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Abstract. Ergodic and combinatorial results obtained in Bergelson and Moreira [Ergodic theorem involving additive and multiplicative groups of a field and $\{x + y, xy\}$ patterns. *Ergod. Th. & Dynam. Sys.* to appear, published online 6 October 2015, doi:10.1017/etds.2015.68], involved measure preserving actions of the affine group of a countable field K. In this paper, we develop a new approach, based on ultrafilter limits, which allows one to refine and extend the results obtained in Bergelson and Moreira, *op. cit.*, to a more general situation involving measure preserving actions of the *non-amenable* affine semigroups of a large class of integral domains. (The results and methods in Bergelson and Moreira, *op. cit.*, heavily depend on the amenability of the affine group of a field.) Among other things, we obtain, as a corollary of an ultrafilter ergodic theorem, the following result. Let K be a number field and let $\mathcal{O}_K$ be the ring of integers of K. For any finite partition $K = C_1 \cup \cdots \cup C_r$, there exists $i \in \{1, \ldots, r\}$ such that, for many $x \in K$ and many $y \in \mathcal{O}_K$, $\{x + y, xy\} \subset C_i$.

1. *Introduction*

1.1. *History.* One of the early results in Ramsey theory, due to Schur [**15**], states that, for any finite partition (or, as it is customary to say, coloring) $\mathbb{N} = C_1 \cup \cdots \cup C_r$ of the natural numbers†, one of the cells C_i contains a triple of the form $\{x, y, x + y\}$. It is not hard to see that any finite coloring $\mathbb{N} = \bigcup_{i=1}^r C_i$ yields also a monochromatic triple of the form $\{x, y, xy\}$ (just observe that the restriction of a coloring of $\mathbb{N}$ to the set $\{2^n : n \in \mathbb{N}\}$ induces a new coloring of $\mathbb{N}$ and apply Schur's theorem).

A famous open conjecture states that, for any finite coloring of $\mathbb{N}$, one finds (many) monochromatic quadruples of the form $\{x, y, x + y, xy\}$‡. Even a weaker version of this conjecture, asking for non-trivial monochromatic configurations of the form $\{x + y, xy\}$

† In this paper we abide by the convention that $\mathbb{N} = \{1, 2, 3, \ldots\}$.
‡ A variant of this conjecture for finite fields was recently proved by Green and Sanders [**13**].

is, so far, quite recalcitrant. The above questions become more manageable if one considers finite partitions of the set of rational numbers $\mathbb{Q}$. An ergodic approach, developed by the authors in [**10**], shows that, actually, any 'large' set in $\mathbb{Q}$ (and, indeed, in any countable field K) contains plenty of configurations of the form $\{x + y, xy\}$.

The results obtained in [**10**] naturally lead to new questions which are addressed in this paper. In order to present the questions (and the answers), we need first to introduce pertinent notation and definitions and formulate some relevant results from [**10**].

1.2. *Ergodic theorem for the affine group of a field.* Let K be a countably infinite field. For each $u \in K$, let $A_u : K \to K$ be the addition map $A_u : x \mapsto x + u$ and, for $u \neq 0$, let $M_u : K \to K$ denote the multiplication map $M_u : x \mapsto ux$. Let $\mathcal{A}_K = \{A_u M_v : x \mapsto vx + u \mid u, v \in K, v \neq 0\}$ denote the affine group of K. A sequence $(F_N)_{N\in\mathbb{N}}$ of finite subsets of K is a *double Følner sequence* if it is asymptotically invariant under any fixed affine transformation $g \in \mathcal{A}_K$. Given any double Følner sequence $(F_N)_{n\in\mathbb{N}}$ in K, one can define the *affinely invariant* upper density $\bar{d}_{(F_N)}(\cdot)$ by the formula

$$\bar{d}_{(F_N)}(E) := \limsup_{N\to\infty} \frac{|E \cap F_N|}{|F_N|}, \quad E \subset K.$$

(The affine invariance means that $\bar{d}_{(F_N)}(E) = \bar{d}_{(F_N)}(f(E))$ for any $f \in \mathcal{A}_K$.) The main ergodic theoretical result in [**10**] is the following analogue of von Neumann's mean ergodic theorem.

THEOREM 1.1. [**10**, Lemma 3.2] *Let K be an infinite countable field, let $(U_g)_{g\in\mathcal{A}_K}$ be a unitary representation of $\mathcal{A}_K$ on a Hilbert space $\mathcal{H}$, let $I = \{f \in \mathcal{H} : (\forall g \in \mathcal{A}_K) U_g f = f\}$ be the invariant subspace and let $P : \mathcal{H} \to I$ be the orthogonal projection onto I. Then, for any $f \in \mathcal{H}$ and any double Følner sequence $(F_N)_{N\in\mathbb{N}}$ in K,*

$$\lim_{N\to\infty} \frac{1}{|F_N|} \sum_{u\in F_N} U_{M_u A_{-u}} f = Pf.$$

From Theorem 1.1, one derives the following results.

THEOREM 1.2. [**10**, Theorem 1.4] *Let K be an infinite countable field, let $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathcal{A}_K})$ be a probability measure preserving system and let $B \in \mathcal{B}$. Then, for any double Følner sequence $(F_N)_{N\in\mathbb{N}}$ in K,*

$$\lim_{N\to\infty} \frac{1}{|F_N|} \sum_{u\in F_N} \mu(T_{A_u}^{-1} B \cap T_{M_u}^{-1} B) \geq \mu(B)^2. \tag{1.1}$$

COROLLARY 1.3. [**10**, Corollary 2.13] *Let K be an infinite countable field, let $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathcal{A}_K})$ be a probability measure preserving system and let $B \in \mathcal{B}$. Then, for any $\delta \in (0, 1)$, the set*

$$\mathcal{R}(B, \delta) := \{u \in K : \mu(T_{M_u}^{-1} B \cap T_{A_u}^{-1} B) > \delta\mu(B)^2\} \tag{1.2}$$

has positive upper density with respect to any double Følner sequence.

Using a version of Furstenberg's correspondence principle (see [**10**, Theorem 2.8]) one deduces, from Theorem 1.2, the following combinatorial corollary.

COROLLARY 1.4. [**10**, Theorem 1.5] *Let K be an infinite countable field, let $(F_N)_{N\in\mathbb{N}}$ be a double Følner sequence in K and let $E \subset K$ be such that $\bar{d}_{(F_N)}(E) > 0$. Then E contains 'many' pairs of the form $\{x + y, xy\}$.*

Theorems 1.1 and 1.2 depend heavily on the amenability of the affine group $\mathcal{A}_K$ and form a sort of ultimate result that can be achieved via Cesàro averages. When trying to obtain analogues of the above results for $\mathbb{Z}$ or other rings, one runs into serious difficulties. The main problem is that the semigroup of affine transformations of $\mathbb{Z}$ is not amenable†. Therefore it is *a priori* not clear what kinds of statement similar to Theorems 1.1 and 1.2 and Corollaries 1.3 and 1.4 can be formulated (and proved) if one replaces fields by more general rings. In particular, one would like to know if the corresponding set $\mathcal{R}(B, \delta)$ is non-empty (and indeed 'large') for any measure preserving action of the affine semigroup $\mathcal{A}_{\mathbb{Z}}$ of $\mathbb{Z}$. As we will see below, an alternative approach, based on convergence along ultrafilters, not only allows one to have reasonable analogues of Theorems 1.1 and 1.2 for actions of $\mathcal{A}_{\mathbb{Z}}$, but also leads to a strong generalization of Corollary 1.3, which guarantees that the sets $\mathcal{R}(B, \delta)$ are not only non-empty but actually possess the filter property (see Theorem 1.7 below for the precise formulation).

1.3. *Khintchine-type recurrence.* Observe that (1.1) resembles a classical result of Khintchine (see, for example, [**2**, Theorem 5.2]) which states that, for any probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $B \in \mathcal{B}$,

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(B \cap T^{-n}B) \geq \mu(B)^2. \tag{1.3}$$

Formula (1.3), in turn, implies the Khintchine's recurrence theorem, stating that the set

$$\mathcal{S}(B, \delta) = \{n : \mu(B \cap T^{-n}B) \geq \delta\mu(B)^2\} \tag{1.4}$$

is syndetic for any $\delta \in (0, 1)$. (A set $E \subset \mathbb{Z}$ is *syndetic* if it has bounded gaps or, equivalently, if a finite number of translates of E cover $\mathbb{Z}$. More generally, a subset E of a group is (left) *syndetic* if a finite number of translates of the form gE cover G.) Motivated by Khintchine's recurrence theorem, one would like to get a similar finite tiling property for sets of the form $\mathcal{R}(B, \delta)$.

Corollary 1.3 states that $\mathcal{R}(B, \delta)$ has positive upper density with respect to any *double* Følner sequence. One can show (see Example 4.3 below) that sets which have positive density along any double Følner sequence are, in general, neither additively syndetic nor multiplicatively syndetic. Nevertheless, they still posses a rather strong tiling property which is revealed via the (*a posteriori* quite natural) notion of affine syndeticity.

Definition 1.5. (Affine syndeticity) Given an infinite field K, a set $S \subset K$ is called *affinely syndetic* if there exists a finite number of affine transformations $g_1, \ldots, g_k \in \mathcal{A}_K$ such that, for any $x \in K$, at least one of the images $g_1(x), \ldots, g_k(x)$ lies in S.

The notion of affine syndeticity is explored in detail in §4. In particular, we have the following proposition (cf. Theorem 4.5 below).

† See Proposition 2.4 below.

PROPOSITION 1.6. *Let K be an infinite countable field. A subset $S \subset K$ is affinely syndetic if and only if it has positive upper density with respect to any double Følner sequence. In particular, the sets $\mathcal{R}(B, \delta)$, defined in (1.2), are affinely syndetic.*

Observe that, in general, affinely syndetic sets do not have the finite intersection property. For example, the subsets of rational numbers defined by

$$E_1 = \bigcup_{n\in\mathbb{Z}} [2n, 2n+1) \subset \mathbb{Q}, \quad E_2 = \bigcup_{n\in\mathbb{Z}} [2n-1, 2n) \subset \mathbb{Q}$$

are both additively (and hence affinely) syndetic, but have empty intersection.

On the other hand, one can show that the sets $\mathcal{S}(B, \delta)$ appearing in (1.4) do have the finite intersection property, although the easiest way of proving this involves either the so-called IP-limits or limits along idempotent ultrafilters (we note, in passing, that these 'non-Cesàrian' limits are also useful when one deals with large returns along polynomials; see [**5**], [**1**, §3] and [**11**]).

1.4. *Statements of the main new results.* The above discussion suggests that the sets $\mathcal{R}(B, \delta)$ may have the finite intersection property as well. The following theorem shows that this is indeed so.

THEOREM 1.7. *Let K be a field, let $t \in \mathbb{N}$ and, for each $i = 1, \ldots, t$, let (Ω_i, μ_i) be a probability space, let $(T_g^{(i)})_{g\in\mathcal{A}_K}$ be a measure preserving action of the affine semigroup $\mathcal{A}_K$ of K on (Ω_i, μ_i) and let $B_i \subset \Omega_i$ be a measurable set with positive measure. Let $\delta \in (0, 1)$ and let $\mathcal{R}(B_i, \delta)$ be defined as in equation (1.2) with respect to the action $(T_g^{(i)})_{g\in\mathcal{A}_K}$. Then the intersection*

$$\mathcal{R}(B_1, \delta) \cap \cdots \cap \mathcal{R}(B_t, \delta) \tag{1.5}$$

is affinely syndetic (and, in particular, non-empty).

