}$ is constant, so are the sequences $\langle n(t,j)\rangle_{t=2}^{\infty}$ and $\langle m(t,j)\rangle_{t=2}^{\infty}$ for all $j<i$. And if $\langle m(t,i)\rangle_{t=2}^{\infty}$ is constant, so are the sequences $\langle n(t,j)\rangle_{t=2}^{\infty}$ for $j\le i$ and $\langle m(t,j)\rangle_{t=2}^{\infty}$ for $j<i$. We also know that the sequence $\langle m(t,k)\rangle_{t=2}^{\infty}$ is strictly increasing.

Therefore we must have either

(1) there is $l\in\{g+1,g+2,\ldots,k\}$ such that $\langle m(t,l)\rangle_{t=2}^{\infty}$ is strictly increasing and $\langle n(t,j)\rangle_{t=2}^{\infty}$ is constant for $j\le l$ and $\langle m(t,j)\rangle_{t=2}^{\infty}$ is constant for $j<l$, if any; or
(2) there is $l\in\{g+1,g+2,\ldots,k\}$ such that $\langle n(t,l)\rangle_{t=2}^{\infty}$ is strictly increasing and $\langle n(t,j)\rangle_{t=2}^{\infty}$ and $\langle m(t,j)\rangle_{t=2}^{\infty}$ are constant for $j<l$, if any.

Assume first that (1) holds. Pick t such that $m(t,l)>m(2,k)$. Then

$$\begin{aligned}y_t-y_2 = &\textstyle\sum_{i=l+1}^{k}(2^{rm(t,i)+i}-2^{rn(t,i)+i})+(2^{rm(t,l)+l}-2^{rm(2,k)+k})\\ &+\textstyle\sum_{i=l+1}^{k}(2^{rn(t,i)+i}-2^{rm(t,i-1)+i-1})\,.\end{aligned}$$

Since each block of 1's in the binary expansion of y_t-y_2 has length divisible by r, by considering the term $2^{rm(t,l)+l}-2^{rm(2,k)+k}$, we conclude that $l=k$. But then, $y_t-y_2=2^{rm(t,l)+l}-2^{rm(2,r)+k}$, so there is only one block of 1's in the binary expansion of y_t-y_2, while $k-g>1$, a contradiction.

Now assume that (2) holds. Pick t such that $n(t,l) > m(2,k)$. Then

$$\begin{aligned} y_t - y_2 = &\textstyle\sum_{i=l+1}^{k}(2^{rm(t,i)+i} - 2^{rn(t,i)+i}) + (2^{rm(t,l)+l} - 2^{rn(t,l)+l} - 2^{rm(2,k)+k}) \\ &+ \textstyle\sum_{i=l+1}^{k}(2^{rn(t,i)+i} - 2^{rm(t,i-1)+i-1}) + 2^{rn(2,l)+l}\,. \end{aligned}$$

Then the binary expansion of $y_t - y_2$ has $2(k-l)+3$ blocks of 1's, one of which has length 1, so $y_t - y_2 \notin A_{r,k,g}$. □

Lemma 6.24. *Let $i \leq r$ in $\mathbb{N}$ and let $\langle y_t\rangle_{t=1}^{\infty}$ be an increasing sequence in $\mathbb{N}$. If $\{y_b - y_a : a,b \in \mathbb{N} \text{ and } a < b\} \subseteq \{2^{rm+i} - 2^{rn+i} : m,n \in \mathbb{N} \text{ and } n < m\} = A_{r,i,i-1}$, then there is some $d \in \mathbb{Z}$ such that $\big\{t \in \mathbb{N} : y_t \in \{2^{rm+i} + d : m \in \mathbb{N}\}\big\}$ is infinite.*

Proof. For each $t \in \mathbb{N}\setminus\{1\}$ pick $n(t) < m(t)$ in $\mathbb{N}$ such that $y_t - y_1 = 2^{rm(t)+i} - 2^{rn(t)+i}$. By thinning the sequences we may assume that the sequence $\langle m(t)\rangle_{t=2}^{\infty}$ is strictly increasing and the sequence $\langle n(t)\rangle_{t=2}^{\infty}$ is either strictly increasing or constant. But if $\langle n(t)\rangle_{t=2}^{\infty}$ were strictly increasing, we could pick t such that $n(t) > m(2)$ so that $y_t - y_2 = 2^{rm(t)+i} - 2^{rn(t)+i} - 2^{rm(2)+i} + 2^{rn(2)+i}$, a number whose binary expansion has three blocks of 1's and is thus not in $A_{r,i,i-1}$. Thus we have some $c \in \mathbb{N}$ such that for each $t \in \mathbb{N}\setminus\{1\}$, $n(t) = c$ and so $y_t = 2^{rm(t)+i} + d$ where $d = y_1 - 2^{rc+i}$. □

