IP Systems, Generalized Polynomials and Recurrence

Vitaly Bergelson, Inger J. Håland Knutson and Randall McCutcheon

Let Ω be an abelian group. A set $R \subset \Omega$ is a *set of recurrence* if for any probability measure preserving system $(X, \mathcal{B}, \mu, \{T_g\}_{g\in\Omega})$ and any $A \in \mathcal{A}$ with $\mu(A) > 0$, $\mu(A \cap T_g A) > 0$ for some $g \in R$. If $(x_i)_{i=1}^{\infty}$ is a sequence in Ω, the set of its finite sums $\{x_{i_1} + x_{i_2} + \cdots + x_{i_k} : i_1 < i_2 < \cdots < i_k\}$ is called an *IP-set*. In [BFM] it is shown that if $p : \mathbf{Z}^d \to \mathbf{Z}^k$ is a polynomial vanishing at zero and F is an IP-set in $\mathbf{Z}^d$ then $\{p(n) : n \in F\}$ is a set of recurrence in $\mathbf{Z}^k$. Here we extend this result to an analagous family of *generalized polynomials*, that is functions formed from regular polynomials by iterated use of the greatest integer function, as a consequence of a theorem establishing a much wider class of recurrence sets occuring in any (possibly non-finitely generated) abelian group. While these sets do in a sense have a distinctively "polynomial" nature, this far-ranging class includes, even in $\mathbf{Z}$, such examples as $\{\sum_{i,j\in\alpha,i<j} 2^i 3^j : \alpha \subset \mathbf{N}, 0 < |\alpha| < \infty\}$, where the connection to conventional polynomials is somewhat distant.

0. Introduction.

A *set of measurable recurrence* in a countable commutative group Ω is a subset $R \subset \Omega$ having the property that for any measure preserving action (T_g) of Ω on a probability space $(X, \mathcal{A}, \mu)$ and any $A \in \mathcal{A}$ with $\mu(A) > 0$, there exists $g \in R$ with $\mu(A \cap T_g A) > 0$. By a correspondence principle of Furstenberg, R is a set of measurable recurrence if and only if R is *density intersective*, that is, if for every $E \subset \Omega$ having positive upper density $\overline{d}(E) = \limsup \frac{|E\cap\Phi_n|}{|\Phi_n}$ with respect to some Følner sequence (Φ_n), $R \cap (E - E) \neq \emptyset$.

A *generalized polynomial* $\mathbf{Z} \to \mathbf{Z}$ is a function, such as $p(x) = [\pi x^2 + .17][[\sqrt{2}x^4]ex]$, built in an iterated manner from regular polynomials, the greatest integer function, additions and multiplications (we shall give a more formal definition later). This project grew out of (and eventually outgrew) the following question: Let p be a generalized polynomial. Is $\{p(n) : n \in \mathbf{N}\}$ a set of measurable recurrence?

If $p \in \mathbf{Z}[x]$ is a regular polynomial with $p(0) = 0$, the answer is *yes*. Sárközy proved this (formulated in terms of density intersectivity) in [S]. In [F], Furstenberg gave a substantially different, ergodic theoretic proof via the spectral theorem. In [B] yet another proof is given, using a method motivated by the difference trick of van der Corput. Complications arise in attempting to adapt these proofs to generalized polynomials.

Much to everyone's surprise, however, a much later (and substantially softer) proof carries over nicely, at least for a large class of generalized polynomials, while at the same time giving somewhat more. The proof referred to is that of [BFM], and the "somewhat more" involves evaluating the given functions along members of *IP-sets*. An IP-set in a commutative group Ω is a set consisting of all finite sums of elements of different indices from a sequence in Ω. Thus an IP-set has the form $\{n_{i_1} + n_{i_2} + \cdots + n_{i_k} : k \in \mathbf{N}, i_1 < i_2 < \cdots < i_k\}$, where $(n_i)_{i=1}^{\infty} \subset \Omega$. By [BFM], if $p(x) \in \mathbf{Z}[x]$ vanishes at zero and F is an IP-set in $\mathbf{Z}$, then $\{p(n) : n \in F\}$ is a set of measurable recurrence.

As we shall show, for many generalized polynomials the same result holds. It bears mentioning that the IP-set feature of the formulation, while offering an obvious strengthening, imposes as well an inherent limitation on the class of generalized polynomials to which it may apply. Consider for example $p(x) = [\pi x]$. One easily sees that for every $k \in \mathbf{N}$, $\{p(n) : n \in \mathbf{N}\}$ has a subset of the form $\{a, 2a, \ldots, ka\}$, and it is, in turn, simple enough to show that any set having this property is a set of measurable recurrence (this line of discussion is pursued further in [BH]). On the other hand, using density of the fractional part of πx on $[0, 1)$, one may easily construct an IP-set F such that for all $n \in F$, $p(n)$ is odd. For such F, $\{p(n) : n \in F\}$ cannot be a set of measurable recurrence. (Simply consider a two point system with transformation T swapping the points.) We give a precise definition of the class we treat here, which we call the *admissible generalized polynomials*, in section 2. For now, suffice it to say that the class of admissible generalized polynomials contains $p(x) = x$, is closed under sums and products, and if p is in the class, r is a real number and $0 < h < 1$, then $q(x) = [rp(x) + h]$ will again be admissible, while $t(x) = [rp(x)]$ need not be (as we have already seen).

Toward the goal of explicating the role of h, it may be instructive to compare the generalized polynomial $q(x) = [\pi x + h]$ to the example $p(x)$ from the previous paragraph, where $0 < h < 1$. Choose a sequence (ϵ_i) of positive reals with $\sum \epsilon_i < \min\{h, 1 - h\}$. Now suppose an IP-set $F = F_1 = \{m_{i_1} + m_{i_2} + \cdots + m_{i_k} : k \in \mathbf{N}, i_1 < i_2 < \cdots < i_k\}$ is given. It is an easy exercise that one may choose $n_1 = m_{j_1} + m_{j_2} + \cdots + m_{j_t} \in F_1$ such that the distance from πn_1 to the nearest integer is less than ϵ_1. Now let F_2 be the IP-set $\{m_{i_1} + m_{i_2} + \cdots + m_{i_k} : k \in \mathbf{N}, j_t < i_1 < i_2 < \cdots < i_k\}$ and choose $n_2 \in F_2$ such that the distance from πn_2 to the nearest integer is less than ϵ_2. (Notice that $n_1 + n_2 \in F$.) Continue in this way until a sequence $(n_i)_{i=1}^{\infty}$ has been chosen such that the IP-set $G = \{\sum_{i\in\alpha} n_i : \alpha \subset \mathbf{N}, 0 < |\alpha| < \infty\}$ is contained in F and such that the distance from each πn_i to the nearest integer is less than ϵ_i. It follows that, for $\alpha \subset \mathbf{N}, 0 < |\alpha| < \infty$, the distance from $\pi \sum_{i\in\alpha} n_i$ to the nearest integer is at most $\sum \epsilon_i$, and from this we get that for any pair of finite subsets $\alpha, \beta \subset \mathbf{N}$, with $\alpha \cap \beta = \emptyset$, $q(\sum_{i\in\alpha} n_i) + q(\sum_{i\in\beta} n_i) = q(\sum_{i\in\alpha\cup\beta} n_i)$. This implies that the set $\{q(n) : n \in G\}$, which is contained in the set $\{q(n) : n \in F\}$, is itself an IP-set, being the set of finite sums of the sequence $\big(q(n_i)\big)_{i=1}^{\infty}$. Since IP-sets are separately known to be measurable sets of recurrence, $\{q(n) : n \in F\}$ is a set of measurable recurrence.

This example, although linear in nature, is representative of the general strategy we employ. Basically, we show that for any admissible generalized polynomial $p(x)$ and any IP-set F, there exists an IP-set $G \subset F$ such that the set $\{p(n) : n \in G\}$ is of a special form that we are able to demonstrate independently to be a set of measurable recurrence[1].

In order to facilitate introduction of this special form, we develop some notation. Denote by $\mathcal{F}$ the set of finite, non-empty subsets of $\mathbf{N}$. An IP-set in a commutative group

[1] If one is interested in mere recurrence, rather than recurrence along every IP-set, it would be enough to find a single G having this property. For our earlier example $p(x) = [\pi x]$, this is easily accomplished, again using density of the fractional part of πx on $[0, 1)$. Using apparatus developed in [Hå], such results can be shown to obtain for many other non-admissible generalized polynomials having sufficient uniform distribution properties. We may treat this issue in a later publication.

with generating sequence $(g_i)_{i=1}^{\infty}$ is naturally indexed by $\mathcal{F}$ by letting $n(\alpha) = \sum_{i\in\alpha} g_i$, $\alpha \in \mathcal{F}$. We call such an indexed IP-set an *IP system*. Among all functions $n : \mathcal{F} \to G$, IP systems are characterized by the linear identity $n(\alpha \cup \beta) = n(\alpha) + n(\beta)$, valid when $\alpha \cap \beta = \emptyset$. Functions $v : \mathcal{F} \to G$ satisfying the quadratic identity

$$v(\alpha \cup \beta \cup \gamma) - v(\alpha \cup \beta) - v(\alpha \cup \gamma) - v(\beta \cup \gamma) + v(\alpha) + v(\beta) + v(\gamma) = 0$$

for α, β, γ pairwise disjoint are called *VIP systems of degree* 2. Higher order identities lead to VIP systems of correspondingly higher degree. One may show that, for example, for any polynomial $p(x) : \mathbf{Z}^d \to \mathbf{Z}$ and any IP system $n(\alpha)$ in $\mathbf{Z}^d$, the function $v(\alpha) = p\big(n(\alpha)\big)$ is a VIP system of degree at most $\deg p$. We call VIP systems arising in this manner *IP polynomials*. A simple example, coming from the polynomial $p(x, y) = xy$ and the IP system $n(\alpha) = \sum_{i\in\alpha}(n_i, m_i)$, is the IP polynomial $v(\alpha) = \sum_{i,j\in\alpha} n_i m_j$.

The method of proof in [BFM] is to show, for example, that for any IP polynomial $v : \mathcal{F} \to \mathbf{Z}$ and any unitary operator U on a Hilbert space $\mathcal{H}$, a weak limit of $U^{v(\alpha)}$ (see below for the sense of convergence) is an orthogonal projection. Applying this to $L^2(X)$, it follows by standard arguments that $\{v(\alpha) : \alpha \in \mathcal{F}\}$ is a set of measurable recurrence. Similar reasoning applies to IP polynomials (analogously defined) in $\mathbf{Z}^k$. On the other hand, a counterexample in [BFM] shows that, in general, for a unitary action (U_g) of a commutative group Ω and a VIP system $v : \mathcal{F} \to \Omega$, a weak limit of $U_{v(\alpha)}$ need *not* be an orthogonal projection.

The argument we use here for admissible generalized polynomials is as follows: Extend the class of IP polynomials to a larger class (the members of which we call *FVIP systems*) of VIP systems along which appropriately defined sequences of unitary actions can be shown to converge to orthogonal projections. This is quite a special property for a VIP system to have, even in $\mathbf{Z}^k$; yet at the same time, the class must be chosen general enough that for any admissible generalized polynomial p and any IP system n, the set $\{g\big(n(\alpha)\big) : \alpha \in \mathcal{F}\}$ can be shown to support such a system. This yields the following consequence of Theorems 2.2 and 2.3 below, a stronger version of which will be given as Corollary 2.9.