Theorem 1.7 is proved in §5, where it is obtained as a corollary of an ultrafilter analogue of Corollary 1.3 (see Theorem 5.14). Roughly speaking, Theorem 5.14 asserts that, given an ultrafilter p with certain rich combinatorial properties and an isometric anti-representation $(U_g)_{g\in\mathcal{A}_K}$ of the affine semigroup $\mathcal{A}_K$ on a Hilbert space $\mathcal{H}$, $p\text{-lim}_u\, U_{M_u A_{-u}} f = Vf$, where† $V : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection. This, in turn, allows us to obtain, as a corollary, the analogue of formulae (1.1) and (1.3) for measure preserving actions $(T_g)_{g\in\mathcal{A}_K}$ of $\mathcal{A}_K$ given by

$$p\text{-}\lim_u \mu(T_{A_u}^{-1} B \cap T_{M_u}^{-1} B) \geq \mu(B)^2.$$

Remark 1.8. To appreciate the power of the ultrafilter approach, one should note that the Cesàro convergence results established in [**10**] imply only the affine syndeticity of the intersections

$$\mathcal{R}(B_1, 0) \cap \cdots \cap \mathcal{R}(B_t, 0) \tag{1.6}$$

of return sets $\mathcal{R}(B_i, 0)$, rather than the affine syndeticity of the intersection of the 'optimal' return sets $\mathcal{R}(B_i, \delta)$, as in (1.5).

† The symbol p-lim denotes the limit along ultrafilter p. See §3 for the relevant background on ultrafilters.

Taking limits along ultrafilters has yet another advantage over Cesàro limits in that it does not require amenability of the underlying (semi)group. We will see, in §5, that Theorem 1.7 is actually true in the much more general situation when one replaces the field K with, say, $\mathbb{Z}$ (and hence replaces the amenable group $\mathcal{A}_K$ with the non-amenable semigroup $\mathcal{A}_\mathbb{Z}$). In fact, the theorem in question holds (and practically with the same proof) for any ring R from the rather large class which we call LID (for large ideal domain) and which is defined in §2. This class includes fields, rings of integers of number fields and rings of polynomials over finite fields. We believe that LIDs form a natural framework for studying dynamics of affine semigroups with a view to combinatorial applications.

Juxtaposing the (still unsolved) problem of finding monochromatic $\{x + y, xy\}$ patterns in $\mathbb{N}$ with the positive result contained in Corollary 1.4, we see that there is a place for an 'intermediate' result which would guarantee, for any finite coloring of $\mathbb{Q}$, the existence of a monochromatic configuration of the form $\{x + n, xn\}$, where $x \in \mathbb{Q}$, $n \in \mathbb{N}$. As we will see, results of this kind can be obtained via ultrafilter methods developed in this paper. Our methods allow us to have this kind of intermediate result not just for the pair $(\mathbb{Q}, \mathbb{N})$ but also for any pair (K, R), where K is a countable field and $R \subset K$ is an LID.

The following theorem lists some special cases of a more general Theorem 5.15, which can be found in §5.

THEOREM 1.9.

(1) *For any finite partition $\mathbb{Q} = C_1 \cup \cdots \cup C_r$ of the rational numbers, there exists a cell $i \in \{1, \ldots, r\}$ and many*† $x \in \mathbb{Q}$, $n \in \mathbb{N}$ *such that* $\{x + n, xn\} \subset C_i$.

(2) *More generally, if K is a number field and $\mathcal{O}_K$ is its ring of integers, for any finite partition $K = C_1 \cup \cdots \cup C_r$, there exists a cell $i \in \{1, \ldots, r\}$ and many $x \in K$, $n \in \mathcal{O}_K$ such that $\{x + n, xn\} \subset C_i$.*

(3) *Let $\mathbb{F}$ be a finite field, let K denote the field of rational functions (i.e. quotients of polynomials) over $\mathbb{F}$ and let $\mathbb{F}[x]$ denote the ring of polynomials. Then, for any finite partition $K = C_1 \cup \cdots \cup C_r$, there exists a cell $i \in \{1, \ldots, r\}$ and many $f \in K$, $g \in \mathbb{F}[x]$ such that $\{f + g, fg\} \subset C_i$.*

The paper is organized as follows. In §2 we define the class of LID rings and present some general facts about affine semigroups. In particular, we prove that the affine semigroup of a countable integral domain R is amenable if and only if R is a field. In §3, we provide the necessary background on ultrafilters, and introduce the notion of DC sets, which will play a fundamental role in the rest of the paper. In §4, we introduce the notions of affinely thick and affinely syndetic, explore some of the properties of these families of sets and connect these notions with DC sets. In §5, we state and prove the main theorems. Finally, in §6, we discuss some notions of largeness pertinent to the study of $\{x + y, xy\}$ patterns and formulate a conjecture which, if true, implies that, for any finite partition of $\mathbb{N}$, one of the cells of the partition contains plenty of configurations $\{x + y, xy\}$.

2. *Preliminaries: large ideal domains, affine semigroups, double Følner sequences*

Throughout this paper, we will work with a special class of rings.

† In Theorem 5.15, we describe more precisely how large is the set of such x and n.

Definition 2.1. A ring R is called a *large ideal domain* (LID) if it is an infinite countable integral domain and, for any $x \in R\backslash\{0\}$, the ideal xR is a finite index additive subgroup of R.

Every field is trivially an LID. The following proposition gives some non-trivial examples of LID rings.

PROPOSITION 2.2. *The following rings are LID.*

(1) *Any integral domain R whose underlying additive group is finitely generated. In particular, the ring of integers $\mathcal{O}_K$ of a number field K satisfies this property.*

(2) *The ring of polynomials $\mathbb{F}[x]$ over a finite field $\mathbb{F}$.*

Proof. (1) Since $(R, +)$ is an infinite finitely generated abelian group, it contains torsion-free elements and therefore the identity 1_R of R has infinite order in $(R, +)$. If some element $x \in R$ had torsion, say, $nx = 0$ for some $n \in \mathbb{N}$, then $(n1_R)x = 0$, which contradicts the absence of zero divisors. Using the classification of finitely generated abelian groups, we can now represent $(R, +)$ as $\mathbb{Z}^d$ for some $d \in \mathbb{N}$.

For any non-zero $x \in R$, the map $\phi : y \mapsto xy$ is an injective endomorphism of $(R, +)$ (injectivity follows from the absence of divisors of zero) whose image $\phi(R)$ is the ideal xR. We claim that the image of any injective homomorphism $\phi : \mathbb{Z}^d \to \mathbb{Z}^d$ has a finite index in $\mathbb{Z}^d$, which will finish the proof.

Indeed, representing ϕ as a matrix, injectivity implies that the determinant of ϕ is non-zero. Therefore it has an inverse ϕ^{-1} with entries in $\mathbb{Q}$. Multiplying ϕ^{-1} by the least common multiple n of its entries, we obtain a matrix $n\phi^{-1}$ with coefficients in $\mathbb{Z}$. Therefore $n\mathbb{Z}^d = (n\phi^{-1})\phi(\mathbb{Z}^d) \subset \phi(\mathbb{Z}^d)$, so $[\mathbb{Z}^d : \phi(\mathbb{Z}^d)] \leq [\mathbb{Z}^d : n\mathbb{Z}^d] = n^d < \infty$, which proves the claim.

(2) Let $f \in \mathbb{F}[x]$ have degree d. For any $g \in \mathbb{F}[x]$, one can divide g by f and obtain $g = fq + r$, where $\deg r < d$. Therefore $g - r$ belongs to the ideal $f\mathbb{F}[x]$. It follows that the set of polynomials r with degree smaller than d form a complete set of coset representatives for $f\mathbb{F}[x]$. Since $\mathbb{F}$ is finite, there are only a finite number of such representatives and hence the index of $f\mathbb{F}[x]$ is finite, as desired. $\square$

Remark 2.3. There are number fields whose ring of integers is not a principal ideal domain (PID). Hence, part (1) of Proposition 2.2 includes some LID which are not PID. We also observe that not every PID is a LID. Indeed, the ring $\mathbb{Q}[x]$ of all polynomials with rational coefficients is a PID, but the ideal $x\mathbb{Q}[x]$ has infinite index as an additive subgroup of $\mathbb{Q}[x]$, so $\mathbb{Q}[x]$ is not an LID.

Some of the results in this paper are true only for fields; we will indicate the distinction in each case and we will use the letter K to denote a field.

Let R be a ring; we denote by R^* the set of its non-zero elements. An *affine transformation* of R is a map $f : R \to R$ of the form $f(x) = ux + v$ with $u \in R^*$, $v \in R$. The *affine semigroup* of R is the semigroup of all affine transformations of R (the semigroup operation being composition of functions) and will be denoted by $\mathcal{A}_R$. Observe that $\mathcal{A}_R$ is a group if and only if R is a field.

For each $v \in R$, the map $x \mapsto x + v$ will be denoted by A_v (add v) and, for each $u \in R^*$, the map $x \mapsto ux$ will be denoted by M_u (multiply by u). Note that the distributive law in R can be expressed as

$$M_u A_v = A_{uv} M_u. \tag{2.1}$$

The affine transformations A_v with $v \in R$ form the *additive subgroup* of $\mathcal{A}_R$, denoted by S_A. The affine transformations M_u with $u \in R^*$ form the *multiplicative sub-semigroup* of $\mathcal{A}_R$, denoted by S_M. Observe that S_A is isomorphic to the additive group $(R, +)$ and S_M is isomorphic to the multiplicative semigroup $(R^*, \cdot)$.

Note that the map $x \mapsto ux + v$ is the composition $A_v M_u$. Thus the sub-semigroups S_M and S_A generate the semigroup $\mathcal{A}_R$. When K is a field, $\mathcal{A}_K$ is the semidirect product of the (abelian) groups S_A and S_M and hence is amenable. However, as was pointed out in [**10**, Remark 6.2], the semigroup $\mathcal{A}_\mathbb{Z}$ is not amenable. In fact we have the following proposition.

PROPOSITION 2.4. *Let R be a countable integral domain. The affine semigroup $\mathcal{A}_R$ is amenable if and only if R is a field.*

Proof. As was explained above, if R is a field, then $\mathcal{A}_R$ is amenable. Assume now that $\mathcal{A}_R$ is amenable. The semigroup $\mathcal{A}_R$ acts naturally on R by affine transformations; therefore the amenability of $\mathcal{A}_R$ implies the existence of a finitely additive mean $\lambda : \mathcal{P}(R) \to [0, 1]$, defined on all the subsets of R, which is invariant under all affine transformations (this means that $\lambda(\{x \in R : g(x) \in E\}) = \lambda(E)$ for any $E \subset R$ and $g \in \mathcal{A}_R$). Given $x \in R^*$, $1 = \lambda(R) = \lambda(xR)$ (because the map $y \mapsto xy$ belongs to $\mathcal{A}_R$).

Assume, for the sake of a contradiction, that R is not a field and let $x \in R^*$ be a non-invertible element. The ideal xR is not the whole ring and hence there is a shift $xR + a$ which is disjoint from xR. The invariance of λ implies that $\lambda(xR) = \lambda(xR + a)$, but disjointness implies that $\lambda(xR \cup (xR + a)) = \lambda(xR) + \lambda(xR + a) = 2\lambda(xR)$. We now conclude that

$$1 = \lambda(xR) = \tfrac{1}{2}\lambda(xR \cup (xR + a)) \leq \tfrac{1}{2}\lambda(R) = \tfrac{1}{2},$$

which gives the desired contradiction. □

When $g \in \mathcal{A}_R$ is an affine transformation of R and $E \subset R$ is any subset, we define

$$\theta_g E = \{g(x) : x \in E\} \quad \text{and} \quad \theta_g^{-1} E = \{x \in R : g(x) \in E\}. \tag{2.2}$$

Throughout this paper, in order to make the notation less cumbersome, and when no confusion can arise, we will adopt the following convention. Let $(T_g)_{g \in \mathcal{A}_R}$ be a measure preserving action of $\mathcal{A}_R$ (on some probability space) and let $(U_g)_{g \in \mathcal{A}_R}$ be a isometric (anti-)representation of $\mathcal{A}_R$ (on some Hilbert space). For $v \in R$ and $u \in R^*$, we will write A_v instead of θ_{A_v}, T_{A_v} or U_{A_v} and M_u instead of θ_{M_u}, T_{M_u} or U_{M_u}.