Theorem 6.25. *Let $g < k \leq r$ in ω and let $v \in \mathbb{N}$. Assume that for each $j \in \{1,2,\ldots,v\}$, $\langle y_{j,t}\rangle_{t=1}^{\infty}$ is an increasing sequence in $\mathbb{N}$ and*

$$\begin{aligned} \{\textstyle\sum_{j=1}^{v}(y_{j,b(j)} - y_{j,a(j)}) &: a(1),b(1),a(2),\ldots,a(v),b(v) \in \mathbb{N} \\ &\text{and } a(1) < b(1) < a(2) < \ldots < a(v) < b(v)\} \subseteq A_{r,k,g}\,. \end{aligned}$$

Then $v = k - g$ and for each $j \in \{1,2,\ldots,v\}$ there exists $d_j \in \mathbb{Z}$ such that $\{t \in \mathbb{N} : y_{j,t} \in \{2^{rm+g+j} + d_j : m \in \mathbb{N}\}\}$ is infinite. In particular, if $v \neq k-g$, then $A_{r,k,g}$ is not a Δ^v set.

Proof. We have by Lemma 6.22 that if $v > k-g$, then $A_{r,k,g}$ is not a Δ^v set. Thus we shall assume that $v \leq k-g$ and prove the statement by induction on $k-g$. Assume first that $k-g = 1$, so that $v = 1$ and Lemma 6.24 applies.

Now assume that $k - g > 1$ and the statement holds for smaller values. We claim that for each $t < s$ in $\mathbb{N}$, there exist $u(t,s) \in \{1,2,\ldots,v-1\}$ and $n(t,s,1) < m(t,s,1) < n(t,s,2) < \ldots < n\big(t,s,u(t,s)\big) < m\big(t,s,u(t,s)\big)$ in $\mathbb{N}$ such that $y_{1,s} - y_{1,t} = \sum_{i=g+1}^{u(t,s)}(2^{rm(t,s,i)+i} - 2^{rn(t,s,i)+i})$. To this end let $t < s$ be given and pick $l \in \mathbb{N}$ such that $2^l > y_{1,s} - y_{1,t}$. For $j \in \{2,3,\ldots,v\}$ pick $a(j)$ and $b(j)$ such that $y_{j,a(j)} \equiv y_{j,b(j)} \pmod{2^{l+1}}$ and $s < a(2) < b(2) < \ldots < a(v) < b(v)$. Then $\sum_{j=2}^{v}(y_{j,b(j)} - y_{j,a(j)}) + (y_{1,s} - y_{1,t}) = \sum_{i=g+1}^{k}(2^{rm(i)+i} - 2^{rn(i)+i})$ for some $n(g+1) < m(g+1) < n(g+2) < \ldots < n(k) < m(k)$. The right hand side of this equation has a binary expansion consisting of $k-g$ blocks of 1's and the binary expansion of the left hand side has a 0 ocurring between the expansion of $(y_{1,s} - y_{1,t})$ and the expansion of $\sum_{j=2}^{v}(y_{j,b(j)} - y_{j,a(j)})$. So $u(t,s)$, $n(t,s,j)$, and $m(t,s,j)$ must exist as claimed.

By Ramsey's Theorem, there must exist some infinite $B \subseteq \mathbb{N}$ and some $u \in \{1,2,\ldots,v-1\}$ such that for all $t < s$ in B, $u(t,s) = u$. Then $\{y_{1,s} - y_{1,t} : s,t \in B \text{ and } t < s\} \subseteq A_{r,u,g}$ so by Lemma 6.23 we must have that $u = g+1$. Further, by Lemma 6.24, we may pick d_1 such that $\{t \in \mathbb{N} : y_{1,t} \in \{2^{rm+g+1} + d_1 : m \in \mathbb{N}\}\}$ is infinite.

