

# On the Density of Coprime Tuples of the Form $(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor)$ , Where $f_1, \dots, f_k$ Are Functions from a Hardy Field

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*It is a well-known theorem of Čebyšev<sup>1</sup> that the probability of the relation  $\gcd(n, m) = 1$  is  $\frac{6}{\pi^2}$ . One can expect this still to remain true if  $m = g(n)$  is a function of  $n$ , provided that  $g(n)$  does not preserve arithmetic properties of  $n$ .*

P. Erdős and G. Lorentz

**Abstract** Let  $k \in \mathbb{N}$  and let  $f_1, \dots, f_k$  belong to a Hardy field. We prove that under some natural conditions on the  $k$ -tuple  $(f_1, \dots, f_k)$  the density of the set

$$\{n \in \mathbb{N} : \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor) = 1\}$$

exists and equals  $\frac{1}{\zeta(k+1)}$ , where  $\zeta$  is the Riemann zeta function.

## 1 Introduction

The above epigraph is a quote from the introduction to a paper by Erdős and Lorentz [12], which establishes sufficient conditions for a differentiable function  $f : [1, \infty) \rightarrow \mathbb{R}$  of sub-linear growth to satisfy

$$d(\{n \in \mathbb{N} : \gcd(n, \lfloor f(n) \rfloor) = 1\}) = \frac{6}{\pi^2}; \quad (1)$$

here  $d(A)$  denotes the natural density of a set  $A \subset \mathbb{N}$ .

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<sup>1</sup>The attribution of this result to Čebyšev (Чебышёв) seems not to be justified; see, however, the very interesting recent preprint [1] where Čebyšev's role in the popularization of this theorem is traced and analyzed. The result itself goes back to Dirichlet (see [10, pp. 51–66] where the equivalent statement  $\sum_{n=1}^N \phi(n) \sim \frac{3}{\pi^2} n^2$  is proven) and was rediscovered multiple times—see, for example, [6, 7, 21, 23, 24]. It is worth noting that it was Cesàro who formulated this result in probabilistic terms [6] and also gave a probabilistic, though not totally rigorous, proof in [7].

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Perhaps the earliest result of this kind is due to Watson [28], who showed that (1) holds for  $f(n) = n\alpha$ , where  $\alpha$  is an irrational number (see also [13, 22]). Other examples of functions for which (1) holds are  $f(n) = n^c$ , where  $c > 0$ ,  $c \notin \mathbb{N}$ , (see [20] for the case  $0 < c < 1$  and [9] for the general case) and  $f(n) = \log^r(n)$  for all  $r > 1$  (see [20] for the case  $r > 2$  and [12] for the general case).

The purpose of this paper is to establish (1) for a large class of smooth functions that naturally includes examples such as  $f(n) = n^c$  or  $f(n) = \log^r(n)$ ; this is the class of functions belonging to a Hardy field.

Let  $G$  denote the set of all germs<sup>2</sup> at  $\infty$  of real valued functions defined on the half-line  $[1, \infty)$ . Note that  $G$  forms a ring under pointwise addition and multiplication, which we denote by  $(G, +, \cdot)$ . Any subfield of the ring  $(G, +, \cdot)$  that is closed under differentiation is called a *Hardy field*. By abuse of language, we say that a function  $f : [1, \infty) \rightarrow \mathbb{R}$  belongs to some Hardy field  $\mathcal{H}$  (and write  $f \in \mathcal{H}$ ) if its germ at  $\infty$  belongs to  $\mathcal{H}$ . See [3–5] and some references therein for more information on Hardy fields.

A classical example of a Hardy field is the class of logarithmico-exponential functions<sup>3</sup> introduced by Hardy in [17, 18]; we denote this class by  $\mathcal{L}$ . It is worth noting that for any Hardy field  $\mathcal{H}$  there exists a Hardy field  $\mathcal{H}'$  such that  $\mathcal{H}' \supset \mathcal{L} \cup \mathcal{H}$ .

If  $\mathcal{H}$  is a Hardy field, then one has the following basic properties:

- If  $f \in \mathcal{H}$ , then  $\lim_{t \rightarrow \infty} f(t)$  exists (as an element in  $\mathbb{R} \cup \{-\infty, \infty\}$ );
- Any non-constant  $f \in \mathcal{H}$  is eventually either strictly increasing or strictly decreasing; any non-linear  $f \in \mathcal{H}$  is eventually either strictly concave or strictly convex.
- If  $f \in \mathcal{H}$ ,  $g \in \mathcal{L}$ , and  $\lim_{t \rightarrow \infty} g(t) = \infty$ , then there exists a Hardy field  $\mathcal{H}'$  containing  $f(g(t))$ .
- If  $f \in \mathcal{H}$ ,  $g \in \mathcal{L}$ , and  $\lim_{t \rightarrow \infty} f(t) = \infty$ , then there exists a Hardy field  $\mathcal{H}'$  containing  $g(f(t))$ .

Some well-known examples of functions coming from Hardy fields are:

$$t^c \ (\forall c \in \mathbb{R}), \log(t), \exp(t), \Gamma(t), \zeta(t), \text{Li}(t), \sin\left(\frac{1}{t}\right), \text{ etc.}$$

Before formulating our main results, we introduce some convenient notation. We use  $\log_n(t)$  to abbreviate the  $n$ -th iteration of logarithms, that is,  $\log_2(t) = \log \log(t)$ ,  $\log_3(t) = \log \log \log(t)$ , and so on. Also, given two functions  $f, g : [1, \infty) \rightarrow \mathbb{R}$  we will write  $f(t) \prec g(t)$  if  $\frac{g(t)}{f(t)} \rightarrow \infty$  as  $t \rightarrow \infty$ .

<sup>2</sup>We define a *germ* at  $\infty$  to be any equivalence class of functions under the equivalence relationship  $(f \sim g) \Leftrightarrow (\exists t_0 > 0 \text{ such that } f(t) = g(t) \text{ for all } t \in [t_0, \infty))$ .

<sup>3</sup>By a *logarithmico-exponential function* we mean any function  $f : (0, \infty) \rightarrow \mathbb{R}$  that can be obtained from constants,  $\log(t)$  and  $\exp(t)$  using the standard arithmetical operations  $+$ ,  $-$ ,  $\cdot$ ,  $\div$ , and the operation of composition.

Let  $\mathcal{H}$  be a Hardy field and let  $f \in \mathcal{H}$ . Consider the following two conditions:

- (A)  $\log(t) \log_4(t) \prec f(t)$ ;
- (B) There exists  $j \in \mathbb{N}$  such that  $t^{j-1} \prec f(t) \prec t^j$ .

We have the following theorem.

**Theorem 1** *Let  $\mathcal{H}$  be a Hardy field and assume that  $f \in \mathcal{H}$  satisfies conditions (A) and (B). Then the natural density of the set*

$$\{n \in \mathbb{N} : \gcd(n, \lfloor f(n) \rfloor) = 1\}$$

*exists and equals  $\frac{6}{\pi^2}$ .*

Examples of sequences  $(f(n))_{n \in \mathbb{N}}$  to which Theorem 1 applies are  $n^c$  (with  $c \notin \mathbb{N}$ ),  $\log^2(n)$ ,  $n^{\sqrt{3}} \log(n)$ ,  $\frac{n}{\log_2(n)}$ ,  $\log(n!)$ ,  $\text{Li}(n)$ ,  $\log(|B_{2n}|)$  (where  $B_n$  denotes the  $n$ -th Bernoulli number), and many more.

We remark that condition (A) is sharp. Indeed, it is shown in [12, Sect. 3] that Theorem 1 does not hold for the function  $f(t) = \log(t) \log_4(t)$ , as well as for many other functions that grow slower than  $\log(t) \log_4(t)$ .

As for condition (B), it can perhaps be replaced by the following:

- (B') There exists  $j \in \mathbb{N}$  such that  $f(t) \prec t^j$  and for all polynomials  $p(t) \in \mathbb{Q}[t]$  we have  $|f(t) - p(t)| \succ \log(t)$ .

Condition (B') is inspired by a theorem of Boshernitzan (cf. [5, Theorem 1.3]). However, proving Theorem 1 under conditions (A) and (B') would certainly necessitate introduction of new ideas.