Definition 2.5. Let K be a field. A *double Følner sequence* in K is a sequence (F_N) of finite subsets of K such that, for every $u \in K^*$,

$$\lim_{N\to\infty} \frac{|F_N \cap (F_N + u)|}{|F_N|} = \lim_{N\to\infty} \frac{|F_N \cap (F_N u)|}{|F_N|} = 1.$$

It follows, from [**10**, Proposition 2.4], that double Følner sequences exist in any countable field K. This fact also follows from Theorem 4.5 below.

Definition 2.6. Let K be a field, let $E \subset K$ and let (F_N) be a double Følner sequence in K. The *upper density* of E with respect to (F_N) is

$$\bar{d}_{(F_N)}(E) := \limsup_{N\to\infty} \frac{|E \cap F_N|}{|F_N|}$$

and the *lower density* of E with respect to (F_N) is

$$\underline{d}_{(F_N)}(E) := \liminf_{N\to\infty} \frac{|E \cap F_N|}{|F_N|}.$$

Several basic properties of the upper and lower densities with respect to a Følner sequence in a group remain true for densities with respect to double Følner sequences, and the proofs carry over to this setting. We list some of these facts in the next lemma.

LEMMA 2.7. *Let K be a field, let (F_N) be a double Følner sequence in K, let $E_1, E_2 \subset K$ and let $g \in \mathcal{A}_K$:*

(1) $\bar{d}_{(F_N)}(\theta_g E) = \bar{d}_{(F_N)}(E)$ *and* $\underline{d}_{(F_N)}(\theta_g E) = \underline{d}_{(F_N)}(E)$*;*
(2) $\bar{d}_{(F_N)}(E_1 \cup E_2) \le \bar{d}_{(F_N)}(E_1) + \bar{d}_{(F_N)}(E_2)$*;*
(3) $\underline{d}_{(F_N)}(E_1 \cup E_2) \ge \underline{d}_{(F_N)}(E_1) + \underline{d}_{(F_N)}(E_2)$*; and*
(4) *if* $E_2 = K \backslash E_1$*, then* $\bar{d}_{(F_N)}(E_1) + \underline{d}_{(F_N)}(E_2) = 1$.

3. *Auxiliary results involving ultrafilters*

To prove Theorems 1.7 and 1.9, we will use ultrafilters on R. For the reader's convenience, we provide, in this section, a brief review of necessary ultrafilter background. For a more detailed account, see [**4**] and, for a comprehensive treatment, see [**14**].

Definition 3.1. Let X be a countable infinite set. An ultrafilter on X is a family p of subsets of X such that:

- $X \in p$;
- if $E_1 \in p$ and $E_1 \subset E_2$, then $E_2 \in p$;
- if $E_1 \in p$ and $E_2 \in p$, then $E_1 \cap E_2 \in p$; and
- $E \in p \iff (R \backslash E) \notin p$.

The set of all ultrafilters on X is denoted by βX.

For any $u \in X$, the *principal ultrafilter* p_u is defined by the rule $E \in p_u \iff u \in E$. By a slight abuse of notation, we will often denote p_u by u.

The set βX of all ultrafilters on X can be identified with the Stone–Čech compactification of the (discrete) set X (see [**14**, Theorem 3.27]). The space βX is a compact Hausdorff space (cf. [**14**, Theorem 3.18]) with the topology generated by the clopen sets

$$\overline{E} := \{p \in \beta X : E \in p\} \quad \text{for all } E \subset X. \tag{3.1}$$

Let $p \in \beta X$ be an ultrafilter, let Y be a compact Hausdorff space and let $f : X \to Y$ be a function. It is not hard to check that there exists a unique point $y \in Y$ such that, for every neighborhood U of y, $\{u \in X : f(u) \in U\} \in p$. We denote this by $p\text{-}\lim_u f(u) = y$ (one can also write $y = \lim_{u \to p} f(u)$, but we stick with the former notation since it is more suggestive of the analogy with Cesàro limits).

Now let $X = R$ be a ring. One can extend the operations of addition and multiplication from R to βR, as follows. Given $p, q \in \beta R$, we define

$$p + q = \{E \subset R : \{u \in R : A_u^{-1} E \in q\} \in p\}, \tag{3.2}$$

$$pq = \{E \subset R : \{u \in R : M_u^{-1} E \in q\} \in p\}. \tag{3.3}$$

The operations defined by (3.2) and (3.3) are associative in βR (cf. [**14**, Theorems 4.1, 4.4 and 4.12]). However (for the rings we deal with), these operations do not commute and fail to satisfy the distributive law. Nevertheless, we have the following proposition.

PROPOSITION 3.2. *Let $u \in R$ and $p, q \in \beta R$. Then:*
- *$u + p = p + u$ and $up = pu$; and*
- *$(p + q)u = pu + qu$.*

One can easily check that, for each $p, q \in \beta R$ (cf. [**14**, Remark 4.2]),

$$p + q = p\text{-}\lim_u (u + q) \quad pq = p\text{-}\lim_u (uq). \tag{3.4}$$

An ultrafilter $p \in \beta R$ is an *additive idempotent* if $p + p = p$, and it is a *multiplicative idempotent* if $pp = p$. Observe that $1 \in \beta R$ is a multiplicative idempotent and $0 \in \beta R$ is both an additive idempotent and a multiplicative idempotent. The following fundamental result due to Ellis (see, for instance, [**1**, Theorem 3.3]) guarantees the existence of idempotents in any compact semigroup.

LEMMA 3.3. *Let $(S, \circ)$ be a compact Hausdorff semigroup such that for each $s \in S$ the function $x \mapsto x \circ s$ from S to itself is continuous. Then there exists $s \in S$ such that $s \circ s = s$.*

In what follows, Lemma 3.3 will be repeatedly applied to closed sub-semigroups of $(\beta R, +)$ and $(\beta(R^*), \cdot)$.

Since R is an integral domain and $\beta(R^*) = (\beta R) \backslash \{0\}$ is closed in βR, it follows, from (3.4), that $\beta(R^*)$ is closed under multiplication. In view of Proposition 3.2 and (3.4), for each $u \in R$, both maps $A_u : p \mapsto p + u$ and $M_u : p \mapsto pu$ are continuous. Therefore we can define topological dynamical systems $(\beta R, S_A)$ and $(\beta(R^*), S_M)$, where S_A and S_M are the additive and multiplicative sub-semigroups of $\mathcal{A}_R$, respectively (cf. §2). Invoking again (3.4), one can check that any closed S_A-invariant subset of βR is a semigroup for addition, and any closed S_M-invariant subset of βR^* is a semigroup for multiplication.

By Zorn's lemma, there exist minimal non-empty compact S_A-invariant subsets of βR and minimal non-empty compact S_M-invariant subsets of $\beta(R^*)$. An *additive minimal idempotent* is a non-principal ultrafilter $p \in \beta R$ which belongs to a minimal compact S_A-invariant set and such that $p + p = p$. A *multiplicative minimal idempotent* is a non-principal ultrafilter $p \in \beta(R^*)$ which belongs to a minimal compact S_M-invariant set and is such that $pp = p$.

Definition 3.4. Let R be a ring. We denote by $\mathcal{AMI}$ the set of all additive minimal idempotents in βR and we denote by $\mathcal{MMI}$ the set of all multiplicative minimal idempotents in $\beta(R^*)$.

A set $C \subset R$ is called *additively central* if there exists $p \in \mathcal{AMI}$ such that $C \in p$. Similarly, any member of an ultrafilter $p \in \mathcal{MMI}$ is called *multiplicatively central*†. In this paper, we are interested in sets $C \subset R$, which are simultaneously additively and multiplicatively central.

Unfortunately, the sets $\mathcal{AMI}$ and $\mathcal{MMI}$ are, in general, disjoint (cf. [**14**, Corollary 13.15]). However, at least when R is an LID, the closure $\overline{\mathcal{AMI}}$ has non-trivial intersection with $\mathcal{MMI}$ (see Proposition 4.7 below).

Definition 3.5.

- Let $\mathcal{G} = \overline{\mathcal{AMI}} \cap \mathcal{MMI}$.
- A set $C \subset R$ is called DC (double central) if there exists an ultrafilter $p \in \mathcal{G}$ such that $C \in p$.
- A set $C \subset R$ is called DC^* if it has non-empty intersection with every DC set‡.

Observe that a set $C \subset R$ is DC^* if and only if it is contained in every ultrafilter $p \in \mathcal{G}$ (this follows directly from Definition 3.5 and the definition of ultrafilters).

We will need four more facts about ultrafilters which do not appear in the literature in the form that we need. Lemma 3.6 is the adaptation of [**7**, Theorem 3.5], where the analogous result is proved for $\mathbb{N}$. The proof carries over to our set-up.

LEMMA 3.6. *Let R be a countable integral domain, let $p \in \mathcal{MMI}$ and let $B \in p$. Then, for every $r \in \mathbb{N}$, there exists a set $Z \subset R$ with cardinality $|Z| = r$ and such that the set of finite sums of Z satisfies*

$$FS(Z) := \left\{ \sum_{i \in Z'} i \,\middle|\, \emptyset \neq Z' \subset Z \right\} \subset B.$$

Proof. Let $T \subset \beta R$ be the collection of all non-principal ultrafilters p such that any member $A \in p$ contains a set of the form $FS(Z)$ with Z having arbitrarily large cardinality (sets A satisfying this property are called IP_0 sets). It follows, from [**14**, Theorem 5.8], that every additive idempotent is in T, so T is non-empty.

Since $p \in \mathcal{MMI}$, there exists some minimal subsystem (Y, S_M) of $(\beta R^*, S_M)$ such that $p \in Y$. We claim that $Y \cap T$ is non-empty.

Let $q \in T$. We have that $up = M_u p \in Y$ for every $u \in R^*$. It follows, from equation (3.4) and the fact that Y is closed, that $qp \in Y$ as well. Let $E \in qp$. By definition, $\{u \in R : M_u^{-1} E \in p\} \in q$. Thus, for each $r \in \mathbb{N}$, there exists $Z \subset K$ with $|Z| = r$ and such that $FS(Z) \subset \{u \in R : M_u^{-1} E \in p\}$. Since $FS(Z)$ is finite, the intersection $\bigcap_{u \in FS(Z)} M_u^{-1} E$ is also in p and hence is infinite. Let a be a non-zero element in that intersection; $a \in$

† The notion of *central set* in $\mathbb{Z}$ was introduced by Furstenberg in topologico-dynamical terms [**12**]. Furstenberg's definition of central sets makes sense in any semigroup (see [**6**, Definition 6.2]). One can show (see [**6**, Theorems 6.8 and 6.11]) that a subset of a countable semigroup is central if and only if it belongs to a minimal idempotent ultrafilter.

‡ We call the reader's attention to the fact that there is no relation between the * in DC^* and the * in R^*.

$M_u^{-1}E$ for every $u \in FS(Z)$ and hence $FS(Z)a = FS(Za) \subset E$. Observe that $|Za| = |Z|$ because there are no divisors of zero. Since $E \subset qp$ and $|Z|$ were chosen arbitrarily, we conclude that $qp \in T$. This proves the claim.

