A Krengel-type theorem for finitely generated nilpotent groups

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Abstract

Let $\{U_g\}_{g\in G}$ be a weakly mixing unitary action of a finitely generated nilpotent group on a Hilbert space $\mathcal{H}$. Extending a known result about abelian groups, we show that weakly wandering vectors are dense in $\mathcal{H}$.

0. Introduction

A unitary $\mathbb{Z}$-action $\{U^n\}_{n\in\mathbb{Z}}$ on a complex Hilbert space $\mathcal{H}$ has *continuous spectrum*, or is *weakly mixing* if the operator U has no eigenvectors. The property of having continuous spectrum can be characterized in a variety of ways. For example, $\{U^n\}_{n\in\mathbb{Z}}$ is weakly mixing if and only if for any $f_1, f_2 \in \mathcal{H}$ and $\varepsilon > 0$ the set $S = \big\{n \in \mathbb{Z} \bigm| |\langle U^n f_1, f_2\rangle| < \varepsilon\big\}$ has density one in $\mathbb{Z}$ with respect to some sequence of intervals $I_k = [a_k, b_k]$ with $b_k - a_k \to \infty$. (This means that $d_{\{I_k\}}(S) = \lim\limits_{k\to\infty} \frac{|S\cap I_k|}{b_k - a_k + 1} = 1$.)

A vector $f \in \mathcal{H}$ is called *weakly wandering* if there is an infinite set $S \subseteq \mathbb{Z}$ such that for any $n, m \in S$, $n \neq m$, one has $\langle U^n f, U^m f\rangle = 0$. The following theorem due to U. Krengel gives a characterization of weak mixing in terms of weakly wandering vectors.

Theorem 1. [K] *If a unitary action $\{U^n\}_{n\in\mathbb{Z}}$ has continuous spectrum, then the weakly wandering vectors are dense in $\mathcal{H}$.*

The notion of a unitary action with continuous spectrum has natural extension to more general groups. While some of the definitions make sense for abelian or, sometimes, amenable groups only, many natural results pertaining to weakly mixing unitary action can be formulated and proved for general locally compact groups (see [BR] for the details). As one can guess, Theorem 1 too, should be extendable to more general group actions. In [G] and [BKM1], Krengel-type results were proved for $\mathbb{Z}^d$-actions. In [BKM2] and in a recent paper [J], Theorem 1 was extended to more general classes of abelian groups, which include, for example, direct sums of cyclic groups with uniformly bounded torsion. Finally, B. Begun in his Ph.D. thesis ([B]) extended Theorem 1 to general countable abelian groups.

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The main theorem in [BKM1], as well as the results in the subsequent papers [BKM2], [J] and [B], have a novel feature. Namely, it is shown there that not only weakly wandering vectors are dense, but that the weak wandering occurs along so-called symmetric IP-sets, which we will presently define.

Definition. Let G be an abelian group. Given an infinite set $A \subseteq G$, *the IP-set* Γ *generated by* A is the set

$$\Gamma = \big\{ x_1 + x_2 + \ldots + x_k \;\big|\; x_i \in A,\, x_i \neq x_j \text{ for } i \neq j,\, 1 \leq i, j \leq k,\, k \in \mathbb{N} \big\}.$$

In other words, Γ is the set of all finite sums of distinct elements from A. The set A is called *the generating set*, or *the set of generators*, for Γ. Finally, the IP-set Γ generated by A is called *symmetric*, if the generating set A is symmetric, that is, $x \in A \iff -x \in A$.

Here is the formulation of the main theorem in [BKM1]:

Theorem 2. ([BKM1]) *A unitary* $\mathbb{Z}^d$*-action* $\{U_g\}_{g \in \mathbb{Z}^d}$ *on a Hilbert space* $\mathcal{H}$ *has continuous spectrum if and only if for any* $f \in \mathcal{H}$ *and any* $\varepsilon > 0$*, there exists* $\tilde{f}$*,* $\|\tilde{f} - f\| < \varepsilon$*, and a symmetric IP-set* $\Gamma \subseteq \mathbb{Z}^d$*, such that* $\langle U_\alpha \tilde{f}, U_\beta \tilde{f} \rangle = 0$ *for all distinct* $\alpha, \beta \in \Gamma$*.*