Now fix $t < s$ in B and pick l such that $2^l > y_{1,s} - y_{1,t}$. By thinning the sequences $\langle y_{j,w}\rangle_{w=1}^{\infty}$ for $j \in \{2, 3, \ldots, v\}$ we may presume that $y_{j,w} \equiv y_{j,z} \pmod{2^{l+1}}$ for all w and z. We claim that

$$\begin{aligned}\{\textstyle\sum_{j=2}^{v}(y_{j,b(j)} - y_{j,a(j)} :&a(2), b(2), a(3), \ldots, a(v), b(v) \in \mathbb{N}\\ &\text{and } s < a(2) < b(2) < a(3) \ldots < a(v) < b(v)\} \subseteq A_{r,k,g+1}\,.\end{aligned}$$

To this end, let $a(2) < b(2) < \ldots < a(v) < b(v)$ be given with $s < a(2)$. Pick $n(g + 1) < m(g + 1) < \ldots < n(k) < m(k)$ such that

$$(y_{1,s} - y_{1,t}) + \textstyle\sum_{j=2}^{v}(y_{j,b(j)} - y_{j,a(j)}) = \sum_{i=g+1}^{k}(2^{rm(i)+i} - 2^{rn(i)+i})\,.$$

Then $y_{1,s} - y_{1,t} = 2^{rm(g+1)+g+1} - 2^{rn(g+1)+g+1}$ so

$$\textstyle\sum_{j=2}^{v}(y_{j,b(j)} - y_{j,a(j)}) = \sum_{i=2}^{k}(2^{rm(i)+i} - 2^{rn(i)+i}) \in A_{r,k,g+1}$$

as claimed. By the induction hypothesis $v - 1 = k - (g + 1)$ and for $j \in \{2, 3, \ldots, v\}$ we may pick d_j such that $\{t \in \mathbb{N} : y_{j,t} \in \{2^{rm+g+j} + d : m \in \mathbb{N}\}\}$ is infinite. □

We see in the following corollary that we have sets whose closure contains almost all of the semigroup generated by $\{-p + p : p \in \mathbb{N}^*\}$.

Corollary 6.26. *Let $k \le r$ in $\mathbb{N}$ and let $B = \mathbb{N} \setminus A_{r,k,0}$. Let T be the subsemigroup of $(\beta\mathbb{N}, +)$ generated by $\{-p + p : p \in \mathbb{N}^*\}$. Then all members of T are in $\overline{B}$ except those of the form $\sum_{i=1}^{k}(-p_i + p_i)$ where for each $i \in \{1, 2, \ldots, k\}$ there exists $d_i \in \mathbb{Z}$ such that $\{2^{rm+i} + d_i : m \in \mathbb{N}\} \cap \mathbb{N} \in p_i$. That is*

$$\begin{aligned}T \setminus \overline{B} = \big\{ \textstyle\sum_{i=1}^{k}(-p_i + p_i) : (\forall i \in \{1, 2, \ldots, k\})\big(&p_i \in \mathbb{N}^* \text{ and}\\ &(\exists d_i \in \mathbb{Z})(\{2^{rm+i} + d_i : m \in \mathbb{N}\} \cap \mathbb{N} \in p_i)\big)\big\}\,.\end{aligned}$$

Proof. First assume that we have $v \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, v\}$ some $p_i \in \mathbb{N}^*$ such that $\sum_{i=1}^{v} \in T \setminus \overline{B}$. By Theorem 6.25, $v = k$. Let $j \in \{1, 2, \ldots, k\}$ and suppose that for all $d \in \mathbb{Z}$, $\{2^{rm+j} + d : m \in \mathbb{N}\} \cap \mathbb{N} \notin p_j$. For $t \in \mathbb{N}$, let $B_{j,t} = \mathbb{N} \setminus \bigcup_{d=-t}^{t}\{2^{rm+j} + d : m \in \mathbb{N}\}$. For $i \in \{1, 2, \ldots, k\} \setminus \{j\}$ and $t \in \mathbb{N}$ let $B_{i,t} = \mathbb{N}$. Pick by Lemma 5.3 for each $i \in \{1, 2, \ldots, k\}$ an injective sequence $\langle y_{i,t}\rangle_{t=1}^{\infty}$ in $\mathbb{N}$ such that for each $t \in \mathbb{N}$, $y_{i,t} \in \bigcap_{l=1}^{t} B_{i,l}$ and

$$\begin{aligned}\{\textstyle\sum_{i=1}^{k}(y_{i,b(i)} - y_{i,a(i)}) : &a(1)bm(1), a(2), \ldots, a(k), m(k) \in \mathbb{N}\\ &\text{and } a(1) < b(1) < a(2) < \ldots < a(k) < b(k)\} \subseteq A_{r,k,0}\,.\end{aligned}$$

Pick by Theorem 6.25 some $d_j \in \mathbb{Z}$ such that $\big\{t \in \mathbb{N} : y_{j,t} \in \{2^{rm+j} + d_j : m \in \mathbb{N}\}\big\}$ is infinite. This is a contradiction, since for all $t \ge d_j$, $y_{j,t} \notin \{2^{rm+j} + d_j : m \in \mathbb{N}\}$.