Theorem A. Let p be an admissible generalized polynomial $\mathbf{Z}^l \to \mathbf{Z}^t$ and suppose F is an IP-set in $\mathbf{Z}^l$. Then $\{p(x) : x \in F\}$ is a density intersective subset of $\mathbf{Z}^t$ (equivalently, is a set of measurable recurrence). Indeed, for any set $E \subset \mathbf{Z}^t$ with $\overline{d}(E) > 0$ and any $\epsilon > 0$, there exists some $x \in F$ such that $\overline{d}\Big(E \cap \big(E - p(x)\big)\Big) > \big(\overline{d}(E)\big)^2 - \epsilon$.

All of this makes for a quite satisfactory treatment of admissible generalized polynomials, however it is at this point in the undertaking that FVIP systems come to be even more interesting in their own right, partly in that one may proceed to define them for arbitrary commutative groups. For example, under this definition, if Ω is a commutative group, (n_i) is a sequence of integers and (y_j) is a sequence in Ω, then $n(\alpha) = \sum_{i\in\alpha} y_i$ and $u(\alpha) = \sum_{i,j\in\alpha, i<j} n_i y_j$ are FVIP systems (the former is just a general IP system; the reader is encouraged to compare the latter to the IP polynomial $v(\alpha) = \sum_{i,j\in\alpha} n_i m_j$ in $\mathbf{Z}$ introduced above). Rather than a mere tool used to prove a special result in $\mathbf{Z}^k$ about generalized polynomials, then, our operator convergence theorem, which states that weak limits of unitary actions along FVIP systems are orthogonal projections, is better thought

of as a joint extension of the $\mathbf{Z}^k$ operator convergence result appearing in [BFM] and a theorem from [FK] involving weak limits of IP systems of unitary operators in arbitrary commutative groups.

We proceed to formulations: for $\alpha, \beta \in \mathcal{F}$, we write $\alpha < \beta$ if $\max\alpha < \min\beta$. Suppose $\alpha_i \in \mathcal{F}$, $i \in \mathbf{N}$, with $\alpha_1 < \alpha_2 < \cdots$. The set $\mathcal{F}^{(1)}$ of non-empty finite unions of the α_i's, is called an *IP ring.* We also set $\mathcal{F}^{(1)}_\emptyset = \mathcal{F}^{(1)} \cup \{\emptyset\}$. $\mathcal{F}^{(1)}$ is the isomorphic image of $\mathcal{F}$ under the map $\beta \to \bigcup_{i\in\beta} \alpha_i$, which we call the *natural isomorphism.* If $\mathcal{F}^{(1)}$ is an IP-ring and $x : \mathcal{F}^{(1)} \to X$ is a function into a topological space, we write $\text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} x(\alpha) = x$ if for every neighborhood U of x, there exists some $\alpha_0 \in \mathcal{F}$ such that $x(\alpha) \in U$ whenever $\alpha \in \mathcal{F}^{(1)}$ and $\alpha > \alpha_0$. As a consequence of a theorem of Hindman, which states, in one of its several formulations, that for any finite partition of an IP-ring there is an IP-sub-ring in one cell of the partition, one may always find, when X is compact metric, an IP-ring $\mathcal{F}^{(1)}$ such that $\text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} x(\alpha)$ exists.

Here now is the aforementioned result from [FK]:

Theorem B$_1$. Let $\mathcal{H}$ be a separable Hilbert space, let Ω be a commutative group and suppose that (U_g) is an action of Ω by unitary operators on $\mathcal{H}$. If n is an IP system in Ω and $\mathcal{F}^{(1)}$ is an IP-ring such that $\text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} U_{n(\alpha)} = P$ exists in the weak operator topology, then P is an orthogonal projection.

And the analogous result from [BFM]:

Theorem B$_2$. Let $\mathcal{H}$ be a separable Hilbert space and suppose (U_n) is an action of $\mathbf{Z}^k$ by unitary operators on $\mathcal{H}$. If v is an IP polynomial in $\mathbf{Z}^k$ and $\mathcal{F}^{(1)}$ is an IP-ring such that $\text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} U_{v(\alpha)} = P$ exists in the weak operator topology, then P is an orthogonal projection.

Finally, the operator convergence theorem we will be proving in section 1.1:

Theorem B. Let $\mathcal{H}$ be a separable Hilbert space, let Ω be a commutative group and suppose that (U_g) is an action of Ω by unitary operators on $\mathcal{H}$. If v is an FVIP system in Ω and $\mathcal{F}^{(1)}$ is an IP-ring such that $\text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} U_{v(\alpha)} = P$ exists in the weak operator topology, then P is an orthogonal projection.

Several of Theorem B's implications for density combinatorics in arbitrary commutative groups are outlined in section 3. Among the more interesting of these involves a correlate of the notion *IP$_r$-set*, which was introduced in [FK]. Given $r \in \mathbf{N}$, an IP$_r$ set in a commutative group Ω is a set having the form $\left\{ \sum_{i\in\alpha} x_i : \emptyset \neq \alpha \subset \{1, \ldots, r\}\right\}$, where $x_1, \ldots, x_r \in \Omega$. In other words, an IP$_r$ set is essentially a finite IP-set.

Obviously no IP$_r$ set, nor any other finite subset of a countable group (to be more precise, no finite subset not containing the identity), is going to be a set of measurable recurrence. On the other hand, it is trivial to show that, given $\delta > 0$, there exists r such that for any measure preserving action (T_g) of a probability space $(X, \mathcal{A}, \mu)$, any measurable A with $\mu(A) > \delta$, and any IP$_r$ set R, $\mu(A \cap T_g A) > 0$ for some $g \in R$. By analogy, one might expect that classes of sets indexed by parameters running through IP-sets might have the property that, given δ, similarly structured finite sets indexed by parameters running through IP_r sets might achieve positive measure returns for sets having measure at least δ. [FK] contains formulations of many such results (formulated in terms of multiple density intersectivity) having this flavor, while [BFM] has none. This is not accidental. What makes the results possible in [FK] is the fact that r can be fixed to work for a "universal" IP system (essentially the finite sums of the generators of an infinite-dimensional free abelian group), whereupon one needs only to check that its properties are passed to homomorphic images. The results of [BFM], limited as they are to IP polynomials in $\mathbf{Z}^k$, exhibit no such universality.

Our results here do, however, have some universal character, and we are able to give, for single recurrence at least, satisfactory polynomial extensions to many of the IP_r results from [FK]. These involve what might be called "IP_r sets having polynomial weights." We shall formulate one simple case here for finite groups, leaving the bulk of the discussion for section 3. By a polynomial $\mathbf{Z} \to \mathbf{Z}$, we mean a polynomial having rational coefficients and taking integers to integers.

Theorem C. Let $\epsilon, \delta > 0$ and let $p : \mathbf{Z} \to \mathbf{Z}$ be a polynomial. There exists $r \in \mathbf{N}$ (depending on ϵ, δ and p) such that if H is any finite commutative group, $u_1, \ldots, u_r \in H$, and $S \subset H$ with $|S| \geq \delta|H|$, then for some non-empty $\alpha \subset \{1, \ldots, r\}$, $|S \cap \big(S - u(\alpha)\big)| \geq (\delta^2 - \epsilon)|H|$, where $u(\{s_1, \ldots, s_b\}) = \sum_{t=1}^{b} p(t) u_{s_t}$.

1. A unitary convergence theorem for FVIP systems.

Let G be an additive abelian group and $v : \mathcal{F}_\emptyset \to G$ a function. If for some d (the least such being called the degree of v), one has

$$v(\alpha) = \sum_{\gamma \subset \alpha, |\gamma| \leq d} f(\gamma),$$

where $f : \{\alpha \in \mathcal{F} : |\alpha| \leq d\} \to G$ with $f(\emptyset) = 0$, then v is called a *VIP system* (of degree d) and f is called the *generating function* of v. (For a proof that this definition is equivalent to that alluded to informally in the introduction, see [M, Proposition 2.5].) The corresponding identity for multiplicative groups is

$$v(\alpha) = \prod_{\gamma \subset \alpha, |\gamma| \leq d} f(\gamma).$$

For $\alpha, \beta \in \mathcal{F}$, we write $\alpha < \beta$ if $\max \alpha < \min \beta$. Given a VIP system v and some fixed $\alpha \in \mathcal{F}$, we define the derivative of v with step α to be the VIP system $D_\alpha v$, where

$$D_\alpha v(\beta) = v(\beta \cup \alpha) v(\beta)^{-1} v(\alpha)^{-1}, \;\; b \in \mathcal{F}, \alpha < \beta.$$

Notice that the VIP system $D_\alpha v$ is not defined for all $\beta \in \mathcal{F}$, but only for $\beta > \alpha$. This leads us to the following digression.

If $\mathcal{F}^{(1)}$ is an IP-ring then any function $\mathcal{F}^{(1)} \to G$ that becomes a VIP system when pulled back to $\mathcal{F}$ via the natural isomorphism will again be called a VIP system. (It is worth noting, though not immediate, that, under this definition, the restriction of any VIP system to an IP ring is a VIP system.) The VIP system $D_\alpha v$ introduced above is defined on the IP ring $\{\beta \in \mathcal{F} : \beta > \alpha\}$.

The family of VIP systems into an abelian group $(G, +)$ itself forms an abelian group under addition. If V is a group of VIP systems having the property that for all $v \in V$ and $\alpha \in \mathcal{F}$, $D_\alpha v$ is equal, on its domain, to some member of V, we say that V is *closed under derivatives*. In this paper we examine groups of VIP systems that are both closed under derivatives and finitely generated.

We now formally define our main object of study. Let Ω be a commutative multiplicative group, let $d \in \mathbf{N}$, let $(y_i)_{i=1}^\infty$ be a sequence in Ω and suppose $(n_i^{(j)})_{i=1}^\infty$ are sequences in $\mathbf{Z}$, $1 \leq j < d$. For $\gamma \in \mathcal{F}$, $\gamma = \{i_1, \ldots, i_d\}$, with $i_1 < i_2 < \cdots < i_d$, put

$$f(\gamma) = y_{i_d}^{n_{i_1}^{(1)} n_{i_2}^{(2)} \cdots n_{i_{d-1}}^{(d-1)}}.$$

Finally for $\alpha \in \mathcal{F}$ set $v(\alpha) = \prod_{\gamma \subset \alpha, |\gamma| = d} f(\gamma)$. Then $v : \mathcal{F} \to \Omega$ is said to be a *simple FVIP system* (of degree, in general, d). If $\mathcal{F}^{(1)}$ is an IP ring and $v : \mathcal{F}^{(1)} \to \Omega$ is a finite product of simple FVIP systems then v is said to be an *FVIP* system, the degree of which is equal to the largest of the degrees of the corresponding simple FVIP systems. In general form:

$$v(\alpha) = \prod_{l=1}^{r} \prod_{\substack{i_1, \ldots, i_{d(l)} \in \alpha \\ i_1 < \cdots < i_{d(l)}}} y_{l, i_{d(l)}}^{n_{l,i_1}^{(l,1)} n_{l,i_2}^{(l,2)} \cdots n_{l,i_{d(l)-1}}^{(l,d(l)-1)}} \tag{1.1}$$

Note that if Ω is written additively instead of multiplicatively, equation (1.1) becomes:

$$v(\alpha) = \sum_{l=1}^{r} \sum_{\substack{i_1, \ldots, i_{d(l)} \in \alpha \\ i_1 < \cdots < i_{d(l)}}} n_{l,i_1}^{(l,1)} n_{l,i_2}^{(l,2)} \cdots n_{l,i_{d(l)-1}}^{(l,d(l)-1)} y_{l, i_{d(l)}}. \tag{1.2}$$

In either case, the sequences $(y_{l,k})_{k=1}^\infty$ in Ω and $\left(n_{l,k}^{(l,j)}\right)_{k=1}^\infty$ in $\mathbf{Z}$ are called the *generating sequences* of v.