Our main goal in this paper is to make a step in a non-commutative direction and establish an extension of Theorem 2 to unitary actions of finitely generated nilpotent groups. The standing assumption throughout this paper will be that the Hilbert spaces we deal with are complex. (We remark in passing that similar results can be established for real Hilbert spaces as well.) To formulate our main result we need to introduce some definitions. Let G be a finitely generated nilpotent group. A sequence of finite sets $\{\Phi_k\}_{k=1}^{\infty}$ in G is called *a (right) Følner sequence* if for any $g \in G$ one has $\lim_{k \to \infty} \frac{|\Phi_k g \triangle \Phi_k|}{|\Phi_k|} = 0$. It is well known that any countable amenable, in particular, nilpotent group has a Følner sequence. A unitary action $\{U_g\}_{g \in G}$ of an amenable group G is called *weakly mixing* if for any $f, g \in \mathcal{H}$ and any $\varepsilon > 0$, the set $S = \big\{ g \in G \;\big|\; |\langle U_g f_1, f_2 \rangle| < \varepsilon \big\}$ has density one with respect to some Følner sequence $\{\Phi_k\}_{k=1}^{\infty}$, that is, $d_{\{\Phi_k\}} = \lim_{k \to \infty} \frac{|S \cap \Phi_k|}{|\Phi_k|} = 1$. (It follows then that S has density one with respect to *any* Følner sequence in G.)

When dealing with noncommutative groups, one has at his disposal a few possibilities for defining an IP-set. Aiming at stronger results (and being lucky to be working with nilpotent groups which are, so to say, close enough to abelian ones) we make the following definition:

Definition. Given a subset $B \subseteq G$ and $c \in \mathbb{N}$, *the symmetric* FP_c*-set generated by* B, $\mathrm{FP}_c(B^{\pm})$, is the set of elements in G which are representable as finite products with entries from $B \cup B^{-1}$ so that every element $b \in B$ participates in the product $\leq c$ times (counting the appearances of b^{-1} as well). Formally,

$$\mathrm{FP}_c(B^{\pm}) = \{e\} \cup \Big\{ b_1^{\epsilon_1} \dots b_k^{\epsilon_k} \;\big|\; k \in \mathbb{N},\ b_1, \dots, b_k \in B,\ \epsilon_1, \dots, \epsilon_k \in \{1, -1\}, \\ \text{and for any } b \in B,\ \#\big\{ i \in \{1, \dots, k\} \;\big|\; b_i = b \big\} \le c \Big\}.$$

Here is the formulation of our main result:

Theorem 3. *Let G be a finitely generated nilpotent group, and let the rank of the (finitely generated abelian) group $G/[G,G]$ equal d. Let U, $g \mapsto U_g$, $g \in G$, be a unitary weakly mixing action of G on a Hilbert space $\mathcal{H}$. Then for any $f \in \mathcal{H}$, $\varepsilon > 0$ and $c \in \mathbb{N}$ there exist $\tilde{f} \in \mathcal{H}$ with $\|f - \tilde{f}\| < \varepsilon$ and an infinite subset B in G such that*
(A) *every d elements of B generate a subgroup of finite index in G;*
(B) *$\langle U_\alpha \tilde{f}, U_\beta \tilde{f} \rangle = 0$ for all $\alpha, \beta \in \mathrm{FP}_c(B^{\pm})$ with $\alpha^{-1}\beta \notin [G,G]$.*

Remark. The restriction $\alpha^{-1}\beta \notin [G,G]$ in (B) is necessary, as the following simple example shows. Let G be the group generated by elements a, b, c satisfying $[a,b] = c$, $[a,c] = [b,c] = \mathbf{1}_G$. ($G$ is isomorphic to the group of upper triangular 3×3 $\mathbb{Z}$-matrices with unit diagonal.) Let $\{\dots, f_{-1}, f_0, f_1, \dots\}$ be an orthonormal basis in a Hilbert space $\mathcal{H}$, and let $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Define an action $\{U_g\}_{g \in G}$ of G on H by $U_c = \lambda \,\mathrm{Id}_{\mathcal{H}}$, $U_a f_j = \lambda^j f_j$, $U_b f_j = f_{j+1}$, $j \in \mathbb{Z}$. It is easy to check that this action is weakly mixing, and is faithful if λ is not a root of unity. The commutator subgroup $[G,G]$ is generated by c and so, for any $\gamma \in [G,G]$ one has $U_\gamma = \lambda^l \,\mathrm{Id}_{\mathcal{H}}$, $l = l(\gamma) \in \mathbb{Z}$. Hence, for any $\alpha, \beta \in G$ with $\alpha^{-1}\beta \in [G,G]$ and for any $\tilde{f}$ we have $U_\alpha \tilde{f} = \lambda^l U_\beta \tilde{f}$, $l = l(\alpha^{-1}\beta)$. But if $B \subseteq G$ has at least two noncommuting elements, then the set $\mathrm{FP}_c(B)$ contains α, β satisfying $\alpha^{-1}\beta \in [G,G] \setminus \{e\}$. Indeed, if $g_1, g_2 \in B$ and $g_1 g_2 \ne g_2 g_1$, let $\alpha = g_2 g_1$ and $\beta = g_1 g_2$; then $\alpha^{-1}\beta = [g_1, g_2] \ne e$.