Now assume that for all $i \in \{1, 2, \ldots, k\}$ we have $p_i \in \mathbb{N}^*$ and $d_i \in \mathbb{Z}$ such that $\{2^{rm+i} + d_i : m \in \mathbb{N}\} \cap \mathbb{N} \in p_i$ and suppose that $\sum_{i=1}^{k}(-p_i + p_i) \in \overline{B}$. For each $i \in \{1, 2, \ldots, k\}$ and each $t \in \mathbb{N}$, let $B_{i,t} = \{2^{rm+i} + d_i : m \in \mathbb{N}\} \cap \mathbb{N}$. Pick by Lemma 5.3 for each $i \in \{1, 2, \ldots, k\}$ an injective sequence $\langle y_{i,t}\rangle_{t=1}^{\infty}$ in $\mathbb{N}$ such that for each $t \in \mathbb{N}$, $y_{i,t} \in \bigcap_{j=1}^{t} B_{i,j}$ and

$$\begin{aligned}\{\textstyle\sum_{i=1}^{k}(y_{i,b(i)} - y_{i,a(i)}) : &a(1)bm(1), a(2), \ldots, a(k), m(k) \in \mathbb{N}\\ &\text{and } a(1) < b(1) < a(2) < \ldots < a(k) < b(k)\} \subseteq B\,.\end{aligned}$$

In particular, $\sum_{i=1}^{k}(y_{i,2i} - y_{i,2i-1}) \in B$. We may presume that each $\langle y_{i,t}\rangle_{t=1}^{\infty}$ is increasing. For $i \in \{1, 2, \ldots, k\}$ pick $m(i)$ and $n(i)$ such that $y_{i,2i} = 2^{rm(i)+i} + d_i$

and $y_{i,2i-1} = 2^{rn(i)+i} + d_i$. Then $\sum_{i=1}^{k}(y_{i,2i} - y_{i,2i-1}) = \sum_{i=1}^{k}(2^{rm(i)+i} - 2^{rn(i)+i}) \in A_{r,k,0}$, a contradiction. □

7. Summary of results about the classes

In this section we list the various classes of subsets of S and classes of subsets of βS which we have discussed, and summarize the main results about each and the relations among them.

Subsets of S

Density intersective set

- Defined for left amenable semigroups. (Definition 3.1.)
- Implied by density recurrent. (Trivial.)
- Same as density recurrent set and set of measurable recurrence if S is countable and left amenable. (Theorem 3.3.)

Density recurrent set

- Defined for left amenable semigroups. (Definition 3.1.)
- Implies density intersective. (Trivial.)
- Same as density intersective set and set of measurable recurrence if S is countable and left amenable. (Theorem 3.3.)

Set of measurable recurrence

- Defined for arbitrary semigroups. (Definition 3.2.)
- Same as density intersective set and density recurrent set if S is countable and left amenable. (Theorem 3.3.)

IP^n set

- Defined for arbitrary semigroups. (Definition 4.1.)
- A is IP^n set if and only if there exist idempotents $p_1, p_2, \ldots, p_n$ in βS such that $A \in p_1 p_2 \cdots p_n$. (Theorem 4.3.)
- Implies IP^{n+1}. (Theorem 4.7.)
- In $(\mathbb{N}, +)$, strictly stronger than IP^{n+1}. (Theorem 4.8.)

IP^{n*} set

- Defined for arbitrary semigroups. (Definition 4.1.)
- A is IP^n set if and only if for all idempotents $p_1, p_2, \ldots, p_n$ in βS, $A \in p_1 p_2 \cdots p_n$. (Corollary 4.4.)
- For left amenable and left cancellative S, if $A \subseteq S$ and $d(A) > 0$, then AA^{-1} is IP^{n*}. (Corollary 4.5.)
- Two combinatorial characterizations. (Theorem 4.9.)
- In $(\mathbb{N}, +)$, strictly weaker than EIP^{2*} for $n \geq 2$. (Theorem 4.11.)
- For countable abelian groups, a recurrence condition sufficient to guarantee IP^{n*}. (Theorem 4.13.)

EIP^{n*} set

- Defined for arbitary semigroups. (Definition 4.10.)
- Implies IP^{n*}. (Trivial.)
- In $(\mathbb{N}, +)$, EIP^{2*} strictly stronger than IP^{n*} for some $n \geq 2$. (Theorem 4.11.)

- For countable abelian groups, a recurrence condition sufficient to guarantee IP^{n*}. (Theorem 4.16.)