Next, let $q \in Y \cap T$ and let $u \in R^*$. We trivially have $uq \in Y$. Furthermore, if $A \in uq$, then $M_u^{-1}A \in q$ and hence it contains $FS(Z)$ for a set Z of arbitrary finite cardinality. But then A contains $M_u FS(Z) = FS(uZ)$ and hence $uq \in T$. This implies that $uq \in Y \cap T$ and hence $(Y \cap T, S_M)$ is a subsystem of $(\beta R^*, S_M)$. Since (Y, S_M) is a minimal system, we conclude that $Y \cap T = Y$. This implies that $Y \subset T$. Hence $p \in T$ and the proof is complete. □

We will also need the following technical lemma.

LEMMA 3.7. *Let G be a group and let $H \subset G$ be a normal subgroup with finite index. Then, for any ultrafilter $p \in \beta G$ in the closure of the idempotents, $H \in p$.*

Proof. The set of ultrafilters containing H is a closed set, and hence we can assume that p is itself an idempotent. Since H has only a finite number of cosets, exactly one of them, say, aH is in p. Therefore, given $g \in G$, $g^{-1}aH \in p$ if and only if $g^{-1}a \in aH$. This is equivalent to $g \in aHa^{-1} = H$ (because H is normal). Since $aH \in p = p + p$, we conclude that

$$\{g \in G : g^{-1}aH \in p\} \in p \iff H \in p. \qquad \square$$

A particular case of Lemma 3.7 is when R is an LID, H is a non-trivial ideal and $p \in \mathcal{G}$. If $p \in \beta(R^*)$ contains an ideal bR for some $b \in R^*$, then one can define an ultrafilter $b^{-1}p$ as the family of sets $E \subset R$ such that $bE \subset p$. Observe that, in this case, $bq = p$.

The following lemma is the analogue of [**6**, Theorem 5.4] (where it is stated and proved for $\mathbb{N}$).

LEMMA 3.8. *Let R be an LID, let $p \in \overline{\mathcal{AMI}}$ and let $u \in R^*$. Then both up and $u^{-1}p$ belong to $\overline{\mathcal{AMI}}$.*

Proof. Since $M_u : p \mapsto up$ and $M_u^{-1} : p \mapsto u^{-1}p$ are continuous (on their respective domains), it suffices to show that if $p \in \mathcal{AMI}$, then also both up and $u^{-1}p$ are in $\mathcal{AMI}$. It follows directly from Proposition 3.2 that $up + up = u(p + p) = up$, so up is an additive idempotent. Checking the definitions easily yields that $u^{-1}p$ is an additive idempotent.

All that it remains to show is that up and $u^{-1}p$ belong to minimal subsystems of $(\beta R, S_A)$.

(1) $u^{-1}p \in \mathcal{AMI}$. Let $X = \overline{\{v + p : v \in R\}}$ be the minimal compact S_A-invariant subset of βR such that $p \in X$. It is not hard to check that the set $u^{-1}X := \{q \in \beta R : bq \in X\}$ is S_A-invariant, compact, and contains $u^{-1}p$.

Since R is an LID, the ideal uR has finite index as an additive subgroup of R. Therefore there exists a finite set $F \subset R$ of coset representatives such that $R = F + uR$. Choose F minimal with this property and such that $F \cap uR = \{0\}$.

If $Z \subset u^{-1}X$ is any compact S_A-invariant subset, then $F + uZ$ is a compact subset of X. We now show that $F + uZ$ is also invariant. Indeed, observe that any $v \in R$ can be decomposed as $v = a + uv'$ with $a \in F$ and $v' \in R$; thus if $a_1 + uz \in F + uZ$ is arbitrary (with $a_1 \in F$ and $z \in Z$) and $v_1 \in R$, then $v_1 + a_1 + uz = a + uv' + uz = a + u(v' + z) \in F + uZ$, by invariance of Z.

Since X is minimal, this implies that $F + uZ$ is either empty (in which case Z is empty) or coincides with X. In the second case, we claim that $Z = u^{-1}X$. Indeed, let $q \in u^{-1}X$, then it satisfies $uq \in X = F + uZ$, and hence $uq = a + uz$ for some $a \in F$ and $z \in Z$. Therefore uR is in both uz and $a + uz$ which implies that $a \in uR \cap uR = \{0\}$. This means that $uq \in uZ$ and hence $q \in Z$, which proves the claim.

It follows that $u^{-1}X$ is a compact minimal S_A-invariant subset of βR. Since $u^{-1}p \in u^{-1}X$, it follows that $u^{-1}p \in \mathcal{AMI}$, as desired.

(2) $up \in \mathcal{AMI}$. Let $Y = \overline{\{v + up : v \in R\}} \subset \beta R$. It suffices to show that Y is itself minimal (compact and S_A-invariant being immediate consequences of its construction). Recalling that $F \subset R$ is a finite set such that $R = F + uR$, we can rewrite

$$Y = \overline{\{(a + uv) + up : a \in F;\ v \in R\}} = F + u\overline{\{v + p : v \in R\}} = F + uX,$$

where, in the second equality, we used Proposition 3.2. Let $Z \subset Y$ be a non-empty compact S_A-invariant subset; we need to show that $Z = Y$. Let $Z_1 = \{q \in X : uq \in Z\} = X \cap u^{-1}Z$.

We claim that $F + uZ_1 = Z$. It is clear that $F + uZ_1 \subset Z$ (because Z is S_A-invariant). Next, let $q \in Z$ be arbitrary; we need to show that $q \in F + uZ_1$. There is exactly one $a \in F$ such that $a + uR \in q$. Let r be the ultrafilter defined by $E \in r \iff a + uE \in q$ (observe that r is, indeed, an ultrafilter because $a + uR \in q$ and hence $R \in r$); we will show that $r \in X$. Indeed, let $E \in r$ and, since $a + uE \in q \in Y$, $v + a + uE \in up$ for some $v \in R$. By definition, this means that $u^{-1}(v + a + uE) \in p$, so $v + a \in uR$ and $u^{-1}(v + a) + E \in p$. Finally, this implies that $E \in -u^{-1}(v + a) + p$ and, since $E \in r$ was arbitrary, it follows that $r \in \overline{\{v' + p : v' \in R\}} = X$, as desired. Next observe that $a + ur = q \in Z$. Since Z is invariant, this implies that $ur \in Z$ as well, and hence $r \in Z_1$, so $q = a + ur \in F + uZ_1$, as desired.

Since Z is non-empty, it follows that Z_1 is non-empty. Next we show that Z_1 is S_A-invariant. For any $v \in R$ and $q \in Z_1$, $u(v + q) = uv + uq \in uv + Z \subset Z$, since Z is invariant, so $v + q \in Z_1$, as desired. Since $Z_1 \subset X$ and X is minimal, $Z_1 = X$. But this means that $Z = F + uZ_1 = F + uX = Y$ and hence Y is minimal, as desired. □

LEMMA 3.9. *Let X be a compact space and let $(x_u)_{u\in R}$ be a sequence in X indexed by a countable ring R. Then, for each $k \in R^*$ and $p \in \beta R$, p-$\lim_u x_{ku} = kp$-$\lim_u x_u$.*

Proof. Let $x = p$-$\lim_u x_{ku}$ and let $U \subset X$ be a neighborhood of x. By definition, the set $E = \{u \in R : x_{ku} \in U\} \in p$. Note that $E = \{u \in R : x_u \in U\}/k$, and hence $\{u \in R : x_u \in U\} \in kp$. Since U is an arbitrary neighborhood of x, we conclude that kp-$\lim_u x_u = x$. □

4. *Affine syndeticity and thickness*

In this section, we will develop the notions of affinely syndetic and affinely thick subsets of R. The definitions and proofs are parallel to the usual notions of syndetic and thick. Recall that, for a discrete semigroup G, a set $S \subset G$ is *syndetic* if a finite number of translates of S cover G. More precisely, S is (left) syndetic in G if there exists a finite set $F \subset G$ such that every $g \in G$ can be written as $g = xs$ with $s \in S$ and $x \in F$.

Recall, from equation (2.2), the notation $\theta_g E = \{g(x) : x \in E\}$ for a set $E \subset R$ and $g \in \mathcal{A}_R$. When $F \subset \mathcal{A}_R$, $S \subset R$ and $x \in R$, we write

$$\theta_F^{-1} S := \bigcup_{g \in F} \theta_g^{-1} S \quad \text{and} \quad \theta_F x := \bigcup_{g \in F} g(x).$$

We slightly generalize here the definition of affine syndeticity, given in the Introduction for fields, to general rings.

Definition 4.1. Let R be a ring. A set $S \subset R$ is *affinely syndetic* if there exists a finite set $F \subset \mathcal{A}_R$ such that $\theta_F^{-1} S = R$.

Observe that if a set $S \subset R^*$ is syndetic in either the group $(R, +)$ or the semigroup $(R^*, \cdot)$, then S is affinely syndetic. Indeed, assume, for instance, that S is syndetic in $(R, +)$ and let $F \subset R$ be a finite set such that $S - F = R$. Then, considering the subset $\{A_u : u \in F\} \subset \mathcal{A}_R$, we deduce that $\theta_F^{-1} S = R$ and hence S is affinely syndetic. On the other hand, S can be affinely syndetic and not be syndetic for either the group $(R, +)$ or the semigroup $(R^*, \cdot)$ (this follows from Example 4.3 and Proposition 4.4 below).

Recall that, for a discrete semigroup G, a set $T \subset G$ is *thick* if it contains a shift of an arbitrary finite set. More precisely, T is (right) thick in G if, for every finite set $F \subset G$, there exists $g \in G$ such that $Fg \subset T$.

Definition 4.2. A set $T \subset R$ is *affinely thick* if, for every finite set $F \subset \mathcal{A}_R$, there exists $x \in R$ such that $\theta_F x \subset T$.

Observe that if $T \subset R$ is affinely thick, then it is thick in both the group $(R, +)$ and the semigroup $(R^*, \cdot)$. The following example shows that there exist sets T which are not affinely thick (even when R is a field) but which are thick in both $(R, +)$ and $(R^*, \cdot)$.

Example 4.3. We take the ring $R = \mathbb{Q}$ of rational numbers. Let (G_N) be an increasing sequence of finite subsets of $\mathbb{Q}$ whose union is $\mathbb{Q}$. For any sequence $(a_N) \subset \mathbb{Q}^*$, the set

$$E = \left(\bigcup_{N=1}^{\infty} (a_{2N-1} + G_{2N-1}) \right) \cup \left(\bigcup_{N=1}^{\infty} (a_{2N} G_{2N}) \right) = \bigcup_{N=1}^{\infty} E_N$$

is additively thick and multiplicatively thick, where $E_N = a_N + G_N$, when N is odd, and $E_N = a_N G_N$, when N is even. However, if (a_N) is growing sufficiently fast, then E is not affinely thick. Indeed, for every point $x \in \mathbb{Q}$, we may have

$$\theta_{\{Id, A_1 M_2\}} x = \{x, x+1, 2x\} \not\subset E.$$

To see this, let $a_0 = 1$ and $E_0 := \{0\}$. Let ΔG_N denote the set defined by $\Delta G_N = \{x_2 - x_1, x_3 - x_2, \ldots, x_k - x_{k-1}\}$, where $x_1 < x_2 < \cdots < x_k$ is an ordering of the elements of G_N. Let $M_N = \min\{|x| : x \in G_N \backslash \{0\}\}$. Define recursively

$$a_N = \begin{cases} 2\max(E_{N-1}) + \max(G_N) - 2\min(G_N) & \text{if } N \text{ is odd,} \\ \dfrac{1}{\min(\Delta G_N)} + \dfrac{2\max(E_{N-1})}{M_N} & \text{if } N \text{ is even.} \end{cases}$$

Note that if N is even and $x \in E_N$, then $x + 1 \notin E_N$. If N is odd and $x \in E_N$, then $x \geq \min(G_N) + a_N$, which implies that $2x > \max(G_N) + a_N$ and hence $2x \notin E_N$. Thus, for any $N \in \mathbb{N}$ and $x \in \mathbb{Q}$, the set $\{x, x+1, 2x\}$ is not a subset of E_N.