Similarly to the proof of Theorem 2 in [BKM1], Theorem 3 is proved by an application of a fixed point argument. However the noncommutativity of the acting group G and the fact that we are aiming at wandering along FP_c-sets complicate things, and necessitate the use of structure theorems on unitary actions of finitely generated nilpotent groups. These structure theorems and some additional auxiliary material are reviewed in Section 1. Section 2 is devoted to the proof of Theorem 3.

1. Some background information

From now on, G will stand for a finitely generated nilpotent group; we will denote the neutral element of G by e.

Given a mapping $\varphi\colon G \longrightarrow G$ and $h \in G$, let $\varphi_h\colon G \longrightarrow G$, $\varphi_h(g) = \varphi(g)^{-1}\varphi(gh)$. We call φ *polynomial* if there is $r \in \mathbb{N}$ such that for any $h_1, \dots, h_r \in G$, the mapping $\big(\dots(\varphi_{h_1})_{h_2} \dots\big)_{h_r}$ is constant. In particular, any self-homomorphism of G is a polynomial mapping: if $\varphi\colon G \longrightarrow G$ is a homomorphism and $h \in G$, then $\varphi_h(g) \equiv \varphi(h)$ for all $g \in G$.

Theorem 4. ([L3], Theorem 5.4) *Polynomial mappings $G \longrightarrow G$ form a group under the element-wise multiplication.*

Corollary 5. *Let $g_1,\dots,g_n \in G$ and $k_1,\dots,k_{n-1} \in \mathbb{Z}$. The mapping $\varphi\colon G \longrightarrow G$, $\varphi(g) = g_1 g^{k_1} g_2 g^{k_2} \dots g_{n-1} g^{k_{n-1}} g_n$ is polynomial.*

A subgroup H of G is said to be *closed* if for every $g \in G \setminus H$, $g^n \notin H$ for all $n \neq 0$. We will need the following fact:

Lemma 6. *If H is a subgroup of G, G contains a subgroup G' of finite index such that $H \cap G'$ is a closed subgroup of G'.*

For a proof, see, for example, [BL1] Proposition 1.17 and [L1] Proposition 2.10.

We will make use of the following theorem, which describes the way the Hilbert space splits under a unitary action of a finitely-generated nilpotent group.

Theorem 7. ([L2]) *Let U, $g \mapsto U_g$, be a unitary action of G on a Hilbert space $\mathcal{H}$. Then there is a decomposition of $\mathcal{H}$, $\mathcal{H} = \bigoplus_{s\in S} \mathcal{L}_s$ into a direct sum of pairwise orthogonal subspaces so that elements of G permute these subspaces: for $g \in G$, $s \in S$ one has $U_g(\mathcal{L}_s) = \mathcal{L}_t$, $t \in S$, and if g, s satisfy $U_g(\mathcal{L}_s) = \mathcal{L}_s$, then U_g is either scalar or weakly mixing on $\mathcal{L}_s$.*

We will call the decomposition described in Theorem 7 *a primitive decomposition of $\mathcal{H}$* (relative to U).

We fix a Følner sequence $\{\Phi_k\}_{k\in\mathbb{N}}$ of finite subsets of G and measure *the density* of subsets of G with respect to this sequence. We will say that a statement is true for *almost all elements of G* if the elements of G for which this statement holds form a subset of density one with respect to $\{\Phi_k\}_{k\in\mathbb{N}}$.

Let V, $g \mapsto V_g$, $g \in G$, be a mapping from G into the group of unitary operators on a Hilbert space $\mathcal{H}$ (V need not to be a homomorphism). Let $\mathcal{L}$ be a subspace of $\mathcal{H}$. We will say that V *is scalar on* $\mathcal{L}$ if for every $g \in G$, V_g is scalar on $\mathcal{L}$: $V_g|_{\mathcal{L}} = a_g \operatorname{Id}_{\mathcal{L}}$, $a_g \in \mathbb{C}$. We will say that V *is weakly mixing on* $\mathcal{L}$ if for every $f_1 \in \mathcal{L}$, $f_2 \in \mathcal{H}$ and every $\varepsilon > 0$, $\bigl|\langle V_g f_1, f_2\rangle\bigr| < \varepsilon$ for almost all $g \in G$.