If v is a VIP system the *derived group* of v is defined to be the group of VIP systems generated by all iterated derivatives of v, specifically the group generated by $\{D_{\alpha_k} D_{\alpha_{k-1}} \cdots D_{\alpha_1} v : k \in \mathbf{N}, \alpha_1 < \cdots < \alpha_k\}$. It is not difficult to show that, if v is an FVIP system, then the derived group is finitely generated. Indeed, if v is given by (1.1), the group generated by the FVIP systems in Ω that can be formed from some combination of the sequences $y_{l,\cdot}$ and $n_{l,\cdot}^{(l,j)}$ is finitely generated and closed under derivatives, and hence contains the derived group of v. We illustrate this by means of an example, leaving the general case to the reader. Suppose that Ω is an additive group and

$$v(\alpha) = \sum_{\substack{i_1, i_2, i_3 \in \alpha \\ i_1 < i_2 < i_3}} n_{i_1}^{(1)} n_{i_2}^{(2)} y_{i_3}.$$

One then easily checks that

$$D_\alpha v(\beta) = \Big(\sum_{i_1\in\alpha} n^{(1)}_{i_1}\Big)\Big(\sum_{i_2,i_3\in\beta, i_2<i_3} n^{(2)}_{i_2} y_{i_3}\Big) + \Big(\sum_{i_1,i_2\in\alpha, i_1<i_2} n^{(1)}_{i_1} n^{(2)}_{i_2}\Big)\Big(\sum_{i_3\in\beta} y_{i_3}\Big).$$

The expressions in the large parentheses are FVIP-systems (in β) of degrees 2 and 1, respectively, each of which is generated by some combination of the sequences $n^{(1)}, n^{(2)}, y$, and $D_\alpha v$ is a linear combination of these.

Indeed, one can guess more from this example. First, $D_\alpha v$ is of lesser degree than v. (Actually $\deg D_\alpha v = (\deg v) - 1$.) Second, the "coefficients" (the expressions appearing inside the smaller parentheses) of the various FVIP systems appearing in $D_\alpha v$ are themselves $\mathbf{Z}$-valued FVIP systems in the variable α. Third, the "expressions in the large parentheses" are elements of the derived group. It is routine to establish these facts in general. The significance of the latter is that it allows for the following proposition.

Proposition 1.1 Let v be an FVIP system and let W be the derived group of v. If $V < W$ is a subgroup with $[W : V] < \infty$ then there exists an IP ring $\mathcal{F}^{(1)}$ such that for all $\alpha \in \mathcal{F}^{(1)}$, $D_\alpha v \in V$.

Proof. First we remark that for any finite family C of VIP systems in $\mathbf{Z}$ and any $n \in \mathbf{N}$, there exists an IP ring $\mathcal{F}^{(1)}$ such that for all $\alpha \in \mathcal{F}^{(1)}$ and all $c \in C$, $c(\alpha)$ is divisible by n. This follows immediately from Lemma 2.6 below, applied to the family of real valued VIP systems $\{c/n : c \in C\}$. (Alternatively, one may derive it from Hindman's theorem and the characterization of VIP systems in terms of identities; for example, suppose c is VIP of degree 2. Passing to an IP ring upon which $c(\alpha)$ is constant mod n, the identity $c(\alpha\cup\beta\cup\gamma) - c(\alpha\cup\beta) - c(\alpha\cup\gamma) - c(\beta\cup\gamma) + c(\alpha) + c(\beta) + c(\gamma) = 0$ immediately implies that this constant value must be zero.)

Choose $n \in \mathbf{N}$ such that $nu \in V$ for all $u \in W$. Now, $D_\alpha v(\beta)$ is, recall, a linear combination of terms having the form $c(\alpha)u(\beta)$, where the "coefficient" c is an FVIP system into $\mathbf{Z}$ and $u \in W$. Choose an IP ring $\mathcal{F}^{(1)}$ such that for all $\alpha \in \mathcal{F}^{(1)}$ and each of these coefficients c in the expansion of $D_\alpha v$, $c(\alpha)$ is divisible by n. Now for fixed $\alpha \in \mathcal{F}^{(1)}$, $D_\alpha v$ is equal to n times a linear combination of terms having the form $\frac{c(\alpha)}{n}u$. Writing $w \in W$ for this linear combination, one has $D_\alpha v = nw \in V$.

□

Some terminology: if G is a commutative group and $E \subset G$ intersects non-trivially the range of every FVIP system into G then E is said to be FVIP*. If n is an IP system in $\mathbf{Z}^d$ and $p : \mathbf{Z}^d \to \mathbf{Z}^l$ is a polynomial mapping with $p(0) = 0$ then one may show that $v(\alpha) = p(n(\alpha))$ defines an FVIP system. We call such systems, which were studied in [BFM], IP polynomials.

The proof of Theorem 1.2 below shows that the FVIP system $v(\alpha) = \sum_{i,j\in\alpha, i<j} 2^i 3^j$ in $\mathbf{Z}$ is not an IP polynomial. Notice, however, that if the restriction $i < j$ in the definition of v were removed, the result would be the IP polynomial

$$v'(\alpha) = p\big(n(\alpha), m(\alpha)\big), \text{ where } p(x,y) = xy, \quad n(\alpha) = \sum_{i\in\alpha} 2^i, \text{ and } m(\alpha) = \sum_{j\in\alpha} 3^j. \quad (1.3)$$

Thus we see that the difference between general FVIP systems and IP polynomials in $\mathbf{Z}$ is somewhat subtle. Indeed, in $\mathbf{Z}$, this subtlety is completely characterized by the occurrence of the condition $i_1 < \cdots < i_{d(l)}$ in the index of summation in equation (1.2), without which the definition reverts to a subclass of FVIP systems that is easily seen to consist precisely of the IP polynomials. For example, the IP polynomial $v'(\alpha)$ defined by (1.3) is an FVIP system, as may be observed by the equation $v'(\alpha) = \sum_{i,j\in\alpha, i<j} 2^i 3^j + \sum_{i,j\in\alpha,i<j} 3^i 2^j + \sum_{i\in\alpha} 2^i 3^i$.

For infinitely generated groups Ω, however, this point is not so subtle. In this case, the definition resulting from the removal of the condition $i_1 < \cdots < i_{d(l)}$ from equation (1.2) yields a class of VIP systems that not only fails to be a subclass of the FVIP systems, but is also, at present, beyond our ability to deal satisfactorily with. For example, if $\pi(\alpha) = \prod_{i\in\alpha} p_i$ is an IP system in $(\mathbf{N}, \cdot)$ and $\sigma(\alpha) = \sum_{i\in\alpha} s_i$ is an IP system in $(\mathbf{N}, +)$, then $v(\alpha) = \prod_{i,j\in\alpha,i<j} p_j^{s_i}$ is an FVIP system in $(\mathbf{N}, \cdot)$, while $u(\alpha) = \pi(\alpha)^{\sigma(\alpha)} = \prod_{i,j\in\alpha} p_j^{s_i}$ is in general not. Indeed, we do not at present know whether or not the range of such a VIP system u need constitute a set of recurrence.

Theorem 1.2 There exists an FVIP system in $(\mathbf{Z}, +)$ that is not an IP polynomial.

Proof. It is an easy exercise that if v and u are VIP systems with $v(\alpha) = u(\alpha)$ for all $\alpha \in \mathcal{F}$, then their degrees and generating functions coincide. Indeed, this is because one can always express the generating function f in terms of values of v. For example, $f(\{i,j,k\}) = v(\{i,j,k\}) - v(\{i,j\}) - v(\{i,k\}) - v(\{j,k\}) + v(\{i\}) + v(\{j\}) + v(\{k\})$. At any rate, this fact is implicit throughout the proof to follow.

Let $v(\alpha) = \sum_{i,j\in\alpha,i<j} 2^i 3^j$. Then v is a simple FVIP system of degree 2. Suppose that v is an IP polynomial. That is, suppose that for some $d \in \mathbf{N}$, some polynomial $p : \mathbf{Z}^d \to \mathbf{Z}$ vanishing at zero, and some IP system n in $\mathbf{Z}^d$, $v(\alpha) = p\big(n(\alpha)\big)$. It does not follow that $\deg p = 2$; however if $\deg p > 2$, we can nevertheless find some $d' \in \mathbf{N}$, a polynomial $p' : \mathbf{Z}^{d'} \to \mathbf{Z}$ having degree 2, and some IP system n' in $\mathbf{Z}^{d'}$ such that $v(\alpha) = p'\big(n'(\alpha)\big)$. We consider this an exercise, so rather than prove it formally, we shall merely illustrate it with an example.

Consider $p(x,y,z) = xyz$ and suppose by chance that for some IP systems n, m, r in $\mathbf{Z}$, $\alpha \to p(n(\alpha), m(\alpha), r(\alpha))$ is VIP of degree 2. Then

$$\begin{aligned} p(n(\alpha), m(\alpha), r(\alpha)) &= \sum_{i,j,k\in\alpha} n_i m_j r_k \\ &= \sum_{|\{i,j,k\}|=3} n_i m_j r_k + \sum_{|\{i,j,k\}|=2} n_i m_j r_k + \sum_{|\{i,j,k\}|=1} n_i m_j r_k \\ &= \sum_{|\{i,j\}|=2} (s_i r_j + t_i m_j + n_j u_i) + \sum_i n_i m_i r_i, \end{aligned}$$

where $s_i = n_i m_i$, $t_i = n_i r_i$ and $u_i = m_i r_i$. (It is the observation of the first paragraph of this proof that allows one to cancel the first summand in moving to the last line.) One can easily check that this can be obtained as an IP polynomial using an appropriate polynomial of degree 2 composed with an appropriate IP system.

Hence without loss of generality, we may assume that p is of degree 2. Supposing this to be the case, we have

$$v(\alpha) = \sum_{i=1}^{t} c_i n^{(i)}(\alpha) m^{(i)}(\alpha) + \sum_{i=1}^{t} b_i k^{(i)}(\alpha)$$

for some IP systems $n^{(i)}$, $m^{(i)}$ and $k^{(i)}$, $1 \le i \le t$, having generating sequences $(n_j^{(i)})_{j=1}^{\infty}$, $(m_j^{(i)})_{j=1}^{\infty}$ and $(k_j^{(i)})_{j=1}^{\infty}$, respectively. In particular, equating generating functions on sets of cardinality two yields $\sum_{r=1}^{t} c_r \big(n_i^{(r)} m_j^{(r)} + m_i^{(r)} n_j^{(r)}\big) = 2^i 3^j$ for $i < j$.