Since $\min\{|x| : x \in E_{N+1}\backslash\{0\}\} > 2\max\{|x| : x \in E_N\}$, if $x \in E_N$, then $2x \notin E_{N+1}$ (and, in fact, $2x \notin E_L$ for any $L > N$) and hence $\{x, x+1, 2x\}$ is not a subset of E for any $x \in \mathbb{Q}$.

The following proposition is an immediate consequence of the definitions.

PROPOSITION 4.4. *A set $S \subset R$ is affinely syndetic if and only if it has non-empty intersection with every affinely thick set. A set $T \subset R$ is affinely thick if and only if it has non-empty intersection with every affinely syndetic set.*

Now we connect affine syndeticity and thickness in countable fields with upper and lower density with respect to double Følner sequences.

THEOREM 4.5. *Let K be a countable field. A set $S \subset K$ is affinely syndetic if and only if for every double Følner sequence (F_N) in K, $\bar{d}_{(F_N)}(S) > 0$. A set $S \subset K$ is affinely thick if and only if there exists a double Følner sequence (F_N) in K such that $\underline{d}_{(F_N)}(T) = 1$.*

Proof. Assume that $S \subset K$ is affinely syndetic and let $F \subset \mathcal{A}_K$ be a finite set such that $\theta_F^{-1} S = K$. Then, for any double Følner sequence (F_N), using parts (1) and (2) of Lemma 2.7,

$$1 = \bar{d}_{(F_N)}(K) = \bar{d}_{(F_N)}\left(\bigcup_{g \in F} \theta_{g^{-1}} S\right) \leq \sum_{g \in F} \bar{d}_{(F_N)}(\theta_{g^{-1}} S) = |F| \bar{d}_{(F_N)}(S)$$

and hence $\bar{d}_{(F_N)}(S) \geq 1/|F| > 0$.

Now assume that $T \subset K$ is affinely thick and let (G_N) be an arbitrary (left) Følner sequence in $\mathcal{A}_K$. For each $N \in \mathbb{N}$, let $x_N \in K$ be such that $F_N := \theta_{G_N} x_N \subset T$ and $|F_N| = |G_N|$. To see why this is possible, note that, for any affine transformations $g_1, g_2 \in \mathcal{A}_K$ with $g_1 \neq g_2$, there is at most one solution $x \in K$ to the equation $g_1(x) = g_2(x)$. Thus there are only a finite number of $x \in K$ such that $g_1 x = g_2 x$ for some pair $g_1 \neq g_2 \in G_N$. On the other hand, since T is affinely thick, there are an infinite number of $x \in K$ such that $\theta_{G_N} x \subset T$ (and, indeed, an affinely thick set of such x).

We now show that (F_N) is a double Følner sequence in K. For any fixed $g \in \mathcal{A}_K$,

$$F_N \cap \theta_g F_N = \theta_{G_N} x_N \cap \theta_g(\theta_{G_N} x_N) \supset \theta_{G_N \cap g G_N} x_N$$

and hence

$$1 \geq \limsup_{N \to \infty} \frac{|F_N \cap g F_N|}{|F_N|} \geq \liminf_{N \to \infty} \frac{|F_N \cap g F_N|}{|F_N|} \geq \lim_{N \to \infty} \frac{|G_N \cap g G_N|}{|G_N|} = 1$$

because (G_N) is a left Følner sequence in $\mathcal{A}_K$. This implies that (F_N) is a double Følner sequence in K. Since, for each $N \in \mathbb{N}$, $F_N \subset T$ we conclude that $\underline{d}_{(F_N)}(T) = 1$.

Now if S is not syndetic, then it follows, from Proposition 4.4, that $K \backslash S$ is thick. Therefore there exists a double Følner sequence (F_N) such that $\underline{d}_{(F_N)}(K \backslash S) = 1$. From part (2.7) of Lemma 2.7, if follows that $\bar{d}_{(F_N)}(S) = 0$.

Finally, if T is not thick, then $K \backslash T$ is syndetic and hence, for every double Følner sequence (F_N), $\bar{d}_{(F_N)}(K \backslash T) > 0$. By part (2.7) of Lemma 2.7, $\underline{d}_{(F_N)}(T) < 1$ for every double Følner sequence in K. □

Remark 4.6. In view of Theorem 4.5, it follows, from (the proof of) [**10**, Theorem 2.5], that the sets of return times $R(B, \epsilon)$, defined in (1.2), are affinely syndetic. The main idea behind the proof of Theorem 1.7 is that the sets $R(B, \epsilon)$ are not only affinely syndetic, but actually DC^*.

In every countable semigroup, any thick set is central. The same phenomenon occurs in our situation.

PROPOSITION 4.7. *Assume that R is an LID. Then every affinely thick set in R is DC (see Definition 3.5).*

Proof. Let $T \subset R$ be an affinely thick set. For $g \in \mathcal{A}_R$, define $\overline{\theta_{g^{-1}}T} \subset \beta R$ by equations (2.2) and (3.1). Note that, for any finite set $F \subset \mathcal{A}_R$,

$$\bigcap_{g\in F} \overline{\theta_{g^{-1}}T} = \overline{\bigcap_{g\in F} \theta_{g^{-1}}T} = \overline{\{x \in R : \theta_F x \subset T\}}.$$

Since T is affinely thick, the family of compact sets $\{\overline{\theta_{g^{-1}}T} : g \in \mathcal{A}_R\}$ has the finite intersection property, and hence the intersection $\mathcal{T} := \bigcap_{g\in\mathcal{A}_R} \overline{\theta_{g^{-1}}T}$ is a non-empty compact subset of βR. We have the description of $\mathcal{T}$ given by

$$p \in \mathcal{T} \iff (\forall g \in \mathcal{A}_R) p \in \overline{\theta_{g^{-1}}T} \iff (\forall g \in \mathcal{A}_R)\theta_{g^{-1}}T \in p.$$

If $p, q \in \mathcal{T}$, we claim that both $p + q \in \mathcal{T}$ and $pq \in \mathcal{T}$. Indeed, for all $g \in \mathcal{A}_R$ and $u \in R$, $A_u^{-1}\theta_{g^{-1}}T = (\theta_g A_u)^{-1}T$. Therefore

$$\theta_{g^{-1}}T \in p + q \iff \{u \in R : A_u^{-1}\theta_{g^{-1}}T \in q\} \in p \iff \{u \in R : (\theta_g A_u)^{-1}T \in q\} \in p.$$

Since $q \in \mathcal{T}$, the set $\{u \in R : (\theta_g A_u)^{-1}T \in q\} = R \in p$, so we conclude that $p + q \in \mathcal{T}$. The same argument, with obvious modifications, implies that $pq \in \mathcal{T}$, which proves the claim.

We now have that $(\mathcal{T}, S_A)$ is a topological dynamical system. Hence, by Zorn's lemma, there exists a minimal subsystem. It follows, from (3.4), that each minimal subsystem is actually an (additive) left ideal in βR, and hence, in view of Lemma 3.3, there exist (additive) minimal idempotents in $\mathcal{T}$. Therefore the intersection $\mathcal{T}_1 := \overline{\mathcal{AMI}} \cap \mathcal{T}$ is a non-empty compact subset of $\mathcal{T}$.

If $u \in R^*$ and $p \in \mathcal{T}_1$, it follows, from Lemma 3.8, that $up \in \overline{\mathcal{AMI}}$, and thus $up \in \mathcal{T}_1$. This means that $(\mathcal{T}_1, S_M)$ is a topological dynamical system and hence, by Zorn's lemma, it has minimal subsystems. By Ellis theorem, each minimal system (=left ideal) contains some multiplicative idempotent. Let p be a multiplicative minimal idempotent in $\mathcal{T}_1$. Since $\mathcal{T}_1 \subset \mathcal{T}$, we conclude that $T \in p$. Since $\mathcal{T}_1 \subset \overline{\mathcal{AMI}}$, we conclude that $p \in \overline{\mathcal{AMI}}$, and hence $p \in \mathcal{G}$. □

Remark 4.8. An immediate consequence of Propositions 4.7 and 4.4 is that every DC^* set is affinely syndetic.

5. *Finite intersection property of sets of return times*

In this section, we study isometric anti-representations† $(U_g)_{g\in\mathcal{A}_R}$ of the affine semigroup $\mathcal{A}_R$ of a ring R on a Hilbert space $\mathcal{H}$ (this means that $\langle U_g\phi, U_g\psi\rangle = \langle\phi, \psi\rangle$ and $U_g(U_h\phi) = U_{hg}\phi$ for any $g, h \in \mathcal{A}_R$ and $\phi, \psi \in \mathcal{H}$).

Recall that if G is a semigroup and $(U_g)_{g\in G}$ is an isometric (anti-)representation of G on a Hilbert space $\mathcal{H}$, then a vector $\phi \in \mathcal{H}$ is called *compact* if the orbit $\{U_g\phi : g \in G\} \subset \mathcal{H}$ is pre-compact in the norm topology. It is easy to see that the set of compact vectors is a closed subspace.

When G is the additive sub-semigroup S_A of the affine semigroup $\mathcal{A}_R$, we denote the orthogonal projection onto the space of compact vectors by V_A and, when G is the multiplicative sub-semigroup S_M of the affine semigroup $\mathcal{A}_R$, we denote the orthogonal projection onto the space of compact vectors by V_M. Our main ergodic-theoretic result is the following analogue of Theorem 1.1, with Cesàro averages (which are unavailable in our current situation) replaced with limits along ultrafilters $p \in \mathcal{G} = \overline{\mathcal{AMI}} \cap \mathcal{MMI}$.

THEOREM 5.1. *Let R be an LID (see Definition 2.1), let $\mathcal{H}$ be a Hilbert space and let $(U_g)_{g\in\mathcal{A}_R}$ be an isometric anti-representation of $\mathcal{A}_R$ on $\mathcal{H}$. Then, for any $\phi, \psi \in \mathcal{H}$ and $p \in \mathcal{G}$ (see Definition 3.5),*

$$p\text{-}\lim_u \langle A_u\phi, M_u\psi\rangle = \langle V_A\phi, V_M\psi\rangle.$$

In this section, we will always work under the assumptions of Theorem 5.1.

5.1. *Projection onto the space of compact vectors.*

We have the following result.

LEMMA 5.2. *If $p \in \mathcal{G}$ (see Definition 3.5) and $\phi \in \mathcal{H}$, then*

$$V_M\phi = p\text{-}\lim_u M_u\phi \text{ in the topology of weak convergence}.$$

If $p \in \overline{\mathcal{AMI}}$ and $k \in R^*$, then

$$V_A\phi = p\text{-}\lim_u A_{ku}\phi \text{ in the topology of weak convergence.}$$

Proof. Since $p \in \mathcal{MMI}$, the first equality follows‡ from [**3**, Corollary 4.6]. By the same corollary, $V_A\phi = q\text{-}\lim_u A_u\phi$ for every additive minimal idempotent q.