We actually prove a multi-dimensional generalization of Theorem 1. Let  $\mathcal{H}$  be a Hardy field and assume  $f_1, \dots, f_k \in \mathcal{H}$ . In addition to conditions (A) and (B) consider the following:

- (C)  $\frac{f_{i+1}}{f_i} \succ \log_2^4(t)$  for all  $i = 1, \dots, k - 1$ .

**Theorem 2** *Let  $\mathcal{H}$  be a Hardy field and assume  $f_1, \dots, f_k \in \mathcal{H}$  satisfy conditions (A)–(C). Then the natural density of the set*

$$\{n \in \mathbb{N} : \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor) = 1\}$$

*exists and equals  $\frac{1}{\zeta(k+1)}$ , where  $\zeta$  is the Riemann zeta function.*

We would like to remark that our proof of Theorem 2 works for (a larger class of) functions which have sufficiently many derivatives and possess some other natural regularity properties. We decided in favor of dealing with Hardy fields since they (a) provide an ample supply of interesting examples and (b) allow for, so to say, cleaner proofs.

The structure of the paper is as follows. In Sect. 2 we prove some differential inequalities for functions from a Hardy field; these inequalities will play a crucial role in the later sections. In Sect. 3 we briefly recall van der Corput's method for

estimating exponential sums. In Sect. 4 we apply van der Corput's method to derive useful estimates for exponential sums involving functions from a Hardy field and in Sect. 5 we use a higher dimensional version of the Erdős-Turán inequality to convert these estimates into discrepancy estimates. In Sect. 6 we use the estimates derived in the previous sections to give a proof of Theorem 2. Finally, in Sect. 7, we formulate some natural open questions.

## 2 Differential Inequalities for Functions from a Hardy Field

In this section we derive some differential inequalities for functions belonging to a Hardy field. Similar inequalities can be found in [14, Sect. 2.1] and in [2, Sect. 2.1].

Given two functions  $f, g : [1, \infty) \rightarrow \mathbb{R}$  we write  $f(t) \ll g(t)$  if there exist  $C > 0$  and  $t_0 \geq 1$  such that  $f(t) \leq Cg(t)$  for all  $t \geq t_0$ . Also, for  $\ell \in \mathbb{N}$  we use  $f^{(\ell)}(t)$  to denote the  $\ell$ -th derivative of  $f(t)$ .

The following lemma appears in [14].

**Proposition 3 (See [14, Corollary 2.3])** *Let  $\mathcal{H}$  be a Hardy field. Suppose  $f \in \mathcal{H}$  satisfies condition (B). Then, for all  $\ell \in \mathbb{N}$ , we have*

$$\frac{f(t)}{t^\ell \log^2(t)} \prec |f^{(\ell)}(t)| \ll \frac{f(t)}{t^\ell}.$$

Next, we derive a series of lemmas (Lemmas 4–7) which are needed for the proof of the main result of this section, Proposition 8.

**Lemma 4** *Let  $m \in \mathbb{N}$  and let  $\mathcal{H}$  be a Hardy field. Suppose  $f, g \in \mathcal{H}$  satisfy  $|f(t)| \prec |g(t)| \prec |f(t)| \log^m(t)$  and  $|\log(|f(t)|)| > \log_2(t)$ . Then*

$$\frac{f'(t)}{f(t)} \sim \frac{g'(t)}{g(t)}.$$

*Proof* Our goal is to show that

$$\frac{\frac{g'(t)}{g(t)}}{\frac{f'(t)}{f(t)}} \xrightarrow{t \rightarrow \infty} 1.$$

First we note that since  $\mathcal{H}$  is a field closed under differentiation, the function  $\frac{g'(t)/g(t)}{f'(t)/f(t)}$  is contained in  $\mathcal{H}$ . From this it follows that  $\lim_{t \rightarrow \infty} \frac{g'(t)/g(t)}{f'(t)/f(t)}$  exists (as a number in  $\mathbb{R} \cup \{-\infty, \infty\}$ ). From L'Hospital's rule we now obtain

$$\lim_{t \rightarrow \infty} \frac{\frac{g'(t)}{g(t)}}{\frac{f'(t)}{f(t)}} = \lim_{t \rightarrow \infty} \frac{\log(|g(t)|)}{\log(|f(t)|)}.$$

To finish the proof we distinguish between the cases  $|f(t)| > 1$  and  $|f(t)| < 1$ . If  $|f(t)| > 1$  then, using  $|f(t)| < |g(t)| < |f(t)| \log^m(t)$  and  $|\log(|f(t)|)| > \log_2(t)$ , we deduce that

$$1 \leq \lim_{t \rightarrow \infty} \frac{\log(|g(t)|)}{\log(|f(t)|)} \leq \lim_{t \rightarrow \infty} \frac{\log(|f(t)|) + m \log_2(t)}{\log(|f(t)|)} = 1.$$

Likewise, if  $|f(t)| < 1$ , then we have

$$1 \geq \lim_{t \rightarrow \infty} \frac{\log(|g(t)|)}{\log(|f(t)|)} \geq \lim_{t \rightarrow \infty} \frac{\log(|f(t)|) + m \log_2(t)}{\log(|f(t)|)} = 1.$$

This finishes the proof. □

**Lemma 5** *Let  $\mathcal{H}$  be a Hardy field and suppose  $f \in \mathcal{H}$  satisfies condition (B). Then  $f^{(\ell)}$  satisfies either  $f^{(\ell)}(t) > 1$  or  $f^{(\ell)}(t) < 1$  for all  $\ell \in \mathbb{N}$ .*

*Proof* By way of contradiction, let us assume that there exist  $\ell \in \mathbb{N}$  and  $c \in \mathbb{R}$  such that  $f^{(\ell)}(t) \sim c$ . Observe that  $c \neq 0$ , because otherwise  $f(t)$  is a polynomial, which contradicts condition (B).

Using Proposition 3 we deduce that

$$\frac{f(t)}{t^\ell \log^2(t)} < |c| \ll \frac{f(t)}{t^\ell},$$

which is equivalent to

$$t^\ell \ll f(t) < t^\ell \log^2(t).$$

It follows from condition (B) that we can replace  $t^\ell \ll f(t)$  with  $t^\ell < f(t)$ . Therefore, we have

$$t^\ell < f(t) < t^\ell \log^2(t).$$

By using induction on  $i$  and by repeatedly applying Lemma 4 to the functions  $f^{(i)}(t)$  and  $t^i$ , we conclude that for all  $i \in \{0, 1, \dots, \ell - 1\}$ ,

$$\frac{f^{(i+1)}(t)}{f^{(i)}(t)} \sim \frac{(\ell - 1)t^{\ell-i-1}}{t^{\ell-i}} = \frac{\ell - i}{t}.$$

In particular, this shows that

$$\frac{f^{(\ell)}(t)}{f(t)} \sim \frac{\ell!}{t^\ell}.$$

Finally, combining  $t^\ell < f(t)$  and  $\frac{f^{(\ell)}(t)}{f(t)} \sim \frac{\ell!}{t^\ell}$  yields  $f^{(\ell)}(t) > 1$ , which contradicts  $f^{(\ell)}(t) \sim c$ .  $\square$

**Lemma 6** *Let  $\mathcal{H}$  be a Hardy field and suppose that  $f \in \mathcal{H}$  satisfies either  $f(t) > 1$  or  $f(t) < 1$ . Also, assume  $|\log(|f(t)|)| \ll \log_2(t)$ . Then*

$$\frac{|f(t)|}{t \log(t) \log_2^2(t)} < |f'(t)| \ll \frac{|f(t)|}{t \log(t)}.$$

*Proof (cf. The Proof of Lemma 2.1 in [14])* By L'Hospital's rule we get

$$\lim_{t \rightarrow \infty} \frac{\frac{f'(t)}{f(t)}}{\frac{1}{t \log(t)}} = \lim_{t \rightarrow \infty} \frac{\log(|f(t)|)}{\log_2(t)} \ll 1.$$

This proves that  $|f'(t)| \ll \frac{|f(t)|}{t \log(t)}$ .