Write

$$x_i = \langle c_1 n_i^{(1)}, c_2 n_i^{(2)}, \ldots, c_s n_i^{(s)}, c_1 m_i^{(1)}, c_2 m_i^{(2)}, \ldots, c_s m_i^{(s)} \rangle \text{ and}$$
$$y_j = \langle m_j^{(1)}, m_j^{(2)}, \ldots, m_j^{(s)}, n_j^{(1)}, n_j^{(2)}, \ldots, n_j^{(s)} \rangle.$$

Then $\langle y_i, x_j \rangle = \langle x_i, y_j \rangle = 2^i 3^j$ for all $i < j$. (Here $\langle a, b\rangle$ is the dot product of a and b.)

For $N \in \mathbf{N}$, let $S_N = \text{span}\,\{x_i : 1 \le i \le N\}$. Then (S_N) is an increasing sequence of subgroups of $\mathbf{Z}^{2s}$. Consequently, there exists $N \in \mathbf{N}$ such that $\{x_i : i \in \mathbf{N}\} \subset S_N$. In particular, $x_{N+3} = \sum_{i=1}^{N} a_i x_i$ for some appropriately chosen a_i's. We now have

$$2^{N+2} 3^{N+3} = \big\langle x_{N+3}, y_{N+2} \big\rangle = \Big\langle \sum_{i=1}^{N} a_i x_i, y_{N+2} \Big\rangle = \sum_{i=1}^{N} a_i 2^i 3^{N+2},$$

which implies that $\sum_{i=1}^{N} a_i 2^i = 2^{N+2} \cdot 3$. On the other hand,

$$2^{N+1} 3^{N+3} = \big\langle x_{N+3}, y_{N+1} \big\rangle = \Big\langle \sum_{i=1}^{N} a_i x_i, y_{N+1} \Big\rangle = \sum_{i=1}^{N} a_i 2^i 3^{N+1},$$

which implies that $\sum_{i=1}^{N} a_i 2^i = 2^{N+1} \cdot 9$, a contradiction. □

Hindman's Theorem ([Hi]) states that if $\mathcal{F}^{(1)}$ is an IP ring and $\mathcal{F}^{(1)} = \bigcup_{i=1}^{r} C_i$ then for some i, $1 \le i \le r$, C_i contains an IP ring $\mathcal{F}^{(2)}$. A natural consequence is the following.

Proposition 1.3 ([FK], Theorem 1.5.) Suppose that, for every $n \in \mathbf{N}$, X_n is a compact metric space and x_n is an $\mathcal{F}$-sequence in X_n. Then there exists an IP ring $\mathcal{F}^{(1)}$ such that

$$\operatorname*{IP-lim}_{\alpha \in \mathcal{F}^{(1)}} x_n(\alpha) = z_n$$

exists for each $n \in \mathbf{N}$.

Lemma 1.4. ([FK]) Suppose x is a bounded $\mathcal{F}$-sequence of vectors in a Hilbert space $\mathcal{H}$ and $\mathcal{F}^{(1)}$ is an IP ring. If

$$\operatorname*{IP-lim}_{\beta \in \mathcal{F}^{(1)}} \operatorname*{IP-lim}_{\alpha \in \mathcal{F}^{(1)}} \big\langle x(\alpha),\ x(\alpha \cup \beta) \big\rangle = 0$$

then for some IP subring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$, $\text{IP-}\lim_{\alpha \in \mathcal{F}^{(2)}} x(\alpha) = 0$ in the weak topology.

Finally we have the following.

Proposition 1.5. ([FK, Theorem 1.7].) Suppose that $\mathcal{H}$ is a Hilbert space and $\{U_\alpha\}_{\alpha \in \mathcal{F}}$ is an IP system of unitary operators on $\mathcal{H}$. If for some IP ring $\mathcal{F}^{(1)}$

$$\text{IP-}\lim_{\alpha \in \mathcal{F}^{(1)}} U_\alpha = P$$

weakly, then P is an orthogonal projection onto a subspace of $\mathcal{H}$.

We now give two lemmas which will help facilitate the proof of Theorem 1.9.

Lemma 1.6. Suppose that $s \in \mathbf{N}$ and that v is an $\mathcal{F}$-sequence in $\mathbf{Z}^s$. Then for any IP ring $\mathcal{F}^{(1)}$ there exists $l \leq s$, an $l-$dimensional subgroup $V \subset \mathbf{Z}^s$, and an IP subring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$, such that $\{v(\alpha) : \alpha \in \mathcal{F}^{(2)}\} \subset V$ and such that (if $l > 0$) whenever $\alpha_1, \cdots, \alpha_l \in \mathcal{F}^{(2)}$ with $\alpha_1 < \cdots < \alpha_l$, the set $\{v(\alpha_1), \cdots, v(\alpha_l)\}$ is linearly independent.

Proof. [BFM, Lemma 1.6]. □

Lemma 1.7. Suppose that $l \in \mathbf{N}$, $\mathcal{F}^{(1)}$ is an IP ring, $\mathcal{H}$ is a Hilbert space and $\{P(\alpha)\}_{\alpha \in \mathcal{F}}$ is an $\mathcal{F}$-sequence of commuting orthogonal projections on $\mathcal{H}$ such that if $\alpha_1, \cdots, \alpha_l \in \mathcal{F}^{(1)}$ with $\alpha_1 < \cdots < \alpha_l$, and $f \in \mathcal{H}$ we have $\big(\prod_{i=1}^{l} P(\alpha_i)\big) f = 0$. Then $\text{IP-}\lim_{\alpha \in \mathcal{F}^{(1)}} \big|\big|P(\alpha) f\big|\big| = 0$.

Proof. [BFM, Lemma 1.7]. □

The following fact is routine.

Proposition 1.8. If v is an FVIP system and $\alpha \in \mathcal{F}$ then the derived group of v contains the derived group of $D_\alpha v$.

This is our main theorem.

Theorem 1.9. Suppose that $\mathcal{H}$ is a Hilbert space and that Γ is a commutative group of unitary operators on $\mathcal{H}$. Let v be an FVIP system in Γ and suppose $\mathcal{F}^{(1)}$ is an IP ring such that for each $f \in \mathcal{H}$,

$$\text{IP-}\lim_{\alpha \in \mathcal{F}^{(1)}} v(\alpha) f = Pf$$

exists in the weak topology. Then P is an orthogonal projection.

Proof. We use induction on $d = \deg v$. The case $d = 1$ is easily seen to follow from Proposition 1.5. Suppose now that the conclusion holds for FVIP systems of degree less than d. Clearly $||P|| \leq 1$. It is not difficult to show that any idempotent Hilbert space operator Q with $\big|\big|Q\big|\big| \leq 1$ satisfies $Q = Q^*$ and is an orthogonal projection onto a subspace of $\mathcal{H}$. Hence we need only show that $Pf = P^2 f$ for an arbitrary $f \in \mathcal{H}$, which we now fix. We also remark that since we need only consider the space spanned by f and its translates under the $v(\alpha)$'s, we can assume without loss of generality that $\mathcal{H}$ is separable.

Let W be the derived group of $\big(v(\alpha)\big)_{\alpha \in \mathcal{F}}$. Being a finitely generated abelian group, W has a subgroup of finite index isomorphic to $\mathbf{Z}^s$ for some s. Since we may by Proposition 1.1 upon restriction to a suitable IP ring assume that $D_\alpha v$ belongs to this torsion-free subgroup, we will simply assume that W itself is torsion-free. Accordingly, by Lemma 1.6 there exists $l \leq s$, an $l-$dimensional subgroup $V \subset W$ and an IP ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that $\{D_\alpha v : \ \alpha \in \mathcal{F}^{(2)}\} \subset V$ and such that whenever $\alpha_i \in \mathcal{F}^{(2)}$, $\alpha_1 < \cdots < \alpha_l$, the set $\{D_{\alpha_i} v : 1 \leq i \leq l\}$ is linearly independent (as a subset of W). By Proposition 1.3, we may further require of $\mathcal{F}^{(2)}$ that all limits we encounter along $\mathcal{F}^{(2)}$ exist.

For $u \in W$, let us define $P_u = \underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ u(\alpha)$, where the limit is in the weak operator topology. These limits are orthogonal projections and commuting by the induction hypothesis. For Y a subgroup of W, we denote by P_Y the orthogonal projection onto $\big\{f : \ P_u f = f \text{ for every } u \in Y\big\}$. Observe that if Y is the $l-$dimensional subgroup of V generated by the linearly independent set $\{D_{\alpha_i} v : \ 1 \leq i \leq l\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_l$, we have $\prod_{i=1}^l P_{D_{\alpha_i} v} = P_Y$.

For every $n \in \mathbf{N}$ let $V_n = \{g^{n!} : g \in W\} \cap V$. Then V_n is an l-dimensional subgroup of V for each n and P_{V_n} is an increasing sequence of orthogonal projections, so that $Q = \lim_{n\to\infty} P_{V_n}$ is an orthogonal projection. Furthermore, we have for every l-dimensional subgroup $Y \subset V$, $V_n \subset Y$ for all n large enough. Hence

$$Q\mathcal{H} = \overline{\{f \in \mathcal{H} : \ P_{V_n} f = f \text{ for some } n \in \mathbf{N}\}},$$

$$(Q\mathcal{H})^\perp = \{f \in \mathcal{H} : \ P_Y f = 0 \text{ for every } l\text{-dimensional subgroup } Y \subset V\}.$$

According to our earlier remarks, all we must show is that for an arbitrarily chosen $f \in \mathcal{H}$, which we now fix, $Pf = P^2 f$. We may assume that $\big|\big|f\big|\big| < 1$. We have $f = g + h$, where $g \in Q\mathcal{H}$ and $h \in (Q\mathcal{H})^\perp$. We claim that $Ph = \underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ v(\alpha)h = 0$. As (per our choice of $\mathcal{F}^{(2)}$) we already know this limit to exist, by Lemma 1.4 it suffices to show that

$$\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \underset{\beta \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \big\langle v(\alpha \cup \beta)h, v(\beta)h\big\rangle = 0.$$

Notice that by the properties ascribed to $\mathcal{F}^{(2)}$ earlier and the fact that $h \in (Q\mathcal{H})^\perp$, we have that whenever $\alpha_i \in \mathcal{F}^{(2)}$, $\alpha_1 < \cdots < \alpha_l$, $\big(\prod_{i=1}^l P_{D_{\alpha_i} v}\big)h = 0$. Therefore by Lemma 1.7 we have

$$\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \big|\big|P_{D_\alpha v} h\big|\big| = 0$$

and

$$\begin{aligned}
\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \underset{\beta \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \Big\langle v(\alpha \cup \beta)h, v(\beta)h\Big\rangle &= \underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \underset{\beta \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \Big\langle D_\alpha v(\beta)h, v(\alpha)^{-1}h\Big\rangle \\
&= \underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \Big\langle P_{D_\alpha v} h, v(\alpha)^{-1}h\Big\rangle \\
&\leq \big|\big|h\big|\big| \ \big(\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \ \big|\big|P_{D_\alpha v} h\big|\big|\big) = 0.
\end{aligned}$$

This establishes our claim.