It follows, from Lemma 3.9, that $p\text{-}\lim_u A_{ku}\phi = kp\text{-}\lim_u A_u\phi$. In view of Lemma 3.8, $kp \in \overline{\mathcal{AMI}}$. Since the map $q \mapsto q\text{-}\lim_u A_u\phi$ is continuous, we conclude that

$$p\text{-}\lim_u A_{ku}\phi = kp\text{-}\lim_u A_u\phi = V_A\phi. \qquad \square$$

LEMMA 5.3. *For every $\phi \in \mathcal{H}$, $V_AV_M\phi = V_MV_A\phi$.*

† We deal here with anti-representations instead of (*a priori* more natural) representations because a measure preserving action $(T_g)_{g\in G}$ of a non-commutative semigroup G induces a natural anti-representation of G by isometries on the corresponding L^2 space. Of course, the results obtained in this section hold true for isometric representations as well.

‡ In [**3**], the results are stated and proved for groups only, but it is easy to check that the proofs work for discrete semigroups as well (as is observed in the first paragraph after the remark following [**3**, Theorem 4.1]).

Proof. Let $p \in \mathcal{G}$. For each $k \in R^*$, it follows, from Lemma 5.2, that

$$M_k V_A \phi = M_k(p\text{-}\lim_u A_u \phi) = p\text{-}\lim_u M_k A_u \phi = p\text{-}\lim_u A_{ku} M_k \phi = V_A M_k \phi.$$

Therefore

$$V_M V_A f = p\text{-}\lim_k M_k V_A \phi = p\text{-}\lim_k V_A M_k \phi = V_A(p\text{-}\lim_k M_k \phi) = V_A V_M \phi. \quad \square$$

In view of Lemma 5.3, the operator $V := V_A V_M$ is an orthogonal projection. This gives the following simple corollary of Lemma 5.3, which will be needed in the proof of Theorem 5.14 below.

COROLLARY 5.4. *Let $\phi, \psi \in \mathcal{H}$ and assume that $U_g \psi = \psi$ for every $g \in \mathcal{A}_R$. Then*

$$\|\psi\|^2 \cdot \langle V_A \phi, V_M \phi \rangle \geq |\langle \phi, \psi \rangle|^2.$$

Proof.

$$\begin{aligned} \|\psi\|^2 \cdot \langle V_A \phi, V_M \phi \rangle &= \|\psi\|^2 \cdot \langle V \phi, \phi \rangle = \|\psi\|^2 \cdot \|V \phi\|^2 \\ &\geq |\langle V \phi, \psi \rangle|^2 = |\langle \phi, V \psi \rangle|^2 = |\langle \phi, \psi \rangle|^2, \end{aligned}$$

where the inequality follows from Cauchy–Schwarz inequality. $\square$

5.2. *Dealing with $V_A \phi$.* The scheme of the proof of Theorem 5.1 is as follows. First, we decompose $\phi = V_A \phi + \phi^\perp$ into its 'additively compact' and 'additively weak-mixing' components. Observe that $V_A(V_A \phi) = V_A \phi$ and $V_A \phi^\perp = 0$. The two main steps are to show that $p\text{-}\lim_u \langle A_u V_A \phi, M_u \psi \rangle = \langle V_A \phi, V_M \psi \rangle$ and that $p\text{-}\lim_u \langle A_u V_A \phi^\perp, M_u \psi \rangle = 0$. In this subsection, we deal with the first step.

LEMMA 5.5. *Let $\phi \in \mathcal{H}$ be additively compact (i.e. such that $V_A \phi = \phi$). Then, for any $p \in \mathcal{G}$,*

$$p\text{-}\lim_u \|A_u \phi - \phi\| = 0.$$

In other words, for all $\epsilon > 0$, the set $S := \{u \in K : \|A_u \phi - \phi\| < \epsilon\}$ is DC^*.

Proof. The orbit closure $\overline{\{A_u \phi : u \in R\}}$ of ϕ is trivially contained in the union $\bigcup_{u \in R} B(A_u \phi, \epsilon/2)$. Hence, by compactness, there exists some finite set $F \subset R$ such that the union $\bigcup_{u \in F} B(A_u \phi, \epsilon/2)$ contains the whole orbit of ϕ under the additive sub-semigroup S_A. Let $r := |F| + 1$.

Let $Z \subset K$ be an arbitrary subset with cardinality $|Z| = r$. We claim that the set of finite sums $FS(Z) \cap S \neq \emptyset$. Indeed, let $Z = \{z_1, \ldots, z_r\}$, let $z_i' = z_1 + \cdots + z_i$ for each $i = 1, \ldots, r$ and note that $z_i - z_j \in FS(Z)$ for each $i > j$. By the pigeonhole principle, there are $1 \leq i < j \leq r$ such that $A_{z_i'} \phi$ and $A_{z_j'} \phi$ are in the same ball $B(A_u \phi, \epsilon/2)$ for some $u \in F$. Thus $\|A_{z_i'} f - A_{z_j'} f\| < \epsilon$ and, since the action of S_A is an isometry of $\mathcal{H}$, we conclude that $\|A_{z_i' - z_j'} \phi - \phi\| < \epsilon$. This implies that $z_i' - z_j' \in S$ and it proves the claim.

By Lemma 3.6, every DC set contains $FS(Z)$ for some set $Z \subset R$ with $|Z| = r + 1$. Therefore S has non-empty intersection with every DC set, and hence S is DC^*, as desired. $\square$

LEMMA 5.6. *For all $p \in \mathcal{G}$ and ϕ, $\psi \in \mathcal{H}$,*

$$p\text{-}\lim_u \langle A_u(V_A\phi), M_u\psi\rangle = \langle V_A\phi, V_M\psi\rangle.$$

Proof. We will assume, without loss of generality, that $\|\phi\|, \|\psi\| \le 1$. In view of Lemma 5.2,

$$p\text{-}\lim_u \langle V_A\phi, M_u\psi\rangle = \langle V_A\phi, (p\text{-}\lim_u M_u\psi)\rangle = \langle V_A\phi, V_M\psi\rangle.$$

Therefore, for every $\epsilon > 0$, the set

$$S_1 = \left\{u \in R : |\langle V_A\phi, M_u\psi\rangle - \langle V_A\phi, V_M\psi\rangle| < \frac{\epsilon}{2}\right\}$$

belongs to p.

Applying Lemma 5.5 with $V_A\phi$, we get that the set $S_2 := \{u \in R : \|A_u V_A\phi - V_A\phi\| < \epsilon/2\}$ is also in p. Using the Cauchy–Schwarz inequality, we have that, for any $u \in S_2$,

$$|\langle V_A\phi, M_u\psi\rangle - \langle A_u V_A\phi, M_u\psi\rangle| < \frac{\epsilon}{2}.$$

Finally, let $S := S_1 \cap S_2 \in p$ and let $u \in S$. We conclude that

$$|\langle A_u V_A\phi, M_u\psi\rangle - \langle V_A\phi, V_M\psi\rangle| < \epsilon,$$

which finishes the proof. □

5.3. *Dealing with $\phi^\perp$ when R is a field.* We now turn our attention to the weak-mixing component $\phi^\perp$. Dealing with this component in the general case requires some technical steps which obscure the main ideas. In order to clarify these ideas, we restrict our attention, in this subsection, to the case where R is a field; the general case is treated in the next subsection. (Of course, the results of this subsection also follow logically from the results in the next one.)

We will use the following version of the van der Corput trick.

PROPOSITION 5.7. (Cf. [**9**, Theorem 2.3]) *Let $p \in \mathcal{G}$, let $\mathcal{H}$ be a Hilbert space and let $(a_u)_{u\in R^*}$ be a bounded sequence in $\mathcal{H}$ indexed by R^*. If $p\text{-}\lim_u \langle a_{bu}, a_u\rangle = 0$ for all b in a co-finite subset of R^*, then $p\text{-}\lim_u a_u = 0$ in the weak topology of $\mathcal{H}$.*

LEMMA 5.8. *Let K be a field, let $\mathcal{H}$ be a Hilbert space, let $(U_g)_{g\in\mathcal{A}_K}$ be a unitary anti-representation of $\mathcal{A}_K$ on $\mathcal{H}$ and let $\phi^\perp$, $\psi \in \mathcal{H}$, where we assume that $V_A\phi^\perp = 0$. Then, for all $p \in \mathcal{G}$,*

$$p\text{-}\lim_u \langle A_u\phi^\perp, M_u\psi\rangle = 0.$$

Proof. Observe that, since we deal with an anti-representation, the distributive law (see (2.1)) takes the form

$$A_v M_u = M_u A_{vu} \tag{5.1}$$

for any $v \in K$ and $u \in K^*$. Let $a_u = M_{1/u}A_u\phi^\perp$. Then, for all $b \in K\backslash\{-1, 0, 1\}$, using (5.1) and the fact that isometries preserve scalar products,

$$\langle a_{ub}, a_u\rangle = \langle M_{1/ub}A_{ub}\phi^\perp, M_{1/u}A_u\phi^\perp\rangle = \langle A_{u(b-1/b)}\phi^\perp, M_b\phi^\perp\rangle.$$

Therefore, it follows, from Lemma 5.2, that, for every $p \in \mathcal{G}$,

$$p\text{-}\lim_u \langle a_{ub}, a_u \rangle = \langle p\text{-}\lim_u A_{u(b-1/b)}\phi^\perp, M_b\phi^\perp \rangle = \langle V_A\phi^\perp, M_b\phi^\perp \rangle = 0.$$

By Proposition 5.7, we conclude that $p\text{-}\lim_u M_{1/u}A_u\phi^\perp = p\text{-}\lim_u a_u = 0$.

$$\begin{aligned} p\text{-}\lim_u \langle A_u\phi^\perp, M_u\psi \rangle &= p\text{-}\lim_u \langle M_{1/u}A_u\phi^\perp, \psi \rangle \\ &= \langle p\text{-}\lim_u M_{1/u}A_u\phi^\perp, \psi \rangle = 0. \end{aligned}$$ □

5.4. *Dealing with $\phi^\perp$ when R is a general LID.* In this subsection, we extend the scope of Lemma 5.8 from the previous subsection to the case when we have a general LID (not necessarily a field). Namely, we will prove the following lemma.

LEMMA 5.9. *Assume R is an LID, let $\mathcal{H}$ be a Hilbert space, let $(U_g)_{g\in\mathcal{A}_R}$ be an isometric anti-representation of $\mathcal{A}_R$ on $\mathcal{H}$ and let $\phi^\perp, \psi \in \mathcal{H}$. Assume that $V_A\phi^\perp = 0$. Then, for all $p \in \mathcal{G}$,*

$$p\text{-}\lim_u \langle A_u\phi^\perp, M_u\psi \rangle = 0.$$

In the proof of this lemma, we will need a few facts about isometric anti-representations of $\mathcal{A}_R$. First, observe that, unlike the case when R is a field, M_u is not necessarily invertible. Thus its adjoint M_u^T (defined so that $\langle M_u\phi, \psi \rangle = \langle \phi, M_u^T\psi \rangle$ for all $\phi, \psi \in \mathcal{H}$) may not be in $\mathcal{A}_R$. However, since A_u is invertible (and hence unitary), we have the following distributivity relation.

LEMMA 5.10. *Under the assumptions of Lemma 5.9,*

$$A_{uv}M_u^T = M_u^T A_v.$$

Proof. For any $\phi, \psi \in \mathcal{H}$,

$$\langle A_{uv}M_u^T\phi, \psi \rangle = \langle \phi, M_uA_{-uv}\psi \rangle = \langle \phi, A_{-v}M_u\psi \rangle = \langle M_u^T A_v\phi, \psi \rangle.$$

This implies the identity in question. □

Another difficulty, which is present in our current context, is the fact that the composition $M_nM_n^T$ is not necessarily the identity map. The following lemma allows us to circumvent this difficulty when R is an LID.