On the other hand, we have

$$\lim_{t \rightarrow \infty} \frac{\frac{f'(t)}{f(t)}}{\frac{1}{t \log(t) \log_2^2(t)}} = \lim_{t \rightarrow \infty} \log(|f(t)|) \log_2(t) = \pm \infty,$$

which shows that  $\frac{|f(t)|}{t \log(t) \log_2^2(t)} < |f'(t)|$ .  $\square$

**Lemma 7** *Let  $m \in \mathbb{N}$ , let  $\mathcal{H}$  be a Hardy field and let  $f, g \in \mathcal{H}$ . Assume that  $f$  satisfies either  $f(t) > 1$  or  $f(t) < 1$  and  $g$  satisfies either  $g(t) > 1$  or  $g(t) < 1$ . Also, assume  $|f(t)| < |g(t)| < |f(t)| \log^m(t)$  and  $|\log(|f(t)|)| \ll \log_2(t)$ . Then*

$$\left| \frac{f'(t)}{f(t)} \right| \frac{1}{\log_2^2(t)} < \left| \frac{g'(t)}{g(t)} \right| < \left| \frac{f'(t)}{f(t)} \right| \log_2^2(t).$$

*Proof* It follows from  $|f(t)| < |g(t)| < |f(t)| \log^m(t)$  and  $|\log(|f(t)|)| \ll \log_2(t)$  that  $|\log(|g(t)|)| \ll \log_2(t)$ . Hence we can apply Lemma 6 to both  $f$  and  $g$  and obtain

$$\frac{1}{t \log(t) \log_2^2(t)} < \left| \frac{f'(t)}{f(t)} \right| \ll \frac{1}{t \log(t)}$$

as well as

$$\frac{1}{t \log(t) \log_2^2(t)} < \left| \frac{g'(t)}{g(t)} \right| \ll \frac{1}{t \log(t)}.$$

We deduce that

$$\left| \frac{g'(t)}{g(t)} \right| \ll \frac{1}{t \log(t)} = \frac{\log_2^2(t)}{t \log(t) \log_2^2(t)} < \left| \frac{f'(t)}{f(t)} \right| \log_2^2(t).$$

Similarly,

$$\left| \frac{g'(t)}{g(t)} \right| > \frac{1}{t \log(t) \log_2^2(t)} \gg \left| \frac{f'(t)}{f(t)} \right| \frac{1}{\log_2^2(t)}.$$

□

**Proposition 8** *Let  $m \in \mathbb{N}$ , let  $\mathcal{H}$  be a Hardy field, let  $f, g \in \mathcal{H}$ , and let  $F : [1, \infty) \rightarrow (0, \infty)$  be an increasing function satisfying  $1 < F(t) < \log^m(t)$ . If  $f$  and  $g$  satisfy condition (B) and  $\frac{g(t)}{f(t)} > F(t)$ , then*

$$\left| \frac{g^{(\ell)}(t)}{f^{(\ell)}(t)} \right| > \frac{F(t)}{\log_2^2(t)}, \quad \forall \ell \in \mathbb{N}.$$

*Proof* Let  $\ell \in \mathbb{N}$  be arbitrary. We distinguish between the following two cases. The first case is  $g(t) > f(t) \log^{m+2}(t)$  and the second case is  $g(t) \ll f(t) \log^{m+2}(t)$ .

We start with the proof of the first case. Using Proposition 3 we obtain the estimate

$$\left| \frac{g^{(\ell)}(t)}{f^{(\ell)}(t)} \right| > \frac{g(t)}{f(t) \log^2(t)},$$

and, since  $g(t) > f(t) \log^{m+2}(t)$ , we get

$$\frac{g(t)}{f(t) \log^2(t)} > \log^m(t) > F(t).$$

Therefore  $\left| \frac{g^{(\ell)}(t)}{f^{(\ell)}(t)} \right| > F(t)$ , which concludes the proof of case one.

Next, we deal with the second case. Consider the product

$$\frac{f(t)}{g(t)} \left| \frac{g^{(\ell)}(t)}{f^{(\ell)}(t)} \right| = \prod_{i=0}^{\ell-1} \left| \frac{g^{(i+1)}(t)}{g^{(i)}(t)} \frac{f^{(i+1)}(t)}{f^{(i)}(t)} \right|.$$

In virtue of Lemma 5,  $f^{(i)}$  satisfies either  $f^{(i)} > 1$  or  $f^{(i)} < 1$ . The same is true for  $g^{(i)}$ . Also, it follows from Proposition 3 that for at most one  $i$  between 1 and  $\ell$  the function  $f^{(i)}$  satisfies  $|\log(|f^{(i)}(t)|)| \ll \log_2(t)$ ; for all other  $i$  between 1 and

$\ell$  the function  $f^{(i)}$  must satisfy  $|\log(|f^{(i)}(t)|)| > \log_2(t)$ . We can therefore apply Lemmas 4 and 7 to deduce that for at most one  $i$  between 1 and  $\ell$  we have

$$\frac{\left| \frac{g^{(i+1)}(t)}{g^{(i)}(t)} \right|}{\left| \frac{f^{(i+1)}(t)}{f^{(i)}(t)} \right|} > \log_2^2(t)$$

and for all other  $i$  we have

$$\frac{\left| \frac{g^{(i+1)}(t)}{g^{(i)}(t)} \right|}{\left| \frac{f^{(i+1)}(t)}{f^{(i)}(t)} \right|} \xrightarrow{t \rightarrow \infty} 1.$$

Therefore

$$\frac{f(t)}{g(t)} \left| \frac{g^{(\ell)}(t)}{f^{(\ell)}(t)} \right| = \prod_{i=0}^{\ell-1} \frac{\left| \frac{g^{(i+1)}(t)}{g^{(i)}(t)} \right|}{\left| \frac{f^{(i+1)}(t)}{f^{(i)}(t)} \right|} > \log_2^2(t).$$

This, together with  $\frac{g(t)}{f(t)} > F(t)$ , implies

$$\left| \frac{g^{(\ell)}(t)}{f^{(\ell)}(t)} \right| > \frac{F(t)}{\log_2^2(t)}.$$

□

### 3 van der Corput's Method for Estimating Exponential Sums

We recall three classical theorems on estimating exponential sums. For proofs and more detailed discussion we refer the reader to Sect. 2 in the book of Graham and Kolesnik [16].

We start with the Kusmin–Landau inequality for exponential sums (cf. [16, Theorem 2.1]).

**Theorem 9 (Estimate Based on 1st Derivative)** *Suppose  $I \subset \mathbb{R}$  is an interval,  $f \in C^1(I)$  and there exists  $\lambda > 0$  such that  $\lambda < |f'(t)| < (1 - \lambda)$  for all  $t \in I$ . Then*

$$\left| \sum_{n \in I} e(f(n)) \right| \ll \lambda^{-1}.$$

The next two theorems are due to van der Corput [26, 27].



**Theorem 10 (Estimate Based on 2nd Derivative)** *Suppose  $I \subset \mathbb{R}$  is an interval,  $f \in C^2(I)$  and there are  $\lambda > 0$  and  $\eta \geq 1$  such that  $\lambda < |f''(t)| \leq \eta\lambda$  for all  $t \in I$ . Then*

$$\left| \sum_{n \in I} e(f(n)) \right| \ll |I| \eta \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}.$$

**Theorem 11 (Estimate Based on 3rd and Higher Derivatives)** *Suppose  $I \subset \mathbb{R}$  is an interval,  $\ell \geq 3$ ,  $f \in C^\ell(I)$  and there are  $\lambda > 0$  and  $\eta \geq 1$  such that  $\lambda < |f^{(\ell)}(t)| \leq \eta\lambda$  for all  $t \in I$ . Let  $Q := 2^{\ell-2}$ . Then*

$$\left| \sum_{n \in I} e(f(n)) \right| \ll |I| (\eta^2 \lambda)^{\frac{1}{4Q-2}} + |I|^{1-\frac{1}{2Q}} \eta^{\frac{1}{2Q}} + |I|^{1-\frac{2}{Q} + \frac{1}{Q^2}} \lambda^{-\frac{1}{2Q}}.$$