Next we show that $Pg = P^2 g$. Let ρ be a metric on the unit ball of $\mathcal{H}$ for the weak topology satisfying $\rho(x,y) \leq \big|\big|x - y\big|\big|$ and let $\epsilon > 0$ be arbitrary. Choose k, $\big|\big|k\big|\big| < 1$, and $n \in \mathbf{N}$, with $P_{V_n} k = k$ and $||g - k|| < \epsilon$. There exists $\alpha_0 \in \mathcal{F}^{(2)}$ such that for every $\alpha \in \mathcal{F}^{(2)}$, $\alpha > \alpha_0$, $\rho\big(v(\alpha)k,\ Pk\big) < \epsilon$ and

$$\rho\Big(v(\alpha)Pk,\ P^2 k\Big) < \epsilon. \tag{1.4}$$

Let $\alpha \in \mathcal{F}^{(2)}$ be chosen with $\alpha > \alpha_0$ and such that $D_\alpha v \in V_n$. (Existence of such an α follows from Proposition 1.1.) For every $\beta \in \mathcal{F}^{(2)}$, $\beta > \alpha$, we have $(\alpha \cup \beta) > \alpha_0$ as well, so that

$$\rho\big(v(\alpha)v(\beta)D_\alpha v(\beta)k, Pk\big) = \rho\big(v(\alpha \cup \beta)k,\ Pk\big) < \epsilon. \tag{1.5}$$

Since $D_\alpha v \in V_n$ there exists $\beta_0 \in \mathcal{F}^{(2)}$, $\beta_0 > \alpha$, such that for every $\beta \in \mathcal{F}^{(2)}$ with $\beta > \beta_0$, $||D_\alpha v(\beta)k - k|| < \epsilon$, which implies that

$$\rho\big(v(\alpha)v(\beta)D_\alpha v(\beta)k, v(\alpha)v(\beta)k\big) < \epsilon. \tag{1.6}$$

We may now fix such a β with the further property that

$$\rho(v(\alpha)v(\beta)k, v(\alpha)Pk) < \epsilon. \tag{1.7}$$

(We have used weak continuity of $v(\alpha)$.) Equations (1.4-7) and the triangle inequality give

$$\rho(Pk, P^2k) < 4\epsilon.$$

Recall that $||Px|| \leq ||x||$ and $\rho(x, y) \leq ||x - y||$. Hence

$$\rho(Pg, Pk) < \epsilon \text{ and } \rho(P^2 g, P^2 k) < \epsilon,$$

which gives us finally $\rho\big(Pg, P^2 g\big) < 6\epsilon$. Since ϵ was arbitrary, we have

$$Pf = Pg = P^2 g = P^2 f.$$

□

2. Applications.

Our first application of Theorem 1.9 is the following generalization of Khinchine's recurrence theorem.

Theorem 2.1. Suppose that (X, μ) is a measure space with $\mu(X) = 1$ and that G is an abelian group of measure preserving transformations on X. Let $\big(v(\alpha)\big)_{\alpha \in \mathcal{F}}$ be an FVIP system in G. Then for any measurable set A, there exists an IP ring $\mathcal{F}^{(1)} \subset \mathcal{F}$ such that

$$\underset{\alpha \in \mathcal{F}^{(1)}}{\text{IP-lim}}\ \mu\big(A \cap v(\alpha)A\big) \geq \mu(A)^2.$$

Proof. By letting H be a countable subgroup of G and restricting attention (if necessary) to the σ-algebra generated by $\{T1_A : T \in H\}$, we may assume that $L^2(X)$ is separable.

Accordingly, one may choose, by using a standard diagonal argument, an IP ring $\mathcal{F}^{(1)}$ with $\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\ v(\alpha)f = Pf$ existing for all $f \in L^2(X)$. By Theorem 1.9, P is an orthogonal projection, so that

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\ \mu\big(A\cap v(\alpha)A\big) = \underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\ \big\langle 1_A, v(\alpha)1_A\big\rangle = \big\langle 1_A, P1_A\big\rangle = ||P1_A||^2 \geq \mu(A)^2.$$

□

Next, we have a combinatorial version of Theorem 2.1.

Theorem 2.2. Let G be a countable abelian group and suppose $E \subset G$ such that (with respect to a given Følner sequence for G), $\overline{d}(E) > 0$. If $\big(v(\alpha)\big)_{\alpha\in\mathcal{F}}$ is an FVIP system into G then there exists an IP ring $\mathcal{F}^{(1)} \subset \mathcal{F}$ such that

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\ \overline{d}\Big(E\cap\big(E - v(\alpha)\big)\Big) \geq \overline{d}(E)^2.$$

Proof. By the so-called Furstenberg correspondence principle, there exists a probability space $(X, \mathcal{A}, \mu)$, a measurable set $A \in \mathcal{A}$ with $\mu(A) \geq \overline{d}(E)$, and a measure preserving G-action T_g of X such that for every $g \in G$, $\overline{d}\big(E\cap(E-g)\big) \geq \mu\big(A\cap T_g(A)\big)$. The result now follows immediately from Theorem 2.1. □

In order to lend significance to the above two theorems, we now give some natural examples of classes of FVIP systems in commutative groups.

For $r \in \mathbf{R}$, let $[r]$ denote the integer part of r, i.e. the greatest integer less than or equal to r. Put also $\{r\} = r - [r]$, the fractional part of r, and $\langle r\rangle = |r - [r + \frac{1}{2}]|$, the distance from r to the nearest integer. For fixed $l \in \mathbf{N}$, the set of *generalized polynomials* $\mathbf{Z}^l \to \mathbf{Z}$ is the smallest set $\mathcal{G}$ that is a function algebra (i.e. is closed under sums and products) containing $\mathbf{Z}[x_1, \ldots, x_l]$ and having the additional property that for all $m \in \mathbf{N}$, $c_1, \ldots, c_m \in \mathbf{R}$ and $p_1, \ldots, p_m \in \mathcal{G}$, the mapping $\mathbf{n} \to [\sum_{i=1}^m c_i p_i(\mathbf{n})]$ is in $\mathcal{G}$.

The *admissible generalized polynomials* $\mathbf{Z}^l \to \mathbf{Z}$ consist of the smallest subset $\mathcal{G}_a$ of the generalized polynomials that includes, for $1 \leq i \leq l$, $(n_1, \ldots, n_l) \to n_i$, is closed under differences, is an ideal in the space of all generalized polynomials, i.e. is such that if $p \in \mathcal{G}_a$ and $q \in \mathcal{G}$ then $pq \in \mathcal{G}_a$, and has the property that for all $m \in \mathbf{N}$, $c_1, \ldots, c_m \in \mathbf{R}$, $p_1, \ldots, p_m \in \mathcal{G}_a$ and $0 < k < 1$, the mapping $\mathbf{n} \to [\sum_{i=1}^m c_i p_i(\mathbf{n}) + k]$ is in $\mathcal{G}_a$.

It is clear from the definition that if $g(x)$ is admissible then $g(\mathbf{0}) = 0$, and that, in fact, usual polynomial mappings $p : \mathbf{Z}^l \to \mathbf{Z}$ are admissible if and only if $p(\mathbf{0}) = 0$. The class of admissible generalized polynomials contains such things as:

$$p(n_1, n_2) = [\sqrt{3}[\sqrt{2}n_1^2 n_2]n_2^5 + \sqrt{17}n_1^3 + \frac{1}{2}][\sqrt{5}n_2].$$

Finally, if $t \in \mathbf{N}$ we say a map $p : \mathbf{Z}^l \to \mathbf{Z}^t$ is an admissible generalized polynomial if its coordinate functions are admissible generalized polynomials.

Theorem 2.3 Let $p(x)$ be an admissible generalized polynomial $\mathbf{Z}^l \to \mathbf{Z}^t$ and suppose $\big(x(\alpha)\big)_{\alpha\in\mathcal{F}}$ is an IP system in $\mathbf{Z}^l$. Then for every IP ring $\mathcal{F}^{(1)}$ there exists an IP ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that $\big(p(x(\alpha))\big)_{\alpha\in\mathcal{F}^{(2)}}$ is an FVIP system in $(\mathbf{Z}, +)$.

Note that Theorem 2.3 is sufficient, when combined with Theorem 2.2, to obtain Theorem A of the introduction. The remainder of this section will be primarily devoted to a proof of Theorem 2.8, which is a more general version of Theorem 2.3. First, however, we note that it implies, by the original argument, the following generalized polynomial version of [BFM, Proposition 2.3].

Theorem 2.4 Suppose (Y, ν) is a measure space with $\nu(Y) \leq \infty$ and that S is a conservative (in case $\nu(Y) = \infty$) measure preserving transformation of Y. Suppose that (X, μ) is a probability space and that T is an invertible measure preserving tranformation of X. Then if $A \subset Y \times X$ with $(\nu \times \mu)(A) > 0$ and $p(x)$ is an admissible generalized polynomial, there exists $n \in \mathbf{N}$ such that

$$(\nu \times \mu)\big(A \cap (S^n \times T^{p(n)})^{-1} A\big) > 0.$$

We now proceed toward our goal of proving a more general version of Theorem 2.3. First, we need two simple lemmas.

Lemma 2.5. Let $(x_i)_{i=1}^\infty \subset \mathbf{Z}$, $(y_i)_{i=1}^\infty \subset \mathbf{R}$, $0 < k < 1$ and suppose that $\sum_{i=1}^\infty |x_i| \langle y_i \rangle < \min\{k, 1-k\}$. Then for every $n \in \mathbf{N}$, $[\sum_{i=1}^n x_i y_i + k] = \sum_{i=1}^n x_i [y_i + \frac{1}{2}]$.

Proof.

$$\Big|\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i [y_i + \frac{1}{2}]\Big| \leq \sum_{i=1}^n |x_i| \cdot |y_i - [y_i + \frac{1}{2}]| = \sum_{i=1}^n |x_i| \langle y_i \rangle \leq \min\{k, 1-k\} < 1.$$

□

The second is [BKM, Lemma 2.2]. We include a proof for convenience.

Lemma 2.6. Let $(v_\alpha^{(i)})_{\alpha \in \mathcal{F}}$ be VIP systems in $\mathbf{R}$, $i \in \mathbf{N}$. For any IP ring $\mathcal{F}^{(1)}$ there exists an IP ring $\mathcal{F}^{(2)}$ such that

$$\operatorname*{IP-lim}_{\alpha \in \mathcal{F}^{(2)}} \big\langle v_i(\alpha) \big\rangle = 0$$

for all $i \in \mathbf{N}$.