LEMMA 5.11. *Under the assumptions of Lemma 5.9, there exists an orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ such that, for every $\phi \in \mathcal{H}$,*

$$p\text{-}\lim_u \|M_uM_u^T\phi - P\phi\| = 0.$$

Proof. Let $P_u = M_uM_u^T$. Since M_u is an isometry, P_u is the orthogonal projection onto the image of M_u. Observe that, in particular, the image of $P_{u_1u_2}$ is contained in the image of each P_{u_i}, $i = 1, 2$.

Let $\{r_1, r_2, \ldots\}$ be an arbitrary enumeration of the elements of R^* and let $u_n = \prod_{i=1}^n r_i$. Let S_n be the image of M_{u_n} so that P_{u_n} is the orthogonal projection onto S_n. Note that $S_{n+1} \subset S_n$. Let $S = \bigcap_{n\geq 1} S_n$ and let $P : \mathcal{H} \to S$ be the orthogonal projection. Let

E_0 be an orthonormal basis for S and, for each $n \geq 1$, let E_n be an orthonormal basis for $S_n \cap (S_{n+1})^\perp$. Thus $E = \bigcup_{n\geq 0} E_n$ is an orthonormal basis for $\mathcal{H}$. Write ϕ in terms of the basis E as $\phi = \sum_{n\geq 0}\sum_{e\in E_n} c_e e$. For a fixed $\epsilon > 0$, let $m \in \mathbb{N}$ be such that $\sum_{n\geq m}\sum_{e\in E_n} |c_e|^2 < \epsilon^2$.

Next, let u be in the ideal $u_m R$. We have that the image of P_u is contained in the image of P_{u_m}, so $P_u h \in S_m$ and hence

$$P_u\phi = \sum_{e\in E_0} c_e e + \sum_{n=m}^{\infty}\sum_{e\in E_n} c_e e = P\phi + \sum_{n=m}^{\infty}\sum_{e\in E_n} c_e e.$$

Therefore $\|P_u\phi - P\phi\| < \epsilon$. Since the ideal $u_m R$ has finite index as an additive group, it follows, from Lemma 3.7, that it belongs to p. We conclude that $p\text{-lim}\, M_n M_n^T\phi = p\text{-lim}\, P_n\phi = P\phi$ in the strong topology, as desired. □

Finally, we need a strengthening of Lemma 5.2.

Definition 5.12. Let R be an integral domain, let $b \in R$ and let $p \in \beta R$. Assume that $bR \in p$. Given a sequence $(x_u)_{u\in R}$ in a compact space X we define $p\text{-lim}_u\, x_{u/b}$ to be the point $x \in X$ such that, for every neighborhood U of x, the set $\{u \in bR : x_{u/b} \in U\} \in p$.

LEMMA 5.13. *Let R be an LID, let $p \in \mathcal{G}$ and let $k, b \in R^*$. For any unitary anti-representation $(U_g)_{g\in\mathcal{A}_R}$ of the semigroup $\mathcal{A}_R$ on a Hilbert space $\mathcal{H}$ and any $\phi \in \mathcal{H}$,*

$$p\text{-}\lim_u A_{ku/b}\phi = V_A\phi \text{ in the weak topology.}$$

Proof. First, observe that the p-lim is well defined since the ideal bR has finite index in R, p belongs to the closure $\overline{\mathcal{AMI}}$ of the additive minimal idempotents and hence, in view of Lemma 3.7, $bR \in p$.

It follows, from Lemma 3.9, that $p\text{-lim}_u\, A_{ku/b}\phi = kp\text{-lim}_u\, A_{u/b}\phi$. Since, in view of Lemma 3.8, $kp \in \overline{\mathcal{AMI}}$, we can, and will, assume that $k = 1$. Next, let $q = b^{-1}p$ be the ultrafilter defined so that $E \in q \iff bE \in p$. It follows, from Lemma 3.8, that $q \in \overline{\mathcal{AMI}}$. Therefore it follows, from Lemma 5.2, that, for any $\psi \in \mathcal{H}$ and $\epsilon > 0$, the set

$$E = \{u \in R : |\langle A_u\phi - V_A\phi, \psi\rangle| < \epsilon\} \in q.$$

We conclude that

$$bE = \{u \in bR : |\langle A_{u/b}\phi - V_A\phi, \psi\rangle| < \epsilon\} \in p. \qquad \square$$

We can now give a proof of Lemma 5.9.

Proof of Lemma 5.9. Let M_u^T denote the adjoint of M_u. Then $\langle A_u\phi^\perp, M_u\psi\rangle = \langle M_u^T A_u\phi^\perp, \psi\rangle$, so the lemma will follow if we show that $p\text{-lim}\, M_u^T A_u\phi^\perp = 0$ (in the weak topology). To do this, we will use the van der Corput trick (Proposition 5.7), so it suffices to show that

$$p\text{-}\lim_u \langle M_{ub}^T A_{ub}\phi^\perp, M_u^T A_u\phi^\perp\rangle = 0 \quad \text{for all } b \in R\backslash\{-1, 0, 1\}. \tag{5.2}$$

Since the operator A_u is unitary, we can rewrite the inner product in (5.2) as $\langle M_{ub}^T A_{ub}\phi^\perp, M_u^T A_u\phi^\perp\rangle = \langle A_{-u}M_u M_{ub}^T A_{ub}\phi^\perp, \phi^\perp\rangle$. By (5.1), $A_{-u}M_u = M_u A_{-u^2}$

(recall this is an anti-representation). Also, assuming that $u \in bR$ and evoking Lemma 5.10, we conclude that

$$\langle M_{ub}^T A_{ub}\phi^\perp, M_u^T A_u\phi^\perp\rangle = \langle M_u M_{ub}^T A_{ub-u/b}\phi^\perp, \phi^\perp\rangle = \langle A_{ub-u/b}\phi^\perp, M_b M_n M_n^T \phi^\perp\rangle.$$

By Lemma 5.13, $p\text{-lim}\, A_{ub-u/b}\phi^\perp = V_A\phi^\perp = 0$ in the weak topology. By Lemma 5.11, $p\text{-lim}_u\, M_b M_u M_u^T \phi^\perp$ exists in the strong topology. Thus we conclude that $p\text{-lim}\langle A_{ub-u/b)}\phi^\perp, M_b M_n M_n^T \phi^\perp\rangle = 0$, which gives (5.2) and finishes the proof. □

5.5. *Proofs of the main results.* We have now gathered all the ingredients necessary for the proofs of the main Theorems of the paper. We start by proving Theorem 5.1.

Proof of Theorem 5.1. Let $\phi^\perp = \phi - V_A\phi$, so that $V_A\phi^\perp = 0$. Using Lemmas 5.6 and 5.9, we deduce that

$$p\text{-lim}\langle A_u\phi, M_u\psi\rangle = p\text{-lim}\langle A_u V_A\phi, M_u\psi\rangle + \langle A_u\phi^\perp, M_u\psi\rangle = \langle V_A\phi, V_M\psi\rangle. \quad \square$$

As a corollary, we now obtain the following theorem.

THEOREM 5.14. *Let R be an LID, let (Ω, μ) be a probability space, let $(T_g)_{g\in\mathcal{A}_R}$ be a measure preserving action of $\mathcal{A}_R$ on Ω, let $B \subset \Omega$ be a measurable set and let $\epsilon > 0$. Then the set*

$$R(B, \epsilon) := \{u \in R : \mu(A_u^{-1}B \cap M_u^{-1}B) \geq \mu(B)^2 - \epsilon\}$$

is DC^* and, in particular, affinely syndetic.

Proof. Let $\mathcal{H} = L^2(\Omega, \mu)$ and, for each $g \in \mathcal{A}_R$, define the operator $(U_g\phi)(x) = \phi(T_g x)$. Observe that $U_g U_h = U_{hg}$, so this induces an isometric anti-representation $(U_g)_{g\in\mathcal{A}_R}$ of $\mathcal{A}_R$ in $\mathcal{H}$. Let $B \subset \Omega$. Observe that

$$1_{T_g^{-1}B}(x) = 1 \iff T_g x \in B \iff 1_B(T_g x) = 1 \iff U_g 1_B(x) = 1.$$

Therefore $\mu(A_u^{-1}B \cap M_u^{-1}B) = \int_\Omega A_u 1_B \cdot M_u 1_B \, d\mu = \langle A_u 1_B, M_u 1_B\rangle$. It follows, from Theorem 5.1, that, for any $\epsilon > 0$, the set

$$\{u \in R : \langle A_u 1_B, M_u 1_B\rangle \geq \langle V_A 1_B, V_M 1_B\rangle - \epsilon\}$$

is DC^*. Finally, it follows, from Corollary 5.4 (applied with $\phi = 1_B$ and $\psi \equiv 1$), that

$$\langle V_A 1_B, V_M 1_B\rangle \geq \mu(B)^2. \quad \square$$

Observe that Theorem 1.7 easily follows from Theorem 5.14. Indeed, given $p \in \mathcal{G}$ it follows, from the definition of DC^* sets and Theorem 5.14, that $R(B_i, \delta) \in p$ for every i. Therefore also the intersection $R = R(B_1, \delta) \cap \cdots \cap R(B_t, \delta)$ belongs to p. Since $p \in \mathcal{G}$ was arbitrary, it follows that R is itself a DC^* set. Finally, Remark 4.8 implies that R must be affinely syndetic.

We now present the main combinatorial corollary of Theorem 5.14.

THEOREM 5.15. *Let K be a countable field and let $R \subset K$ be a sub-ring which is an LID. Let $E \subset K$ with $\bar{d}_{(F_N)}(E) > 0$ for some double Følner sequence (F_N) and let $\epsilon > 0$. Then the set*

$$\{u \in R : \bar{d}_{(F_N)}((E - u) \cap (E/u)) > \bar{d}_{(F_N)}(E)^2 - \epsilon\} \tag{5.3}$$

is DC^* and, in particular, affinely syndetic in R.

Proof. Using the correspondence principle [**10**, Theorem 2.8], one can construct a measure preserving action $(T_g)_{g\in\mathcal{A}_K}$ of $\mathcal{A}_K$ on a probability space $(\Omega, \mathcal{B}, \mu)$ and a set $B \in \mathcal{B}$ such that $\mu(B) = \bar{d}_{(F_N)}(E)$ and, for each $u \in K^*$,

$$\bar{d}_{(F_N)}((E-u)\cap(E/u)) \geq \mu(A_u^{-1}B \cap M_u^{-1}B).$$

The result now follows from Theorem 5.14. □

One can deduce parts (2) and (3) of Theorem 1.9 from Theorem 5.15 using the fact that for any finite partition of a countable field, one of the cells of the partition has positive upper density with respect to a double Følner sequence. Then using that cell C_i of the partition as E, for any element n of the (non-empty) set defined in (5.3) and for any x in the (non-empty) intersection $(C_i - n)\cap(C_i/n)$, $\{x+n, xn\} \subset C_i$.

To deduce part (1) of Theorem 1.9, one needs an additional fact.