### 4 Deriving Estimates for Exponential Sums Involving Functions from Hardy Fields

**Proposition 12** *Let  $\mathcal{H}$  be a Hardy field and assume  $f_1, \dots, f_k : [1, \infty) \rightarrow \mathbb{R}$  are in  $\mathcal{H}$ . For  $t \in [1, \infty)$  define  $f(t) := (f_1(t), \dots, f_k(t))$  and  $E(t) := \min\{|f_1(t)|, \dots, |f_k(t)|\}$ . Suppose we have*

- (i) *for all  $i \in \{1, \dots, k\}$  the function  $f_i$  satisfies condition (B);*
- (ii)  *$\log_2(t) \prec \log(f_i(t))$  for all  $i = 1, \dots, k$ ;*
- (iii) *after reordering  $f_1, \dots, f_k$  if necessary, we have  $\frac{f_{i+1}}{f_i} \succ \log_2^4(t)$  for all  $i = 1, \dots, k-1$ ;*

*Then there exists a constant  $C > 0$  such that for all  $M \in \mathbb{N}$ , all  $r, s \in [1, \min\{M \log^2(M), \frac{E(M)}{\log^4(M)}\}]$  with  $r \geq s$  and all  $\tau \in [-\log_2^2(M), \log_2^2(M)]^k \cap \mathbb{Z}^k$  with  $\tau \neq (0, \dots, 0)$  we have*

$$\left| \sum_{n=M}^{2M} e\left(\frac{1}{r} \langle f(sn), \tau \rangle\right) \right| \leq \frac{CM}{\log^2(M)}.$$

*Proof* Let  $\epsilon > 0$ ,  $r, s \in [1, \min\{M \log^2(M), \frac{E(M)}{\log^4(M)}\}]$  with  $r \geq s$  and  $\tau \in [-\log_2^2(M), \log_2^2(M)]^k \cap \mathbb{Z}^k$  with  $\tau = (\tau_1, \dots, \tau_k) \neq (0, \dots, 0)$  be arbitrary. Let  $b(t) := \frac{1}{r} \langle f(st), \tau \rangle$ . Our goal is to estimate

$$\left| \sum_{n=M}^{2M} e(b(t)) \right|$$

by using van der Corput's method of estimating exponential sums. We therefore have to find convenient estimates for the derivatives of  $b(t)$  on the interval  $[M, 2M]$ .

Let us pick  $i_0 \in \{1, \dots, k\}$  such that  $\tau_{i_0} \neq 0$  and  $\tau_i = 0$  for all  $i > i_0$ . Define  $E_0(t) := |\tau_{i_0} f_{i_0}(t)|$ . Using condition (iii) we deduce that  $\tau_{i_0} f_{i_0}(t)$  is the dominating term in the sum  $\tau_1 f_1(t) + \dots + \tau_{i_0} f_{i_0}(t)$  and therefore

$$E_0(t) \ll |\langle f(t), \tau \rangle| \ll E_0(t), \quad (2)$$

where the implied constants depend neither on  $t$  nor on the value of  $\tau_1, \dots, \tau_{i_0}$ . A similar argument also applies to the derivatives of  $\langle f(t), \tau \rangle$ . Indeed, it follows from condition (iii) and Proposition 8 (with  $F(t) = \log^2(t)$ ) that  $\tau_{i_0} f_{i_0}^{(\ell)}(t)$  is the dominating term in  $\tau_1 f_1^{(\ell)}(t) + \dots + \tau_{i_0} f_{i_0}^{(\ell)}(t)$  and therefore

$$|\tau_{i_0} f_{i_0}^{(\ell)}(t)| \ll |\langle f^{(\ell)}(t), \tau \rangle| \ll |\tau_{i_0} f_{i_0}^{(\ell)}(t)|. \quad (3)$$

Next let  $u := \inf\{c \in [0, \infty) : f_{i_0}(t) \prec t^c\}$  and pick  $d \in \mathbb{R}$  such that  $s = (M \log^2(M))^d$  and  $h \in \mathbb{R}$  such that  $r = (M \log^2(M))^h$ . From the conditions on  $r$  and  $s$  we deduce that  $d \in [0, \min\{1, u\}]$  and  $h \in [d, \min\{1, u\}]$ . We now define

$$\ell := \left\lceil u + du - h + \frac{1}{2} \right\rceil$$

and set  $x := \ell - u - ud + h$ . Note that  $x \in [\frac{1}{2}, \frac{3}{2})$ .

In view of Proposition 3 we have

$$\frac{f_{i_0}(t)}{t^\ell \log^2(t)} \ll |f_{i_0}^{(\ell)}(t)| \ll \frac{f_{i_0}(t)}{t^\ell}. \quad (4)$$

By combining Eqs. (2)–(4) we obtain

$$\frac{E_0(t)}{t^\ell \log^2(t)} \ll |\langle f^{(\ell)}(t), \tau \rangle| \ll \frac{E_0(t)}{t^\ell}.$$

Hence the minimum of the function  $b^{(\ell)}(t)$  on the interval  $[M, 2M]$  is at least

$$\gg \frac{E_0(2sM)}{r(2M)^\ell \log^2(2sM)}$$

whereas the maximum is at most

$$\ll \frac{E_0(sM)}{rM^\ell}.$$

Since  $E_0(t)$  is eventually increasing, we have  $E_0(2sM) \gg E_0(sM)$ . Also, since  $E_0(t)$  has polynomial growth and  $s \leq E_0(M)$  we can estimate  $\log(2sM) \ll \log(M)$ .

Therefore

$$\frac{E_0(2sM)}{r(2M)^\ell \log^2(2sM)} \gg \frac{E_0(sM)}{rM^\ell \log^2(M)}.$$

If we choose

$$\lambda := \frac{E_0(sM)}{rM^\ell \log^2(M)} \quad \text{and} \quad \eta := \log^2(M)$$

then it follows that

$$\lambda \ll b^{(\ell)}(t) \ll \eta\lambda, \quad \forall t \in [M, 2M]. \tag{5}$$

We now distinguish between the cases  $\ell = 1$ ,  $\ell = 2$ , and  $\ell \geq 3$ .

**The Case  $\ell = 1$**  The case  $\ell = 1$  only occurs if

$$f_{i_0}(t) \prec t^{\frac{1}{2}}.$$

Therefore

$$b'(t) \ll \eta\lambda = \frac{E_0(sM)}{rM} \leq \frac{|\tau_{i_0} f_{i_0}(sM)|}{rM} \leq \frac{\log_2^2(M) f_{i_0}(sM)}{sM} \prec 1.$$

This means we can apply Theorem 9 and obtain

$$\left| \sum_{n=M}^{2M} e(b(n)) \right| \ll \lambda^{-1}.$$

For  $\lambda^{-1}$  we have

$$\lambda^{-1} = \frac{rM \log^2(M)}{E_0(sM)} \leq \frac{rM \log^2(M)}{E_0(M)} \leq \frac{E(M)M}{E_0(M) \log^2(M)}.$$

Finally, since  $\frac{E(M)}{E_0(M)} \leq 1$  we have

$$\lambda^{-1} \ll \frac{M}{\log^2(M)}.$$

**The Case  $\ell = 2$**  If  $\ell = 2$ , then invoking Theorem 10 yields the estimate

$$\left| \sum_{n=M}^{2M} e(b(n)) \right| \ll M\eta\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}. \tag{6}$$

Using  $(sM)^{u-\beta} < f_{i_0}(sM) < (sM)^{u+\beta}$  for all  $\beta > 0$  we can bound  $\lambda$  from above and below,

$$\frac{|\tau_{i_0}|s^{u-\beta}}{rM^{2-u+\beta}\log^2(M)} \ll \lambda \ll \frac{|\tau_{i_0}|s^{u+\beta}}{rM^{2-u-\beta}\log^2(M)}, \quad (7)$$

taking into account that the implied constants in the above equation depend on our choice of  $\beta$ .