Proof. We prove the result for a single VIP system $\big(v(\alpha)\big)_{\alpha \in \mathcal{F}}$, whereupon the general result follows by a standard diagonal argument. It is clear from the definitions that $\big(\{v(\alpha)\}\big)_{\alpha \in \mathcal{F}^{(1)}}$ is a VIP system on the group $[0, 1)$ with operation addition modulo 1. Choose $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that $\big(\{v(\alpha)\}\big)_{\alpha \in \mathcal{F}^{(2)}}$ is convergent in the torus topology (in which 0 and 1 are identified) and call the limit x. If d is the degree of the system, then by [M, Proposition 2.5] one has, for $\alpha_0 < \alpha_1 < \cdots < \alpha_d$, $\alpha_i \in \mathcal{F}^{(2)}$,

$$\sum_{\substack{\{\beta_1, \cdots, \beta_t\} \subset \{\alpha_0, \cdots, \alpha_d\} \\ \beta_i \neq \beta_j,\ 1 \leq i < j \leq t}} (-1)^t v(\beta_1 \cup \cdots \cup \beta_t) = 0.$$

Taking the limit along $\mathcal{F}^{(2)}$ in the preceding equation yields 2^d terms of $-x$ and $2^d - 1$ terms of x, and these sum to zero. That is, $-x = 0$. □

The following lemma constitutes the bulk of the proof of Theorem 2.8 below. For $\alpha \in \mathcal{F}$, we write $\alpha^k_<$ for the set of k-tuples $(i_1, \ldots, i_k) \in \alpha^k$ satisfying $i_1 < i_2 < \ldots < i_k$.

Lemma 2.7. Suppose $\big(v_i(\alpha)\big)_{\alpha\in\mathcal{F}}$ are FVIP systems into $\mathbf{Z}$, $1 \le i \le m$. One has the following:

(1) $\big(v_1(\alpha) + v_2(\alpha)\big)_{\alpha\in\mathcal{F}}$ is an FVIP system.

(2) For any $n \in \mathbf{Z}$, $\big(v_1(\alpha)(v_2(\alpha) + n)\big)_{\alpha\in\mathcal{F}}$ is an FVIP system.

(3) If $c_1, \ldots, c_m, k \in \mathbf{R}$ with $0 < k < 1$ then there exists an IP ring $\mathcal{F}^{(1)}$ such that $\big([\sum_{i=1}^m c_i v_i(\alpha) + k]\big)_{\alpha\in\mathcal{F}^{(1)}}$ is an FVIP system.

Proof. (1) is obvious; we have of course been assuming all along that FVIP systems form a group under addition. For (2), we may by iterated use of (1) assume without loss of generality that $n = 0$ and that $\big(v_1(\alpha)\big)_{\alpha\in\mathcal{F}}$ and $\big(v_2(\alpha)\big)_{\alpha\in\mathcal{F}}$ are simple FVIP systems:

$$v_1(\alpha) = \sum_{(i_1,\ldots,i_d)\in\alpha^d_<} \prod_{j=1}^{d} n^{(j)}_{i_j}, \qquad v_2(\alpha) = \sum_{(l_1,\ldots,l_c)\in\alpha^c_<} \prod_{j=1}^{c} m^{(j)}_{l_j}. \tag{2.1}$$

Then

$$v_1(\alpha)v_2(\alpha) = \sum_{\substack{(i_1,\ldots,i_d)\in\alpha^d_< \\ (l_1,\ldots,l_c)\in\alpha^c_<}} \Big(\prod_{j=1}^{d} n^{(j)}_{i_j}\Big)\Big(\prod_{t=1}^{c} m^{(t)}_{l_t}\Big).$$

The right hand side is a sum of simple FVIP systems, one corresponding to each possible ordering of $\{i_1, \ldots, i_d, l_1, \ldots, l_c\}$ respecting $i_1 < \cdots < i_d$ and $l_1 < \cdots < l_c$. (For example, taking $d = c = 2$, $i_1 < l_1 < l_2 < i_2$ and $l_1 < i_1 < i_2 = l_2$ are two such orderings. In the latter case the corresponding simple FVIP system is of degree at most three, being generated by the sequences $(n^{(1)}_j)$, $(m^{(1)}_j)$ and $(n^{(2)}_j m^{(2)}_j)$.)

For (3), it will be convenient notationally to assume that $m = 1$. Accordingly we will suppress subscripts, writing $v(\alpha)$ for $v_1(\alpha)$ and c for c_1. The case of general m does not follow from the case $m = 1$ directly; the proof however is no different conceptually (merely more complicated).

For an arbitrary sequence $\alpha_1 < \alpha_2 < \cdots$, for $\beta = \{i_1, \ldots, i_M\} \in \mathcal{F}$ we put

$$\begin{aligned} s(\beta) &= v\Big(\bigcup_{i\in\beta} \alpha_i\Big) = v(\alpha_{i_1} \cup \cdots \cup \alpha_{i_M}) \\ &= \sum_{\substack{(a_1,\ldots,a_t) \\ 1\le a_i\le d \\ a_1+\cdots+a_t=d}} \ \sum_{(j_1,\ldots,j_t)\in\beta^t_<} \prod_{i=1}^{t} \ \sum_{(r_1,\ldots,r_{a_i})\in(\alpha_{j_i})^{a_i}_<} \prod_{y=1}^{a_i} n^{(a_1+\cdots+a_{i-1}+y)}_{r_y}. \end{aligned}$$

We write

$$w^{(a_1,\ldots,a_t,i)}(\gamma) = \sum_{(r_1,\ldots,r_{a_i})\in\gamma^{a_i}_<} \prod_{y=1}^{a_i} n^{(a_1+\cdots+a_{i-1}+y)}_{r_y}.$$

Then

$$v\big(\bigcup_{i\in\beta}\alpha_i\big) = \sum_{\substack{(a_1,\dots,a_t)\\ 1\le a_i\le d\\ a_1+\cdots+a_t=d}} \sum_{(j_1,\dots,j_t)\in\beta^t_<} \prod_{i=1}^{t} w^{(a_1,\dots,a_t,i)}(\alpha_{j_i}), \tag{2.2}$$

while if we fix $a_1,\dots,a_t$, and put $m^{(i)}_j = w^{(a_1,\dots,a_t,i)}(\alpha_j)$, the inner sum in this expression becomes

$$\sum_{(j_1,\dots,j_t)\in\beta^t_<} \prod_{i=1}^{t} m^{(i)}_{j_i},$$

which is a simple FVIP system. This establishes, in particular, that $s(\beta)$ is an FVIP system.

Let us go back, however, to (2.2). We claim that the sequence $(\alpha_i)_{i=1}^\infty$ may be chosen so that

$$\sum_{\substack{(a_1,\dots,a_t)\\ 1\le a_i\le d\\ a_1+\cdots+a_t=d}} \sum_{\substack{(j_1,\dots,j_t)\\ j_1<\cdots<j_t}} \Big(\prod_{i=1}^{t-1} w^{(a_1,\dots,a_t,i)}(\alpha_{j_i})\Big)\langle cw^{(a_1,\dots,a_t,t)}(\alpha_{j_t})\rangle < \min\{k,1-k\}.$$

Indeed, this is immediate. Simply choose α_1 with $\langle cw^{(d,1)}(\alpha_1)\rangle = \langle cv(\alpha_1)\rangle < \frac{\min\{k,1-k\}}{2}$ and, having chosen $\alpha_1,\dots,\alpha_{M-1}$, choose $\alpha_M > \alpha_{M-1}$ such that

$$\sum_{\substack{(a_1,\dots,a_t)\\ 1\le a_i\le d\\ a_1+\cdots+a_t=d}} \sum_{\substack{(j_1,\dots,j_t)\\ j_1<\cdots<j_t=M}} \Big(\prod_{i=1}^{t-1} w^{(a_1,\dots,a_t,i)}(\alpha_{j_i})\Big)\langle cw^{(a_1,\dots,a_t,t)}(\alpha_M)\rangle < \frac{\min\{k,1-k\}}{2^M}.$$

This is possible because of Lemma 2.6; the sum is finite and $cw^{(a_1,\dots,a_t,t)}$ is a VIP system. It follows by Lemma 2.5 that

$$\begin{aligned}[cs(\beta)+k] &= [cv\big(\bigcup_{i\in\beta}\alpha_i\big)+k]\\ &= \sum_{\substack{(a_1,\dots,a_t)\\ 1\le a_i\le d\\ a_1+\cdots+a_t=d}} \sum_{(j_1,\dots,j_t)\in\beta^t_<} \Big(\prod_{i=1}^{t-1} w^{(a_1,\dots,a_t,i)}(\alpha_{j_i})\Big)[cw^{(a_1,\dots,a_t,t)}(\alpha_{j_t})+\frac{1}{2}].\end{aligned}$$

Fixing $a_1,\dots,a_t$, if we put $m^{(i)}_j = w^{(a_1,\dots,a_t,i)}(\alpha_j)$, $1\le i\le t-1$, and let $m^{(t)}_j = [cw^{(a_1,\dots,a_t,t)}(\alpha_{j_t})+\frac{1}{2}]$, the inner sum in this expression becomes

$$\sum_{(j_1,\dots,j_t)\in\beta^t_<} \prod_{i=1}^{t} m^{(i)}_{j_i},$$

which is a simple FVIP system. This establishes that $([cs(\beta)+k])_{\beta\in\mathcal{F}}$ is an FVIP system, i.e. $\big([cv(\alpha)+k]\big)_{\alpha\in\mathcal{F}^{(1)}}$ is an FVIP system. $\square$

Here now is the more general version of Theorem 2.3 mentioned earlier.

Theorem 2.8. Let $p(x)$ be a generalized polynomial $\mathbf{Z}^l\to\mathbf{Z}^t$ and suppose $\big(x(\alpha)\big)_{\alpha\in\mathcal{F}}$ is an FVIP system in $\mathbf{Z}^l$. Then for every IP ring $\mathcal{F}^{(1)}$ there exists an IP ring $\mathcal{F}^{(2)}\subset\mathcal{F}^{(1)}$ and some $n\in\mathbf{Z}^t$ such that $\big(p(x(\alpha))+n\big)_{\alpha\in\mathcal{F}^{(2)}}$ is FVIP. If p is admissible then $n=0$.

Proof. Clearly it suffices to consider the case $t=1$. Let $\mathcal{H}$ be the set of generalized polynomials for which the weak conclusion (n not necessarily zero) holds. $\mathcal{H}$ clearly contains $\mathbf{Z}[x_1\dots,x_l]$ and is closed under sums and products by Lemma 2.7 (1) and (2), respectively. Let $m\in\mathbf{N}$, $c_1,\dots,c_m\in\mathbf{R}$ and $p_1,\dots,p_m\in\mathcal{H}$. We claim that $q(x)=[\sum_{i=1}^m c_ip_i(x)]\in\mathcal{H}$.