PROPOSITION 5.16. *The subset $\mathbb{N}$ of the ring $\mathbb{Z}$ belongs to every non-principal multiplicative idempotent.*

Proof. Let $p \in \beta\mathbb{Z}$ be a non-principal multiplicative idempotent. Assume, for the sake of a contradiction, that $\mathbb{N} \notin p$. Then $-\mathbb{N} \in p = pp$, which, by definition, implies that $\{n \in \mathbb{Z}^* : -\mathbb{N}/n \in p\} \in p$. Observe that

$$-\mathbb{N}/n = \{a \in \mathbb{Z}^* : an \in -\mathbb{N}\} = \begin{cases} \mathbb{N} & \text{if } n \in -\mathbb{N}, \\ -\mathbb{N} & \text{if } n \in \mathbb{N}. \end{cases}$$

Therefore $\{n \in \mathbb{Z}^* : -\mathbb{N}/n \in p\} = \mathbb{N} \notin p$, which is the desired contradiction. □

To deduce part (1) of Theorem 1.9, one applies Theorem 5.15 with $K = \mathbb{Q}$, $R = \mathbb{Z}$ and E being a cell of the partition with positive upper density with respect to a double Følner sequence. The set S, defined by (5.3), is DC^* in $\mathbb{Z}$, which means that, for any $p \in \mathcal{G}$, $S \in p$. Since any $p \in \mathcal{G}$ is a non-principal multiplicative idempotent, it follows, from Proposition 5.16, that also $\mathbb{N} \in p$, and therefore $S \cap \mathbb{N} \in p$ and hence is non-empty. For any n in that intersection, the set $(E-n)\cap(E/n)$ is non-empty and any x in this intersection yields $\{x+n, xn\} \subset E$.

6. *Notions of largeness and configurations* $\{xy, x+y\}$ *in* $\mathbb{N}$

In this section, we discuss notions of largeness which guarantee the presence of configurations of the form $\{x+y, xy\}$.

It is a trivial observation that the set of odd numbers in $\mathbb{N}$ or in $\mathbb{Z}$ does not contain pairs $\{x+y, xy\}$. Therefore, additively syndetic sets (i.e. sets which are syndetic with respect to the additive semigroup) do not contain, in general, configurations $\{x+y, xy\}$. It is thus somewhat surprising that multiplicatively syndetic subsets in any integral domain do contain such patterns.

THEOREM 6.1. *Let R be an infinite countable integral domain and let $S \subset R^*$ be multiplicatively syndetic (i.e. syndetic as a subset of the semigroup $(R^*, \cdot)$). Then S contains (many) pairs of the form $\{x+y, xy\}$.*

Proof. Let $F \subset R^*$ be a finite set such that $R^* = \bigcup_{n \in F} S/n$ (the existence of such F is equivalent, by definition, to the statement that S is multiplicatively syndetic). Thus R^* is finitely partitioned into multiplicative shifts of S and hence there exist (many) $a, b \in R^*$ such that $a + bF \subset S$†. Since $ab \in R^* = \bigcup_{n \in F} S/n$, there exist some $n \in F$ such that $abn \in S$. We conclude that $\{a + bn, a(bn)\} \subset S$, as desired. □

While it is not hard to see that there exist partitions of $\mathbb{N}$ or $\mathbb{Z}$ with none of the cells of the partition being multiplicatively syndetic, it is a classical fact that, for any finite partition of a semigroup, one of the cells is piecewise syndetic‡. One could then hope that any multiplicatively piecewise syndetic subset of R^* contains a pattern $\{x + y, xy\}$. Unfortunately, the next example refutes this assertion.

THEOREM 6.2. *There exists a set $E \subset \mathbb{N}$ which is additively thick and multiplicatively thick (and so, in particular, E is a multiplicatively piecewise syndetic subset of $\mathbb{N}$) but does not contain a pair $\{x + y, xy\}$ with $x, y > 2$.*

Proof. Let (p_N) be a sequence of primes such that $p_1 = 5$ and, for each $N \in \mathbb{N}$, $p_{N+1} > 4(Np_N)^4$. For each $N \in \mathbb{N}$, let

$$E_{2N-1} = p_N[1, N] \quad \text{and} \quad E_{2N} = [(Np_N)^2 + 1, 2(Np_N)^2 - 3],$$

where we use the notation $[a, b]$ to denote the set $\{a, a + 1, \ldots, b\}$. Let $E = \bigcup E_N$. It follows directly from the construction that E is additively thick as a subset of either $\mathbb{N}$ or $\mathbb{Z}$ and is multiplicatively thick as a subset of $\mathbb{N}$. Moreover, $E \cup (-E)$ is a multiplicatively thick subset of $\mathbb{Z}^*$. Since $\mathbb{N}$ is a multiplicatively syndetic subset of $\mathbb{Z}^*$, it follows that E is a multiplicative piecewise syndetic subset of $\mathbb{Z}^*$.

We first show that no set E_{2N} contains a pair $\{x + y, xy\}$: assume that $a = x + y \in E_{2N}$ and $x, y \geq 2$. Let $b = xy$. Then $b \geq 2(a - 2) \geq 2[(Np_N)^2 + 1 - 2] = 2(Np_N)^2 - 2$, so b is too large to be in E_{2N}.

Next we show that no set E_{2N-1} contains such a pair. Assume $xy \in E_{2N-1}$, say, $xy = np_N$. Then, without loss of generality, $x = p_N d$ and $y = n/d$ for some divisor d of n. But then $x + y < p_N(d + 1)$ because $n/d \leq N < p_N$. Hence $x + y \notin E_{2N-1}$.

For each $N \in \mathbb{N}$, $(\max E_{2N-1})^2 = (Np_N)^2 < (Np_N)^2 + 1 = \min E_{2N}$ and $(\max E_{2N})^2 = (2(Np_N)^2 - 3)^2 < 4(Np_N)^4 < p_{N+1} = \min E_{2N+1}$. Fix a pair $x, y \in \mathbb{N}$ with both $x, y \geq 2$, let $a = xy$ and $b = x + y$. We observe that $b \leq a \leq (b/2)^2$.

If $b \in E$, say, $b \in E_n$, then $\min E_n \leq b \leq a \leq (b/2)^2 < [(\max E_n)/2]^2 < \min E_{n+1}$ so a can not be in E_m for any $m \neq n$. Since we have already shown that $a \notin E_n$ (otherwise E_n would contain $\{b, a\} = \{x + y, xy\}$), we conclude that $a \notin E$ and this finishes the proof. □

We observe that the complement $\tilde{E} = \mathbb{N} \backslash E$ of the set constructed in Theorem 6.2 is also rather large. In particular, $\bar{d}(\tilde{E}) = 1$, where, as usual, for a subset $S \subset \mathbb{N}$, $\bar{d}(S)$ denotes the

† This is a well-known extension of van der Waerden's theorem in arithmetic progressions. One way to prove this is to apply the Hales–Jewett theorem, as in the proof of [**8**, Proposition 4.4], where a stronger statement is proved.

‡ A subset E of a commutative semigroup is called *piecewise syndetic* if it is the intersection of a syndetic set and a thick set.

upper density

$$\bar{d}(S) = \limsup_{N\to\infty} \frac{|S \cap \{1, \ldots, N\}|}{N}.$$

The next result shows that sets having upper density 1 are large not only additively, but also multiplicatively.

THEOREM 6.3. *Let $E \subset \mathbb{N}$ satisfy $\bar{d}(E) = 1$. Then E is affinely thick.*

Proof. Since $\bar{d}$ is the upper density with respect to an additive Følner sequence, it is not hard to see that $\bar{d}((E - n) \cap E) = 1$ for any $n \in \mathbb{N}$. We claim that, also, $\bar{d}((E/n) \cap E) = 1$ for any $n \in \mathbb{N}$.

Assuming the claim for now, let $F = \{g_1, \ldots, g_k\} \subset \mathcal{A}_{\mathbb{N}}$ be an arbitrary finite set. We can write each g_i as the map $g_i : x \mapsto a_i x + b_i$. Let $E_0 = E$ and, for each $i = 1, \ldots, k$, let $A_i = ((E_{i-1} - b_i) \cap E_{i-1})$ and $E_i = ((A_i/a_i) \cap A_i)$. It follows, by induction, that each of the sets E_i, A_i satisfies $\bar{d}(E_i) = \bar{d}(A_i) = 1$. Take $x \in E_k$; we will show that $g_i(x) \in E$ for every i. Indeed, $x \in E_k \subset E_i = ((A_i/a_i) \cap A_i)$, so $a_i x \in A_i = ((E_{i-1} - b_i) \cap E_{i-1})$ and hence $a_i x + b_i = g_i(x) \in E_{i-1} \subset E$, as desired.

Now we prove the claim. We will write $[1, x]$ to denote the set $\{1, 2, \ldots, \lfloor x \rfloor\}$, where $\lfloor x \rfloor$ is the largest integer no bigger than x.

Let $n \in \mathbb{N}$ and take $\epsilon > 0$ arbitrary. For some arbitrarily large $N \in \mathbb{N}$,

$$|E \cap [1, N]| > \left(1 - \frac{\epsilon}{2n}\right) N = N - \frac{\epsilon N}{2n}.$$

This implies that

$$|nE \cap [1, N]| = \left| E \cap \left[1, \tfrac{N}{n}\right] \right| > \frac{N}{n} - \frac{\epsilon N}{2n}.$$

Using the general fact that $|X \cup Y| + |X \cap Y| = |X| + |Y|$, we deduce that $nE \cap E \cap [1, N] = (nE \cap [1, N]) \cap (E \cap [1, N])$ has cardinality

$$\begin{aligned}
|nE \cap E \cap [1, N]| &= |E \cap [1, N]| + |nE \cap [1, N]| - |(nE \cap [1, N]) \cup (E \cap [1, N])| \\
&\geq N - \frac{\epsilon N}{2n} + \frac{N}{n} - \frac{\epsilon N}{2n} - N \\
&= \frac{N}{n}(1 - \epsilon).
\end{aligned}$$

Dividing by n (and observing that every number in the intersection $nE \cap E \cap [1, N]$ is divisible by n) we deduce that

$$|E \cap (E/n) \cap [1, N/n]| = |nE \cap E \cap [1, N]| \geq \frac{N}{n}(1 - \epsilon).$$

As N can be taken arbitrarily large and ϵ arbitrarily small, we conclude that $\bar{d}(E \cap (E/n)) = 1$, which proves the claim. □

It is clear that, for any $y \in \mathbb{N}$, any affinely thick set contains configurations of the form $\{x + y, xy\}$. This observation applies, in particular, to the complement $\tilde{E}$ of the set E constructed in Theorem 6.2.

Now recall the notion of DC set (see Definition 3.5) and observe that, for any finite partition of $\mathbb{N}$, one of the cells is a DC set. It follows, from (the proof of) [**6**, Corollary 5.5], that any DC set is both additively piecewise syndetic and multiplicatively piecewise syndetic. For a partition of $\mathbb{N}$ into two cells, one has the following dichotomy: either one of the cells has upper density one (in which case Theorem 6.3 assures us that it contains configurations $\{x + y, xy\}$) or both cells have positive lower density. In view of this observation, we make the following conjecture.

Conjecture 6.4. Let $E \subset \mathbb{N}$ be additively and multiplicatively piecewise syndetic and have positive lower density. Then E contains many configurations of the form $\{x + y, xy\}$.

While Conjecture 6.4 implies that, for any partition of $\mathbb{N}$ into two cells, one of the cells contains many configurations $\{x + y, xy\}$, the property of having positive lower density is not stable under partitions. Indeed, it is not hard to construct a partition of $\mathbb{N}$ into two sets, both with zero lower density. However, for any finite partition of a DC set, one of the cells is still a DC set. Observe that the example E, constructed in the proof of the Theorem 6.2, can be split into two sets $E = E_A \cup E_M$ such that E_A is additively thick, but has density zero with respect to any multiplicative Følner sequence, and E_M is multiplicatively thick, but has density zero with respect to any additive Følner sequence. Therefore E is very far from being a DC set. This observation leads to the following conjecture.

Conjecture 6.5. Every DC set in $\mathbb{N}$ contains a configuration $\{x + y, xy\}$.

Observe that Conjecture 6.5 implies that, for any finite partition of $\mathbb{N}$, one of the cells contains plenty of configurations $\{x + y, xy\}$.

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