Furthermore, since  $x = \ell - u - ud + h$ ,  $s = (M \log^2(M))^d$  and  $r = (M \log^2(M))^h$  we obtain from (7) that

$$\frac{1}{M^{x+2\beta}\log^q(M)} \ll \lambda \ll \frac{\log^q(M)}{M^{x-2\beta}}. \quad (8)$$

for some sufficiently large constant  $q > 1$ . We can use (8) to further estimate (6) and obtain

$$M\eta\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} \ll \frac{\log^{2+\frac{q}{2}}(M)M}{M^{\frac{x-2\beta}{2}}} + \frac{M^{\frac{x+2\beta}{2}}}{\log^{\frac{q}{2}}(M)}.$$

Finally, by choosing  $\beta$  sufficiently small and taking into account that  $x \in [\frac{1}{2}, \frac{3}{2}]$ , we have

$$\frac{\log^{2+\frac{q}{2}}(M)M}{M^{\frac{x-2\beta}{2}}} + \frac{M^{\frac{x+2\beta}{2}}}{\log^{\frac{q}{2}}(M)} \ll \frac{M}{\log^2(M)}.$$

**The Case  $\ell \geq 3$**  The case  $\ell \geq 3$  can be dealt with analogously to the case  $\ell = 2$ , only one must use Theorem 11 instead of Theorem 10. With  $Q = 2^{\ell-2}$ , we have

$$\begin{aligned} \left| \sum_{n=M}^{2M} e(b(n)) \right| &\ll M(\eta^2\lambda)^{\frac{1}{4Q-2}} + M^{1-\frac{1}{2Q}}\eta^{\frac{1}{2Q}} + M^{1-\frac{2}{Q}+\frac{1}{Q^2}}\lambda^{-\frac{1}{2Q}}, \\ &\ll \frac{M}{\log^2(M)}, \end{aligned}$$

which finishes the proof.  $\square$

**Theorem 13** *Let  $\mathcal{H}$  be a Hardy field and assume  $f_1, \dots, f_k : [1, \infty) \rightarrow \mathbb{R}$  are in  $\mathcal{H}$ . For  $t \in [1, \infty)$  define  $f(t) := (f_1(t), \dots, f_k(t))$  and  $E(t) := \min\{|f_1(t)|, \dots, |f_k(t)|\}$ . Suppose we have*

- (i) for all  $i \in \{1, \dots, k\}$  the function  $f_i$  satisfies condition (B);
- (ii)  $\log_2(t) < \log(f_i(t))$  for all  $i = 1, \dots, k$ ;
- (iii) after reordering  $f_1, \dots, f_k$  if necessary, we have  $\frac{f_{i+1}}{f_i} > \log_2^4(t)$  for all  $i = 1, \dots, k-1$ ;

Then there exists a constant  $C > 0$  such that for all  $N \in \mathbb{N}$ , all  $r, s \in \left[1, \min \left\{ N, \frac{E(N)}{\log^5(N)} \right\} \right]$  with  $r \geq s$  and all  $\tau \in [-\log_2^2(M), \log_2^2(M)]^k \cap \mathbb{Z}^k$  with  $\tau \neq (0, \dots, 0)$  we have

$$\left| \sum_{n=1}^N e \left( \frac{1}{r} \langle f(sn), \tau \rangle \right) \right| \leq \frac{CN}{\log(N)}.$$

*Proof* First we note that

$$\left| \sum_{n=1}^N e \left( \frac{1}{r} \langle f(sn), \tau \rangle \right) \right| \leq \frac{N}{\log(N)} + \left| \sum_{\substack{N \\ \log(N)} \leq n \leq N} e \left( \frac{1}{r} \langle f(sn), \tau \rangle \right) \right|,$$

so it suffices to estimate the expression

$$\left| \sum_{\substack{N \\ \log(N)} \leq n \leq N} e \left( \frac{1}{r} \langle f(sn), \tau \rangle \right) \right|.$$

Dissect the interval  $\left[ \frac{N}{\log(N)}, N \right]$  into  $\log_2(N)$ -many intervals of the form  $[M, 2M]$ .

If  $M \in \left[ \frac{N}{\log(N)}, N \right]$ , then  $N < M \log^2(M)$  and  $\frac{N}{\log^3(N)} < \frac{M}{\log^4(M)}$  and therefore

$$\left[ 1, \min \left\{ N, \frac{N}{\log^5(N)} \right\} \right] \subset \left[ 1, \min \left\{ M \log^2(M), \frac{M}{\log^4(M)} \right\} \right].$$

Hence applying Proposition 12 to each of the  $\log_2(N)$ -many intervals of the form  $[M, 2M]$  we get

$$\begin{aligned} \left| \sum_{\substack{N \\ \log(N)} \leq n \leq N} e \left( \frac{1}{r} \langle f(sn), \tau \rangle \right) \right| &\ll \log_2(N) \frac{M}{\log^2(M)} \\ &\ll \frac{N}{\log(N)}. \end{aligned}$$

This finishes the proof. □

## 5 Discrepancy Estimates

The following higher dimensional version of the classical Erdős-Turán inequality was discovered by Szűsz [25] and independently by Koksma [19].

**Theorem 14** (See [8, 15, 25] or [11, Theorem 1.21]) *Let  $k \geq 1$ , let  $\vartheta_n \in [0, 1)^k$ ,  $n \in \mathbb{N}$ , let  $N \in \mathbb{N}$ , and let  $a_1, \dots, a_k, b_1, \dots, b_k \in [0, 1)$  with  $0 \leq a_i < b_i < 1$ . Then*

$$\frac{|\{n \leq N : \vartheta_n \in [a_1, b_1] \times \dots \times [a_k, b_k]\}|}{N} = \prod_{i=1}^k (b_i - a_i) + R_{N,k}$$

and where for all  $H \geq 1$ ,

$$|R_{N,k}| \leq C_k \left( \frac{1}{H+1} + \sum_{\substack{\tau \in [-H, H]^k \\ \tau \neq (0, \dots, 0)}} \left( \prod_{i=1}^k \frac{1}{1 + |\tau_i|} \right) \left| \frac{1}{N} \sum_{n=1}^N e((\vartheta_n, \tau)) \right| \right).$$

Here,  $C_k$  is a constant which depends only on  $k$ .

**Theorem 15** *Suppose  $f_1, \dots, f_k : [1, \infty) \rightarrow \mathbb{R}$  and  $E : [0, \infty) \rightarrow (0, \infty)$  are as in the statement of Theorem 13. Then there exists a constant  $C > 0$  such that for all  $N \in \mathbb{N}$  and all  $d \in \left[1, \min \left\{N, \frac{E(N)}{\log^2(N)}\right\}\right]$ ,*

$$\left| \left| \{n \leq N : d \mid \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor)\} \right| - \frac{N}{d^{k+1}} \right| \leq \frac{CN}{d \log^2(N)}.$$

*Proof* Define  $\vartheta_{d,n} := \left( \left\{ \frac{f_1(dn)}{d} \right\}, \dots, \left\{ \frac{f_k(dn)}{d} \right\} \right)$ , where  $\{x\}$  denotes the fractional part of a real number  $x$ . We first observe that

$$\begin{aligned} & |\{1 \leq n \leq N : d \mid \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor)\}| \\ &= \sum_{n=1}^N 1_{d\mathbb{Z}}(n) 1_{d\mathbb{Z}}(\lfloor f_1(n) \rfloor) \cdots 1_{d\mathbb{Z}}(\lfloor f_k(n) \rfloor) \\ &= \sum_{n \leq N/d} 1_{d\mathbb{Z}}(\lfloor f_1(dn) \rfloor) \cdots 1_{d\mathbb{Z}}(\lfloor f_k(dn) \rfloor) \\ &= \left| \left\{ 1 \leq n \leq \frac{N}{d} : \vartheta_{d,n} \in \left[0, \frac{1}{d}\right)^k \right\} \right|. \end{aligned}$$

From Theorem 13 we get that

$$\left| \sum_{n=1}^N e\left(\frac{1}{d}\langle f(dn), \tau \rangle\right) \right| \leq \frac{CN}{\log(N)},$$

for all  $d \in \left[1, \frac{E(N)}{\log^5(N)}\right]$  and  $\tau \in [-\log_2^2(N), \log_2^2(N)]^k \cap \mathbb{Z}^k$  with  $\tau \neq (0, \dots, 0)$ .