By hypothesis there are an IP ring $\mathcal{F}^{(2)}\subset\mathcal{F}^{(1)}$ and some FVIP systems $\big(v_i(\alpha)\big)_{\alpha\in\mathcal{F}^{(2)}}$ such that for all $\alpha\in\mathcal{F}^{(2)}$, $q\big(x(\alpha)\big)=[\sum_{i=1}^m c_ip_i\big(x(\alpha)\big)]=[\sum_{i=1}^m c_i\big(v_i(\alpha)-n_i\big)]=[\sum_{i=1}^m c_iv_i(\alpha)+k]-n$, where $n\in\mathbf{Z}$ and $0\le k<1$. If $0\neq k$ then, by Lemma 2.7 (3), $q\big(x(\alpha)\big)+n=[\sum_{i=1}^m c_iv_i(\alpha)+k]$ is FVIP upon passing to a sub-ring, so that $q\in\mathcal{H}$ as desired. Otherwise, Hindman's theorem may be used to extract a sub-ring of $\mathcal{F}^{(2)}$ along which $[\sum_{i=1}^m c_iv_i(\alpha)]$ is either always equal to (i) $[\sum_{i=1}^m c_iv_i(\alpha)+\frac{1}{2}]$, which implies that $\Big(q\big(x(\alpha)\big)+n\Big)$ is FVIP along a further subring as before, or (ii) $[\sum_{i=1}^m c_iv_i(\alpha)+\frac{1}{2}]-1$, which implies that $\Big(q\big(x(\alpha)\big)+n+1\Big)$ is FVIP along a further subring. In either case, we have $q(x)\in\mathcal{H}$.

The argument so far establishes that $\mathcal{H}$ contains all of the generalized polynomials, as required. Next let $\mathcal{H}'$ be the subset of $\mathcal{H}$ satisfying the stronger conclusion ($n=0$). $\mathcal{H}'$ clearly contains the maps $(n_1,\dots,n_l)\to n_i$, $1\le i\le l$. By Lemma 2.7 (1), $\mathcal{H}'$ is closed under sums. By Lemma 2.7 (2), together with the part of this theorem that has already been proved, $\mathcal{H}'$ is an ideal. Finally, Lemma 2.7 (3) shows that if $m\in\mathbf{N}$, $c_1,\dots,c_m\in\mathbf{R}$, $p_1,\dots,p_m\in\mathcal{H}'$ and $0<k<1$ then $n\to[\sum_{i=1}^m c_ip_i(n)+k]$ is in $\mathcal{H}'$. These facts together establish that $\mathcal{H}'$ contains the admissible generalized polynomials. $\square$

Combining Theorem 2.8 with Theorem 2.2, we get the following strengthening of Theorem A from the introduction.

Corollary 2.9. Let p be an admissible generalized polynomial $\mathbf{Z}^l\to\mathbf{Z}^t$. Then for any set $E\subset\mathbf{Z}^t$ with $\overline{d}(E)>0$ and any $\epsilon>0$, the set

$$\big\{x\in\mathbf{Z}^l:\overline{d}\big(E\cap(E-p(x))\big)>(\overline{d}(E))^2-\epsilon\big\}$$

is *FVIP**.

3. Actions of $\mathbf{Z}^\infty$ and IP_r systems.

We denote by $\mathbf{Z}^\infty$ the free abelian semigroup on generators $\{e_j:j\in\mathbf{N}\}$. We identify $\mathbf{Z}^\infty$ with the set of all functions $\mathbf{N}\to\mathbf{N}\cup\{0\}$ such that $j\to0$ for all but finitely many j. Thus a typical element of $\mathbf{Z}^\infty$ may be represented as $(n_1,n_2,\dots,n_k,0,0,\dots)$. For $l,r\in\mathbf{N}$, we write $\mathbf{Z}(l,r)$ for the set of all members of $\mathbf{Z}^\infty$ having representation $(n_1,\dots,n_r,0,0,\dots)$

with $0 \le n_i < l$, $1 \le i \le r$. Notice that for sequences (l_n) and (r_n) increasing to ∞, $\big(\mathbf{Z}(l_n, r_n)\big)$ is a Følner sequence for $\mathbf{Z}^\infty$.

For the remainder of this section, our convention will be that when we write, for example, $\alpha = \{s_1, s_2, \dots, s_r\}$, we stipulate that $s_1 < s_2 < \cdots < s_r$. Also, recall that by a polynomial $p : \mathbf{Z} \to \mathbf{Z}$ we mean some $p(x) \in \mathbf{Q}[x]$ such that $p(\mathbf{Z}) \subset \mathbf{Z}$.

Theorem 3.1 Let $p : \mathbf{Z} \to \mathbf{Z}$ be a polynomial, and for $\alpha = \{s_1, s_2, \dots, s_r\} \in \mathcal{F}$ let $v(\alpha) = \sum_{t=1}^r p(t) e_{s_t} \in \mathbf{Z}^\infty$. Then v is an FVIP system.

Proof. Let d be the degree of p, put $n_i = 1$ for every $i \in \mathbf{N}$ and for $\alpha = \{s_1, \dots, s_r\}$ set:

$$v_0(\alpha) = \sum_{t=1}^r e_{s_t} = \sum_{i \in \alpha} e_i$$

$$v_1(\alpha) = \sum_{t=2}^r (t-1) e_{s_t} = \sum_{i,j \in \alpha, i<j} n_i e_j$$

$$v_2(\alpha) = \sum_{t=3}^r \binom{t-1}{2} e_{s_t} = \sum_{\substack{i,j,k \in \alpha \\ i<j<k}} n_i n_j e_k.$$

$$\vdots$$

$$v_d(\alpha) = \sum_{t=d+1}^r \binom{t-1}{d} e_{s_t} = \sum_{\substack{i_1, \dots, i_{d+1} \in \alpha \\ i_1 < \cdots < i_{d+1}}} n_{i_1} n_{i_2} \cdots n_{i_d} e_{i_{d+1}}.$$

The proof is now completed by the well-known fact that the polynomials $\binom{t-1}{i}$, $0 \le i \le d$, form a basis for the polynomials $\mathbf{Z} \to \mathbf{Z}$ of degree at most d. Or, to be a bit more precise, we have $p(t) = \sum_{i=0}^d a_i \binom{t-1}{i}$, where $a_i = \sum_{k=1}^{i+1} p(k)(-1)^{i-k+1}\binom{i}{k-1}$. Hence $v = \sum_{i=0}^d a_i v_i$, so that v is an FVIP system. □

We now have the following corollary.

Corollary 3.2. Let $p : \mathbf{Z} \to \mathbf{Z}$ be a polynomial and for $\{s_1, s_2, \dots, s_r\} = \alpha \in \mathcal{F}$ let $v(\alpha) = \sum_{t=1}^r p(t) e_{s_t} \in \mathbf{Z}^\infty$.

a. Suppose $E \subset \mathbf{Z}^\infty$ such that (with respect to a given Følner sequence) $\overline{d}(E) > 0$. There exists an IP ring $\mathcal{F}^{(1)} \subset \mathcal{F}$ such that

$$\underset{\alpha \in \mathcal{F}^{(1)}}{\text{IP-lim}}\ \overline{d}\Big(E \cap \big(E - v(\alpha)\big)\Big) \ge \overline{d}(E)^2.$$

b. For all $\epsilon, \delta > 0$ there exist L and R (depending on ϵ, δ and p) such that if $l \ge L$, $r \ge R$ and $E \subset \mathbf{Z}(l, r)$ with $|E| \ge \delta |\mathbf{Z}(l, r)|$ then for some non-empty set $\alpha \in \{1, 2, \dots, R\}$, $|E \cap \big(E - v(\alpha)\big)| \ge (\delta^2 - \epsilon)|\mathbf{Z}(l, r)|$.

Proof. a. follows immediately from Theorem 3.1 and Theorem 2.2. Suppose for contradiction that b. fails. Then there exist sequences l_n, R_n and r_n going to ∞, with $r_n \ge R_n$, and sets $E_n \subset \mathbf{Z}(l_n, r_n)$ with $|E_n| \geq \delta|\mathbf{Z}(l_n, r_n)|$ but such that for every non-empty $\alpha \in \{1, 2, \ldots, R_n\}$, $|E_n \cap \big(E_n - v(\alpha)\big)| < (\delta^2 - \epsilon)|\mathbf{Z}(l_n, r_n)|$. Now, for any (rapidly enough) increasing function $\pi : \mathbf{N} \to \mathbf{N}$, the set $E = E_{\pi(1)} \cup \bigcup_{n=1}^{\infty} \big(E_{\pi(n+1)} \setminus \mathbf{Z}(l_{\pi(n)}, r_{\pi(n)})\big)$ satisfies $\overline{d}(E) \geq \delta$ with respect to the Følner sequence $\big(\mathbf{Z}(l_{\pi(n)}, r_{\pi(n)})\big)$. Moreover it is not too difficult to see, again if π increases rapidly, that for every $\alpha \in \mathcal{F}$, $\overline{d}\big(E \cap (E - v(\alpha)\big) \leq \delta^2 - \epsilon$, which contradicts a. $\square$

The following theorem contains Corollary 3.2 as a special case.

Theorem 3.3. Let $\epsilon, \delta > 0$ and let $p : \mathbf{Z} \to \mathbf{Z}$ be a polynomial. There exist $\nu > 0$ and $R \in \mathbf{N}$ (depending on ϵ, δ and p) such that if H is any commutative group, $u_1, \ldots, u_R \in H$, $J \subset H$ a finite subset satisfying $|J \triangle \big(J + u_i\big)| < \nu|J|$, $1 \leq i \leq R$, and $S \subset J$ with $|S| \geq \delta|J|$ then for some non-empty $\alpha \subset \{1, \ldots, R\}$, $|S \cap \big(S - u(\alpha)\big)| \geq (\delta^2 - \epsilon)|J|$, where $u(\{s_1, \ldots, s_b\}) = \sum_{t=1}^{b} p(t) u_{s_t}$.

Remark. Notice that if $J = H$ is itself a finite group, the almost invariance condition $|J \triangle \big(J + u_i\big)| < \nu|J|$ is automatically satisfied and we get Theorem C from the introduction.

Proof. Let ϵ, δ and p be given. Let $\delta_1 < \delta$ and $\epsilon_1 > 0$ with $\delta_1^2 - \epsilon_1 > \delta^2 - \epsilon$ and choose $R = R(\epsilon_1, \delta_1, p)$ as in Corollary 3.2 b. Put $M = \max_{1 \leq i \leq R} |p(i)|$ and choose $l \geq L(\epsilon_1, \delta_1, p)$ so large that $(\frac{l - 2M - 1}{l})^R > 1 - \frac{\delta_1^2 - \epsilon_1 - \delta^2 + \epsilon}{2}$. Let $\nu = \min\{\frac{\delta - \delta_1}{2 l^{R+1} R}, \frac{\delta_1^2 - \epsilon_1 - \delta^2 + \epsilon}{2 l^{R+1} R (2M+1)^R}\}$ and suppose H, $u_1, \ldots, u_R$, J and S are given. Let $r \geq R$ be large enough that $l^r \frac{\delta - \delta_1}{2} > l^R |J|$.

Write $X = \{(n_1, \ldots, n_r, 0, 0, \ldots) \in \mathbf{Z}(l, r) : n_1 = n_2 = \cdots = n_R = 0\}$. Choose a dummy element $x \notin H$ and let $\pi : X \to J \cup \{x\}$ satisfy $|\pi^{-1}(x)| < |J|$ and $|\pi^{-1}(j)| = k$, where $\frac{l^{r-R}}{|J|} - 1 < k \leq \frac{l^{r-R}}{|J|}$, for all $j \in J$. Extend π to a function $\pi_1 : \mathbf{Z}(l, r) \to J \cup \{x\}$ by letting $\pi_1(n_1, n_2, \ldots, n_r, 0, 0, \ldots) = \sum_{i=1}^{R} n_i u_i + \pi(0, 0, \ldots, 0, n_{R+1}, \ldots, n_r, 0, 0, \ldots)$ if this sum is defined and in J, and $\pi_1(n_1, n_2, \ldots, n_r, 0, 0, \ldots) = x$ otherwise.