Δ^n set

- Defined for groups and $(\mathbb{N},+)$. (Definition 5.1.)
- A is IP^n set if and only if there exist $p_1, p_2, \ldots, p_n$ in S^* such that $A \in \prod_{i=1}^n (p_i^{-1} p_i)$. (Theorem 5.4.)
- Partition regular. (Corollary 5.7.)
- For each n, there is a subset of $(\mathbb{N},+)$ which is a Δ^n set but not a Δ^k set for any $k \neq n$. (Theorem 6.25.)

Δ^{n*} set

- Defined for groups and $(\mathbb{N},+)$. (Definition 5.1.)
- A is IP^{n*} set if and only if for all $p_1, p_2, \ldots, p_n$ in S^*, $A \in \prod_{i=1}^n (p_i^{-1} p_i)$. (Corollary 5.5.)
- If S is an amendable group or $(\mathbb{N},+)$, and $d(A) > 0$, then AA^{-1} is Δ^{n*} for each n. (Corollary 5.6.)
- For $n \in \mathbb{N}$ there is a set $B \subseteq \mathbb{N}$ such that $d(B) > 0$ and $\{x \in \mathbb{N} : B \cap (B - x) \cap (B - 2x) \neq \emptyset\}$ is not a Δ^{n*} set. (Theorem 6.19.)

Polynomial n-recurrent set

- Defined for $(\mathbb{N},+)$. (Definition 6.4.)
- Examples. (Corollary 6.11 and Theorem 6.13.)

Subsets of βS

$\mathcal{DI}(S)$

- Defined for left amenable semigroups. (Definition 3.1.)
- Contains $\mathcal{DR}(S)$. (Trivial.)
- Equal to $\mathcal{DR}(S)$ if S is countable and left amenable. (Corollary 3.4.)
- If $d(A) > 0$, then contained in $\overline{AA^{-1}}$. (Theorem 3.15.)

$\mathcal{DR}(S)$

- Defined for left amenable semigroups. (Definition 3.1.)
- Contained in $\mathcal{DI}(S)$.
- Equal to $\mathcal{DI}(S)$ if S is countable and left amenable. (Corollary 3.4.)
- Contains $\Gamma_{<\omega}(S)$ if S left cancellative and left amenable. (Lemma 3.8.)
- Properly contains $\Gamma_{<\omega}(\mathbb{N},+))$. (Theorem 3.9.)
- If S is amenable group or $(\mathbb{N},+)$, then includes $p^{-1}p$ for all $p \in S^*$. (Lemma 3.13.)
- If S is left cancellative, then is semigroup and includes $q^{-1}p$ for all $q, p \in \mathcal{DR}(S)$. (Theorem 3.14.)
- $\mathcal{DR}(\mathbb{N},+)$ is a left ideal of $\mathcal{DR}(\mathbb{N},\cdot)$. (Theorem 6.1.)

$\Gamma(S)$

- Defined for arbitrary semigroups. (Definition 4.1.)
- Is contained in $\Gamma_{<\omega}(S)$. (Trivial.)
- $\Gamma(\mathbb{N},+)$ not a semigroup. (Two paragraphs before Lemma 3.6.)

$\Gamma_{<\omega}(S)$

- Defined for arbitrary semigroups. (Definition 4.1.)

- Contains $\Gamma(S)$. (Trivial.)
- Is a semigroup if S is commutative. (Three paragraphs before Lemma 3.6.)
- Is contained in $\mathcal{DR}(S)$ if S left cancellative and left amenable. (Lemma 3.8.)
- Is properly contained in $\mathcal{DR}(\mathbb{N})$. (Theorem 3.9.)

$\mathcal{PR}_n$

- Defined for $(\mathbb{N},+)$. (Definition 6.4.)
- Subsemigroup of $(\beta\mathbb{N},+)$ containing the idempotents. (Theorem 6.5.)
- Closed under subtraction from the left. (Theorem 6.6.)
- Left ideal of $(\beta\mathbb{N},\cdot)$. (Theorem 6.7.)
- There is $p \in \mathbb{N}^*$ such that $-p+p \notin \mathcal{PR}_2$. (Corollary 6.20.)

$\mathcal{PR}$

- Defined for $(\mathbb{N},+)$. (Definition 6.4.)
- Subsemigroup of $(\beta\mathbb{N},+)$ containing the idempotents. (Theorem 6.5.)
- Closed under subtraction from the left. (Theorem 6.6.)
- Left ideal of $(\beta\mathbb{N},\cdot)$. (Theorem 6.7.)

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