We now apply Theorem 14 with  $H = \log_2^2(N)$  and obtain

$$\begin{aligned} & \left| \left\{ 1 \leq n \leq \frac{N}{d} : \vartheta_{d,n} \in \left[0, \frac{1}{d}\right)^k \right\} - \frac{N}{d^{k+1}} \right| \\ & \leq \frac{C_k}{d} \left( \frac{N}{\log_2^2(N)} + \sum_{\substack{\tau \in [-\log_2^2(N), \log_2^2(N)]^k \\ \tau \neq (0, \dots, 0)}} \left( \prod_{i=1}^k \frac{1}{1 + |\tau_i|} \right) \left| \sum_{n=1}^N e(\langle \vartheta_n, \tau \rangle) \right| \right) \\ & \leq \frac{C_k}{d} \left( \frac{N}{\log_2^2(N)} + \frac{CN}{\log(N)} \sum_{\substack{\tau \in [-\log_2^2(N), \log_2^2(N)]^k \\ \tau \neq (0, \dots, 0)}} \left( \prod_{i=1}^k \frac{1}{1 + |\tau_i|} \right) \right) \\ & \ll \frac{N}{d \log_2^2(N)}. \end{aligned}$$

□

## 6 Proving Theorem 2

**Proposition 16** *Let  $k \in \mathbb{N}$ . Let  $\xi_1, \xi_2, \dots$  be a sequence of positive integers and let  $E : [1, \infty) \rightarrow (0, \infty)$  be a function that satisfies  $E(N) \geq \max\{\xi_n : 1 \leq n \leq N\}$  and  $\log_2(t) \prec \log(E(t))$  and assume that  $E(N)$  has polynomial growth (i.e., there exists  $j \in \mathbb{N}$  such that  $E(t) \prec t^j$ ). If*

$$\left| \{n \leq N : d \mid \xi_n\} - \frac{N}{d^{k+1}} \right| \ll \frac{N}{d \log_2^2(N)}, \quad \forall d \in \mathbb{N} \cap \left[1, \frac{E(N)}{\log^5(N)}\right] \tag{9}$$

and

$$|\{n \leq N : p \mid \xi_n\}| \ll \frac{N}{p}, \text{ for all primes } p \in \left(\frac{E(N)}{\log^5(N)}, E(N)\right], \tag{10}$$

then the natural density of  $\{n \in \mathbb{N} : \xi_n = 1\}$  exists and equals  $\frac{1}{\zeta(k+1)}$ .

Our proof of Proposition 16 is similar to the proof of the main result in [9].

*Proof* Define  $G(N) := \log^4(t)$ . Let  $D(N)$  be a slow growing function in  $N$  and let  $\Pi$  denote the *primorial* of  $D(N)$ , that is,

$$\Pi := \prod_{\substack{p \text{ prime,} \\ p \leq D(N)}} p.$$

Here, by “slow growing function” we mean that  $D(N)$  converges to  $\infty$  as  $N \rightarrow \infty$ , but slowly enough so that the inequality  $\Pi \leq \min \left\{ \frac{E(N)}{\log^5(N)}, \log_2^2(N) \right\}$  is satisfied for all  $N \geq 1$ .

Let  $\mu(n)$  denote the classical Möbius function: For  $n \in \mathbb{N}$  define

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^j & \text{if } n = p_1 \cdot p_2 \cdot \dots \cdot p_j \text{ where } p_1, \dots, p_j \text{ are distinct primes;} \\ 0 & \text{otherwise.} \end{cases}$$

Using the identity

$$\sum_{d|a} \mu(d) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

we obtain

$$|\{n \leq N : \xi_n = 1\}| = \sum_d \mu(d) |\{n \leq N : d \mid \xi_n\}|.$$

It will be convenient to decompose the right-hand side of the above equation into two sums  $\Sigma_1 + \Sigma_2$  where

$$\Sigma_1 := \sum_{d|\Pi} \mu(d) |\{n \leq N : d \mid \xi_n\}|$$

and

$$\Sigma_2 := \sum_{d \nmid \Pi} \mu(d) |\{n \leq N : d \mid \xi_n\}|.$$

Our goal is to show that  $\lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_1 = \frac{1}{\zeta(k+1)}$  and that  $\lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_2 = 0$ ; this will finish the proof.

First, let us show  $\lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_1 = \frac{1}{\zeta(k+1)}$ . Recall that  $\Pi \leq \frac{E(N)}{\log^5(N)}$ . Invoking condition (9) it thus follows that

$$\left| \frac{1}{N} \Sigma_1 - \sum_{d|\Pi} \frac{\mu(d)}{d^{k+1}} \right| \ll \frac{1}{\log_2^2(N)} \sum_{d|\Pi} \frac{1}{d}.$$



Since  $\Pi \leq \log_2^2(N)$  it follows that  $\sum_{d|\Pi} \frac{1}{d} = \mathcal{O}(\log_3(N))$  and hence

$$\frac{1}{\log_2^2(N)} \sum_{d|\Pi} \frac{1}{d} \ll \frac{\log_3(N)}{\log_2^2(N)} \xrightarrow{N \rightarrow \infty} 0.$$

Therefore

$$\lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_1 = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{k+1}} = \frac{1}{\zeta(k+1)}.$$

For  $\Sigma_2$  we obtain the estimate

$$\begin{aligned} |\Sigma_2| &= \left| \sum_{\substack{p \text{ prime,} \\ p > D(N)}} \sum_{\substack{1 \leq d \leq E(N), \\ p|d}} \mu(d) |\{n \leq N : d \mid \xi_n\}| \right| \\ &= \left| \sum_{n=1}^N \sum_{\substack{p \text{ prime,} \\ p > D(N)}} \sum_{\substack{d|\xi_n, \\ p|d}} \mu(d) \right| \\ &\leq \sum_{n=1}^N \sum_{\substack{p \text{ prime,} \\ p > D(N)}} \left| \sum_{\substack{d|\xi_n, \\ p|d}} \mu(d) \right|. \end{aligned}$$

It is well known (and easy to show) that

$$\sum_{\substack{d|a, \\ p|d}} \mu(d) = \begin{cases} 1 & \text{if } a = p^j \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$\begin{aligned} \sum_{n=1}^N \sum_{\substack{p \text{ prime,} \\ p > D(N)}} \left| \sum_{\substack{d|\xi_n, \\ p|d}} \mu(d) \right| &\leq \sum_{n=1}^N \sum_{\substack{p \text{ prime,} \\ p > D(N), \\ p|\xi_n}} 1 \\ &\leq \sum_{\substack{p \text{ prime,} \\ p > D(N)}} |\{n \leq N : p \mid \xi_n\}|. \end{aligned}$$

Putting everything together we obtain

$$|\Sigma_2| \leq \sum_{\substack{p \text{ prime,} \\ p > D(N)}} |\{n \leq N : p \mid \xi_n\}|.$$

Again, we split the right-hand side of the above equation into two more manageable sums  $\Sigma_{2,1} + \Sigma_{2,2}$ , where

$$\Sigma_{2,1} := \sum_{\substack{p \text{ prime,} \\ D(N) < p \leq \frac{E(N)}{\log^3(N)}} |\{n \leq N : p \mid \xi_n\}|$$

and

$$\Sigma_{2,2} := \sum_{\substack{p \text{ prime,} \\ \frac{E(N)}{\log^3(N)} < p \leq E(N)}} |\{n \leq N : p \mid \xi_n\}|.$$

Using condition (9) for the sum  $\Sigma_{2,1}$  and using condition (10) for the sum  $\Sigma_{2,2}$  we obtain the estimates

$$\frac{1}{N} \Sigma_{2,1} \ll \sum_{\substack{p \text{ prime,} \\ D(N) < p \leq \frac{E(N)}{\log^3(N)}} \frac{1}{p^2} + \frac{1}{\log_2^2(N)} \sum_{p \leq E(N)} \frac{1}{p}$$

and

$$\frac{1}{N} \Sigma_{2,2} \leq \sum_{\substack{p \text{ prime,} \\ \frac{E(N)}{\log^3(N)} < p \leq E(N)}} \frac{1}{p}.$$

Using

$$\lim_{N \rightarrow \infty} \left| \sum_{\substack{p \text{ prime,} \\ p \leq N}} \frac{1}{p} - \log_2 N \right| = M,$$