Let

$$B = \{j \in J : \exists (n_1, \ldots, n_R) \in \{0, 1, \ldots, l - 1\}^R \text{ with } j - \sum_{i=1}^{R} n_i u_i \notin J\},$$

and notice that for $j \in J \setminus B$, $|\pi_1^{-1}(j)| = l^R k$, while for $j \in B$, $|\pi_1^{-1}(j)| < l^R k$. (A consequence of the former is that for any set $A \subset \pi_1^{-1}(J)$, $|\pi_1(A)| \geq \frac{|A|}{l^R k}$. We shall make use of this momentarily.) Moreover, $|B| < \frac{\delta - \delta_1}{2}|J|$ and $|\pi_1^{-1}(x)| < \frac{\delta - \delta_1}{2}|\mathbf{Z}(l, r)|$. These facts, taken together, imply that

$$|\pi_1^{-1}(S \setminus B)| = l^R k |S \setminus B| > l^R (\frac{l^{r-R}}{|J|} - 1)(\delta_1 + \frac{\delta - \delta_1}{2})|J|$$

$$> l^r (\delta_1 + \frac{\delta - \delta_1}{2}) - l^R |J| > l^r \delta_1 = \delta_1 |\mathbf{Z}(l, r)|.$$

Accordingly, for some non-empty set $\alpha \subset \{1, 2, \ldots, R\}$,

$$|\pi_1^{-1}(S \setminus B) \cap \big(\pi_1^{-1}(S \setminus B) - v(\alpha)\big)| \geq (\delta_1^2 - \epsilon_1)|\mathbf{Z}(l, r)|, \tag{3.1}$$

where for $\{s_1, s_2, \ldots, s_b\} = \alpha$, we have written $v(\alpha) = \sum_{t=1}^{b} p(t) e_{s_t}$.

Let $C = \{y \in \mathbf{Z}(l,r) : y + v(\alpha) \in \mathbf{Z}(l,r), \pi_1(y) \neq x, \pi_1\big(y + v(\alpha)\big) \neq \pi_1(y) + u(\alpha)\}$. Then $|C| \leq (\delta_1^2 - \epsilon_1 - \delta^2 + \epsilon)|\mathbf{Z}(l,r)|$, so by (3.1) we get

$$\left|\Big(\pi_1^{-1}(S \setminus B) \cap \big(\pi_1^{-1}(S \setminus B) - v(\alpha)\big)\Big) \setminus C\right| \geq (\delta^2 - \epsilon)|\mathbf{Z}(l,r)|.$$

Applying π_1 to this, and using the fact alluded to parenthetically above,

$$\left|\pi_1\bigg(\Big(\pi_1^{-1}(S \setminus B) \cap \big(\pi_1^{-1}(S \setminus B) - v(\alpha)\big)\Big) \setminus C\bigg)\right| \geq \frac{(\delta^2 - \epsilon)}{l^R k}|\mathbf{Z}(l,r)| \geq (\delta^2 - \epsilon)|J|.$$

We claim that

$$\pi_1\bigg(\Big(\pi_1^{-1}(S \setminus B) \cap \big(\pi_1^{-1}(S \setminus B) - v(\alpha)\big)\Big) \setminus C\bigg) \subset \Big(S \cap \big(S - u(\alpha)\big)\Big). \qquad (3.2)$$

This will suffice for the proof, as it shows together with the previous display that $\big|S \cap \big(S - u(\alpha)\big)\big| \geq (\delta^2 - \epsilon)|J|$.

We establish (3.2). Let $y \in \Big(\pi_1^{-1}(S \setminus B) \cap \big(\pi_1^{-1}(S \setminus B) - v(\alpha)\big)\Big) \setminus C$. We must show that $\pi_1(y) \in \Big(S \cap \big(S - u(\alpha)\big)\Big)$. Since $y \in \mathbf{Z}(l,r)$, $y + v(\alpha) \in \mathbf{Z}(l,r)$, $\pi_1(y) \neq x$ but $y \notin C$, it must be the case that $\pi_1(y) + u(\alpha) = \pi_1\big(y + v(\alpha)\big) \in S$. Also $\pi_1(y) \in S$, so $\pi_1(y) \in \Big(S \cap \big(S - u(\alpha)\big)\Big)$, as required. $\square$

We now give a reformulation of the previous Theorem in $\mathbf{Z}$.

Corollary 3.4. Let $\epsilon, \delta > 0$ and let $p : \mathbf{Z} \to \mathbf{Z}$ be a polynomial. There exist $N, R \in \mathbf{N}$ such that if $u_1, \ldots, u_R \in \mathbf{Z}$ and $n > N \max_{1 \leq i \leq R} |u_i|$ then for any $E \subset \{1, \ldots, n\}$ with $|E| \geq \delta n$ there exists a non-empty $\alpha \subset \{1, \ldots, R\}$ having the property that $\big|E \cap \big(E - v(\alpha)\big)\big| > (\delta^2 - \epsilon)n$, where $v(\{s_1, s_2, \ldots, s_b\}) = \sum_{t=1}^{b} p(t) u_{s_t}$.

Proof. Let ν and R be as guaranteed by Theorem 3.3 and let $N > \frac{2}{\nu}$. $\square$

Next, we give a formulation in $\mathbf{R}$.

Corollary 3.5. Let $\epsilon, \delta > 0$ and let $p : \mathbf{Z} \to \mathbf{Z}$ be a polynomial. There exist $R \in \mathbf{N}$ and $\lambda > 0$ such that if $A \subset [0,1]$ is measurable with $m(A) \geq \delta$ and $x_1, \ldots, x_R$ are any real numbers with $|x_i| \leq \lambda$, $1 \leq i \leq R$, then for some non-empty $\alpha \subset \{1, \ldots, R\}$ one has $m\big(A \cap (A - u(\alpha))\big) > \delta^2 - \epsilon$, where $u(\{s_1, s_2, \ldots, s_b\}) = \sum_{i=1}^{b} p(i) x_{s_i}$.

Proof. Choose $\delta_1 < \delta$ and $\epsilon_1 > 0$ with $\delta_1^2 - \epsilon_1 > \delta^2 - \epsilon$. Let N and R be as guaranteed by Corollary 3.4 for ϵ_1, δ_1 and p. Let $\lambda = \frac{1}{2N}$. Suppose A and $x_1, \ldots, x_R$ are given. Let $M = \max_{1 \leq i \leq R} |p(i)|$ and choose $\beta > 0$ so small that $(\delta_1^2 - \epsilon_1)\big(1 - \beta(RM + 2)\big) > \delta^2 - \epsilon$ and $\delta - 2\beta > \delta_1$.

It is routine to show via Lebesgue points of density that for any large enough $t \in \mathbf{N}$, if one partitions $[0,1]$ into t sub-intervals $I_1, \ldots, I_t$ of equal length $\frac{1}{t}$ then for all but perhaps βt exceptional indices j, one will have $tm(A \cap I_j) \in \big([0, \beta] \cup [1 - \beta, 1]\big)$. Using the fact that

$m(A) \geq \delta$, it is then easy to see that if $E = \{j \in \{1, \ldots, t\} : tm(A \cap I_j) \geq 1 - \beta\}$ then $|E| \geq t(\delta - 2\beta) > t\delta_1$.

Another standard fact is that every point on the R-torus is recurrent under a shift map. In particular, letting $T(y_1, \ldots, y_R) = (y_1 + x_1, \ldots, y_R + x_R) \pmod 1$, there exist arbitrarily large t such that the distance from $T^t\mathbf{0}$ to $\mathbf{0}$ is less than β in each coordinate. What this implies is that for $j = 1, \ldots, R$, the distance from tx_j to the nearest integer u_j, is less than β. Choose such a t which is also large in the sense of the last paragraph.

Notice that $|u_j| < \lambda t + \beta$, which implies $N|u_j| < t$, $1 \leq j \leq R$. Hence by the choice of N, R, the conclusion of Corollary 3.4, gives a non-empty $\alpha \subset \{1, \ldots, R\}$ having the property that $\left|E \cap \big(E - v(\alpha)\big)\right| > (\delta_1^2 - \epsilon_1)t$, where $v(\{s_1, s_2, \ldots, s_b\}) = \sum_{i=1}^b p(i)u_{s_i}$. Now, for $j \in E \cap \big(E - v(\alpha)\big)$, one has $m(A \cap I_j) > \frac{1-\beta}{t}$ and $m(A \cap I_{j+v(\alpha)}) > \frac{1-\beta}{t}$, which is to say that $m(A \cap [\frac{j-1}{t}, \frac{j}{t}]) > \frac{1-\beta}{t}$ and $m(A \cap [\frac{j+v(\alpha)-1}{t}, \frac{j+v(\alpha)}{t}]) > \frac{1-\beta}{t}$. One easily checks that $|v(\alpha) - tu(\alpha)| \leq \beta R \max_{1 \leq i \leq R} |p(i)| = \beta RM$, from which it follows that

$$m(A \cap [\frac{j + tu(\alpha) - 1}{t}, \frac{j + tu(\alpha)}{t}]) > \frac{1 - \beta(RM + 1)}{t},$$

which is equivalent to

$$m\Big(\big(A - u(\alpha)\big) \cap [\frac{j-1}{t}, \frac{j}{t}]\Big) > \frac{1 - \beta(RM + 1)}{t}.$$

Now we finally obtain

$$m\Big([\frac{j-1}{t}, \frac{j}{t}] \cap A \cap \big(A - u(\alpha)\big)\Big) > \frac{1 - \beta(RM + 2)}{t}.$$

But this holds for all $j \in E \cap \big(E - v(\alpha)\big)$, so

$$m\Big(A \cap \big(A - u(\alpha)\big)\Big) > (\delta_1^2 - \epsilon_1)t\frac{1 - \beta(RM + 2)}{t} > \delta^2 - \epsilon.$$

□

The previous corollary, situated as it is in the uncountable group $\mathbf{R}$, suffers somewhat from an approach we have adopted since Theorem 2.2, namely restriction to countable groups G. An artifact of this approach is that the "polynomial weights" $p(n)$ appearing there are unnecessarily required to be integer-valued. In fact, the corollary remains valid for polynomials $p : \mathbf{Z} \to \mathbf{R}$, though to prove this one may need to go all the way back to Theorem 2.2 and redo everything for sets of measurable density in uncountable groups. For ease of presentation (and in keeping with a historical precedent set forth in [FK]) we have chosen against this approach.

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Vitaly Bergelson
Department of Mathematics
The Ohio State University
Columbus, OH 43210 USA
vitaly@math.ohio-state.edu

Inger J. Håland Knutson
Department of Mathematics
Agder University College
N-4604 Kristiansand, Norway
Inger.J.Knutson@hia.no

Randall McCutcheon
Dept. of Math. Sciences
University of Memphis
Memphis, TN 38152 USA
rmcctchn@memphis.edu