(where  $M$  is the Meissel–Mertens constant), we can estimate

$$\frac{1}{\log_2^2(N)} \sum_{p \leq E(N)} \frac{1}{p} \ll \frac{\log_2(E(N))}{\log_2^2(N)} \ll \frac{1}{\log_2(N)},$$

and

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_{2,2} &\ll \log(\log(E(N))) - \log\left(\log\left(\frac{E(N)}{\log^5(N)}\right)\right) \\ &\ll -\log\left(1 - \frac{\log(\log^5(N))}{\log(E(N))}\right) \\ &\ll \frac{\log_2(N)}{\log(E(N))}. \end{aligned}$$

Therefore

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\Sigma_2| \leq \lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_{2,1} + \lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_{2,2} = 0.$$

□

**Corollary 17** *Let  $\mathcal{H}$  be a Hardy field and suppose  $f_1, \dots, f_k \in \mathcal{H}$  satisfy conditions (A) and (B) and  $\log_2(t) < \log(f_i(t))$  for all  $i = 1, \dots, k$ . Also, assume that  $\frac{f_{i+1}}{f_i} > \log_2^4(t)$  for all  $i = 1, \dots, k - 1$ . Then the natural density of the set*

$$\{n \in \mathbb{N} : \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor) = 1\}$$

*exists and equals  $\frac{1}{\zeta(k+1)}$ .*

*Proof* We define

$$\xi_n := \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor)$$

and

$$E(t) := \min\{|f_1(t)|, \dots, |f_k(t)|\}.$$

Trivially,  $(\xi_n)_{n \in \mathbb{N}}$  satisfies condition (10). Moreover, it follows from Theorem 15 that  $(\xi_n)_{n \in \mathbb{N}}$  satisfies condition (9). Therefore, in view of Proposition 16, we have  $d(\{n \in \mathbb{N} : \xi_n = 1\}) = \frac{1}{\zeta(k+1)}$ . □

**Lemma 18** *Let  $\mathcal{H}$  be a Hardy field and suppose  $f \in \mathcal{H}$  satisfies*

$$\log(t) \log_4(t) < f(t) < \frac{t}{\log_2(t)}.$$

*Define  $S(N, d) := |\{1 \leq n \leq N : d \mid \gcd(n, \lfloor f(n) \rfloor)\}|$ . Then*

$$\lim_{D \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} S(N, p) = 0. \tag{11}$$

We remark that Lemma 18 can be derived from the proof of Theorem 2 in [12]. For the convenience of the reader we include a separate proof here, where we follow the arguments used by Erdős and Lorentz in [12].

*Proof* Let  $N \in \mathbb{N}$  be arbitrary. Define  $I_m := \{1 \leq t \leq N : f(t) \in [m, m + 1)\}$  and note that  $I_m$  is a finite subinterval of  $\mathbb{R}$ . Let  $k_m$  denote the length of  $I_m$ . Note that the contribution of  $I_m \cap \mathbb{N}$  to the size of  $S(N, p)$  is given by  $I_m \cap (p\mathbb{N})$ . Hence this contribution is zero if  $p \nmid m$  and it does not exceed  $\frac{k_m}{p} + 1$  otherwise. It follows that

$$\begin{aligned} \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} S(N, p) &\leq \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \sum_{\substack{1 \leq m \leq f(N) \\ p \mid m}} \frac{k_m}{p} + 1 \\ &\leq \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \sum_{1 \leq m \leq \frac{f(N)}{p}} \frac{k_{pm}}{p} + 1 \\ &\leq \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \sum_{1 \leq m \leq \frac{f(N)}{p}} \frac{k_{pm}}{p} + \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \frac{f(N)}{p}. \end{aligned}$$

Define  $\ell_0 := \lfloor \frac{f(N)}{p} \rfloor$ . We can write

$$\sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \sum_{1 \leq m \leq \frac{f(N)}{p}} \frac{k_{pm}}{p} + \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \frac{f(N)}{p} = \Sigma_1 + \Sigma_2 + \Sigma_3$$

where

$$\begin{aligned} \Sigma_1 &:= \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \frac{f(N)}{p}, \\ \Sigma_2 &:= \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \sum_{1 \leq m \leq \ell_0 - 1} \frac{k_{pm}}{p}, \\ \Sigma_3 &:= \sum_{\substack{p \text{ prime} \\ D < p \leq f(N)}} \frac{k_{p\ell_0}}{p}. \end{aligned}$$

We have to estimate each of the sums  $\Sigma_1$ ,  $\Sigma_2$ , and  $\Sigma_3$  individually.

We start with  $\Sigma_1$ . Since  $f(t) < \frac{t}{\log_2(t)}$ , we conclude that

$$\Sigma_1 \leq f(N) \sum_{p \leq N} \frac{1}{p} \ll f(N) \log_2(N) < N$$

and therefore

$$\lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_1 = 0. \tag{12}$$

Next, we derive an estimate for  $\Sigma_2$ . Since  $f(t)$  is eventually strictly increasing, the inverse function of  $f^{-1}$  is well defined on some half-line  $[t_0, \infty)$ . We have the identity  $k_m = f^{-1}(m + 1) - f^{-1}(m)$ . This means that using the inverse function theorem and the mean value theorem we see that there exists a number  $\xi_m \in I_m$  such that

$$k_m = \frac{1}{f'(\xi_m)}.$$

In view of Proposition 3, the derivative of  $f$  is eventually decreasing and therefore

$$k_m \ll k_l$$

for  $l \geq m$ . From this we derive that

$$\sum_{1 \leq m \leq \ell_0 - 1} \frac{k_{pm}}{p} \ll \frac{1}{p} \left( \frac{k_p}{p} + \frac{k_{p+1}}{p} + \frac{k_{p+2}}{p} + \dots + \frac{k_{\ell_0 p - 1}}{p} \right) \ll \frac{N}{p^2}.$$

Hence

$$\Sigma_2 \ll \sum_{p > D} \frac{N}{p^2}$$

which proves that

$$\lim_{D \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_2 = 0. \tag{13}$$

Next, let  $\epsilon > 0$  be arbitrary and define  $P' := \{p \text{ prime} : D < p < f(N) + 1 - \frac{\log_2(f(N))}{\epsilon}\}$  and  $P'' := \{p \text{ prime} : f(N) + 1 - \frac{\log_2(f(N))}{\epsilon} \leq p \leq f(N)\}$ . We now split the sum  $\Sigma_3$  into two more sums  $\Sigma'_3$  and  $\Sigma''_3$ , where

$$\Sigma'_3 := \sum_{p \in P'} \frac{k_p \ell_0}{p}$$

and

$$\Sigma''_3 := \sum_{p \in P''} \frac{k_p \ell_0}{p}.$$

Arguing as before we have for all  $p \in P'$ ,

$$\begin{aligned} k_{p\ell_0} &\ll \frac{1}{f(N) - p\ell_0 + 1} (k_{p\ell_0} + k_{p\ell_0+1} + \cdots + k_{f(N)}) \\ &\ll \frac{N}{f(N) - p\ell_0 + 1} \\ &= \frac{N\epsilon}{\log_2(f(N))}. \end{aligned}$$

This yields the following estimate for  $\Sigma'_3$ :

$$\Sigma'_3 \ll \sum_{p \leq f(N)} \frac{N\epsilon}{\log_2(f(N))p} \ll N\epsilon.$$

Let  $\Pi := \prod_{p \in P''} p$ . Certainly,

$$\Pi \leq (f(N))^{\frac{\log_2(N)}{\epsilon}}.$$

Using the well-known estimate

$$\sum_{p|n} \frac{1}{p} \ll \log_3(n)$$

we obtain

$$\Sigma''_3 \leq k_{f(N)} \sum_{p|\Pi} \frac{1}{p} \ll \frac{1}{f'(N)} \log_3(\Pi) \ll \frac{1}{f'(N)} \log_3(f(N)).$$

Using L'Hospital's rule, we see that

$$\frac{tf'(t)}{\log_3(f(t))} \gg \frac{f(t)}{\log(t) \log_3(f(t))} \gg \frac{f(t)}{\log(t) \log_3(\log^k(t))} \gg \frac{f(t)}{\log(t) \log_4(t)},$$

and therefore, using  $\log(t) \log_4(t) < f(t)$ , we get

$$\frac{1}{f'(N)} \log_3(f(N)) < N.$$

Putting everything together yields

$$\frac{1}{N} \Sigma_3 = \frac{1}{N} \Sigma'_3 + \frac{1}{N} \Sigma''_3 \ll \epsilon.$$

Since  $\epsilon$  was chosen arbitrarily, we deduce that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \Sigma_3 = 0. \tag{14}$$

Finally, combining Eqs. (12)–(14) completes the proof.  $\square$

**Lemma 19** *Let  $\mathcal{H}$  be a Hardy field and suppose  $f_1, \dots, f_k \in \mathcal{H}$  satisfy condition (B). Also assume that  $f_1(t) > \log(t)$  and  $\frac{f_i+1}{f_i} > 1$  for all  $i = 1, \dots, k - 1$ . For  $D \in \mathbb{N}$  define*

$$A_{D,N} := \{1 \leq n \leq N : \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor, D!) = 1\}.$$

Then

$$\lim_{N \rightarrow \infty} \frac{|A_{D,N}|}{N} = \sum_{d|D!} \frac{\mu(d)}{d^{k+1}}.$$

For the proof of Lemma 19 we need the following Proposition, which is an immediate corollary of [5, Theorem 1.8].

**Proposition 20** *Let  $k \in \mathbb{N}$ , let  $\mathcal{H}$  be a Hardy field, and let  $g_1, \dots, g_k \in \mathcal{H}$  satisfy  $g_1(t) > \log(t)$  and  $\frac{g_i+1}{g_i} > 1$  for all  $i = 1, \dots, k - 1$ . Then the sequence  $(\{g_1(n)\}, \dots, \{g_k(n)\})$ ,  $n \in \mathbb{N}$ , is equidistributed in  $[0, 1)^k$ .*

*Proof of Lemma 19* Define  $\vartheta_{d,n} := \left( \left\{ \frac{f_1(dn)}{d} \right\}, \dots, \left\{ \frac{f_k(dn)}{d} \right\} \right)$ , where  $\{x\}$  denotes the fractional part of a real number  $x$ . We first observe that

$$\begin{aligned} |A_{D,N}| &= \sum_{d|D!} \mu(d) \left| \left\{ 1 \leq n \leq N : d \mid \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor) \right\} \right| \\ &= \sum_{d|D!} \sum_{n=1}^N 1_{d\mathbb{Z}}(n) 1_{d\mathbb{Z}}(\lfloor f_1(n) \rfloor) \cdots 1_{d\mathbb{Z}}(\lfloor f_k(n) \rfloor) \\ &= \sum_{d|D!} \sum_{n \leq N/d} 1_{d\mathbb{Z}}(\lfloor f_1(dn) \rfloor) \cdots 1_{d\mathbb{Z}}(\lfloor f_k(dn) \rfloor) \\ &= \sum_{d|D!} \left| \left\{ 1 \leq n \leq \frac{N}{d} : \vartheta_{d,n} \in \left[ 0, \frac{1}{d} \right)^k \right\} \right|. \end{aligned}$$

Applying Proposition 20 to the functions  $g_1(t) = \frac{f_1(dt)}{d}, \dots, g_k(t) = \frac{f_k(dt)}{d}$ , we deduce that

$$\lim_{N \rightarrow \infty} \left| \left\{ 1 \leq n \leq \frac{N}{d} : \vartheta_{d,n} \in \left[ 0, \frac{1}{d} \right)^k \right\} \right| = \frac{1}{d^{k+1}},$$

which proves the claim.  $\square$

**Theorem 21** *Let  $\mathcal{H}$  be a Hardy field and suppose  $f_1, \dots, f_k \in \mathcal{H}$  satisfy conditions (A) and (B). Also assume that  $f_1(t) < \frac{t}{\log_2(t)}$  and  $\frac{f_i+1}{f_i} > 1$  for all  $i = 1, \dots, k-1$ . Then the natural density of the set*

$$\{n \in \mathbb{N} : \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor) = 1\}$$

*exists and equals  $\frac{1}{\zeta(k+1)}$ .*

*Proof* Let  $D \in \mathbb{N}$ . We define

$$A_N := \{1 \leq n \leq N : \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor) = 1\}$$

and

$$A_{D,N} := \{1 \leq n \leq N : \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor, D!) = 1\}.$$

Certainly,  $A_N \subset A_{D,N}$ . It follows directly from Lemma 19 that

$$\lim_{D \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{|A_{D,N}|}{N} = \frac{1}{\zeta(k+1)}.$$

Hence, it suffices to show that

$$\lim_{D \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{|A_{D,N} \setminus A_N|}{N} = 0.$$

Let

$$S'(N, d) := |\{1 \leq n \leq N : d \mid \gcd(n, \lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor)\}|$$

and let

$$S(N, d) := |\{1 \leq n \leq N : d \mid \gcd(n, \lfloor f_1(n) \rfloor)\}|.$$

It is clear that

$$|A_{D,N} \setminus A_N| \leq \sum_{\substack{p \text{ prime} \\ D < p \leq \lfloor f_1(n) \rfloor}} S'(N, p).$$

However, we have that

$$\sum_{\substack{p \text{ prime} \\ D < p \leq \lfloor f_1(n) \rfloor}} S'(N, p) \leq \sum_{\substack{p \text{ prime} \\ D < p \leq \lfloor f_1(n) \rfloor}} S(N, p)$$



and it follows from Lemma 18 that

$$\lim_{D \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{p \text{ prime} \\ D < p \leq \lfloor f_1(n) \rfloor}} S(N, p) = 0.$$

□

*Proof of Theorem 2* Let  $f_1, \dots, f_k \in \mathcal{H}$  satisfy conditions (A)–(C).

If  $f_1(t) \prec \frac{t}{\log_2(t)}$ , then the conclusion of Theorem 2 follows from Theorem 21.

On the other hand, if  $f_1(t) \gg \frac{t}{\log_2(t)}$ , then the conclusion of Theorem 2 follows from Corollary 17. □

## 7 Some Open Questions

We end this paper with formulating some open questions.

**7.1** The first question concerns a natural extension of Watson’s [28] original result.

**Question 1** Let  $\alpha_1, \dots, \alpha_k$  be  $k$  irrational numbers. Is it true that the natural density of the set

$$\{n \in \mathbb{N} : \gcd(n, \lfloor n\alpha_1 \rfloor, \lfloor n^2\alpha_2 \rfloor, \dots, \lfloor n^k\alpha_k \rfloor) = 1\}$$

exists and equals  $\frac{1}{\xi(k+1)}$ ?

**7.2** Let  $\mathcal{H}$  be a Hardy field, let  $f_1, \dots, f_k \in \mathcal{H}$  and consider the condition:

$$(C') \quad \frac{f_{i+1}}{f_i} > 1 \text{ for all } i = 1, \dots, k-1.$$

**Question 2** In the statement of Theorem 2, can one replace condition (C) with condition (C')?

**7.3** By slightly generalizing the methods used by Estermann in [13], one can prove the following theorem:

**Theorem 22** For any  $k$ -tuple  $(\alpha_1, \dots, \alpha_k)$  of rationally independent irrational numbers the natural density of the set

$$\{n \in \mathbb{N} : \gcd(\lfloor n\alpha_1 \rfloor, \dots, \lfloor n\alpha_k \rfloor) = 1\}$$

exists and equals  $\frac{1}{\xi(k)}$  here and in Question 3.

This leads to the following question.

**Question 3** Let  $\mathcal{H}$  be a Hardy field. Suppose  $f_1, \dots, f_k \in \mathcal{H}$  satisfy conditions (A), (B), and (C'). Is it true that the natural density of the set

$$\{n \in \mathbb{N} : \gcd(\lfloor f_1(n) \rfloor, \dots, \lfloor f_k(n) \rfloor) = 1\}$$

exists and equals  $\frac{1}{\xi(k)}$ ?

**7.4** We conclude this paper with a question which addresses a possible weakening of condition (B) in Theorems 1 and 2.

**Question 4** Can condition (B) be replaced by condition (B') (see page 111) in Theorems 1 and 2?

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