Now, let $\varphi\colon G \longrightarrow G$ be a polynomial mapping with $\varphi(e) = e$, let U be a unitary action of G on a Hilbert space $\mathcal{H}$ and let $\mathcal{H} = \bigoplus_{s\in S} \mathcal{L}_s$ be a primitive decomposition of $\mathcal{H}$ relative to U. Let V be the composition of U and φ: $V_g = U_{\varphi(g)}$. Fix $s \in S$, let $H = \bigl\{g \in G \bigm| U_g(\mathcal{L}_s) = \mathcal{L}_s\bigr\}$ and let $E = \bigl\{g \in H \bigm| U_g \text{ is scalar on } \mathcal{L}_s\bigr\}$.

Theorem 8. *Assume that H and E are closed in G. Then the following holds: If $\varphi(G) \not\subseteq \overline{H}$, then for any $t \in S$, $V_g(\mathcal{L}_s) \perp \mathcal{L}_t$ for almost all $g \in G$. If $\varphi(G) \subseteq H \setminus \overline{E}$, then V preserves $\mathcal{L}_s$ and is weakly mixing on $\mathcal{L}_s$. If $\varphi(G) \subseteq E$, then V is scalar on $\mathcal{L}_s$.*

The third statement of this theorem is trivial, the first two statements are given by Theorem 5.3 in [L3].

2. Proof of Theorem 3

As before, let G be a finitely generated nilpotent group with neutral element e. We may assume that both G and $G/[G,G]$ are torsion-free. This follows from the fact that if Theorem 3 holds for a nilpotent group $\hat{G}$ such that G is a homomorphic image of $\hat{G}$ and $\mathrm{rank}\big(\hat{G}/[\hat{G},\hat{G}]\big) = \mathrm{rank}(G/[G,G])$, then Theorem 3 also holds for G. To see that such a nilpotent group $\hat{G}$ exists, one argues as follows. Assume that $G/[G,G]$ is generated by $h_1[G,G],\ldots,h_d[G,G]$. Then $h_1,\ldots,h_d$ generate G. (This is so since, as it is easy to see, the commutators $[h_i,h_j]$ generate $[G,G]$ modulo $\big[[G,G],G\big]$, the commutators $\big[[h_i,h_j],h_l\big]$ generate $\big[[G,G],G\big]$ modulo $\big[[[G,G],G],G\big]$, and so on, and for a nilpotent G this process is finite.) Consider the free group $\tilde{G}$ generated by $h_1,\ldots,h_d$, and let $\tilde{G} = \tilde{G}_1 \supset \tilde{G}_2 \supset \ldots$ be its lower central series: $\tilde{G}_{i+1} = [\tilde{G}_i,\tilde{G}]$. Assume that G has nilpotency class q; then G is a factor of the group $\hat{G} = \tilde{G}/\tilde{G}_{q+1}$, the "free nilpotent group of class q with d generators"; $\hat{G}$ has no torsion and $\hat{G}/[\hat{G},\hat{G}] \simeq \mathbb{Z}^d$.

Let $\mathcal{H} = \bigoplus_{s\in S}\mathcal{L}_s$ be the primitive decomposition of $\mathcal{H}$ relative to U. Let $f \in \mathcal{H}$, $f \neq 0$. Without loss of generality we may assume that

$$f = r_1 f_1 + \ldots + r_m f_m, \tag{1}$$

where $r_1,\ldots,r_m \in \mathbb{R}$, $f_1 \in \mathcal{L}_{s_1},\ldots,f_m \in \mathcal{L}_{s_m}$ satisfy $\|f_1\| = \ldots = \|f_m\| = 1$, and $s_1,\ldots,s_m$ are distinct elements of S. For each $i = 1,\ldots,m$ let $H_i = \big\{U_g \in G \;\big|\; U_g(\mathcal{L}_{s_i}) = \mathcal{L}_{s_i}\big\}$, $E_i = \big\{g \in H \;\big|\; U_g \text{ is scalar on } \mathcal{L}_{s_i}\big\}$ and $K_i = E_i \cap [G,G]$. Notice that K_i is in the center of H_i.

By Lemma 6, for each $i = 1,\ldots,m$, G contains a subgroup G_i of finite index such that $H_i \cap G_i$ and $E_i \cap G_i$ are closed in G_i. Let us replace G by $\bigcap_{i=1}^m G_i$; after this, we may assume that, for every $i = 1,\ldots,m$, H_i and E_i are closed subgroups of G.

We will prove Theorem 3 in two steps. First, we will show how to find a sequence $B = \{g_1,g_2,\ldots\}$ satisfying the condition (A) of the theorem and such that for any $\lambda \in \mathrm{FP}_{2c}(B^\pm)\setminus[G,G]$ (and even for any $\lambda \in \mathrm{FP}_{6c}(B^\pm)\setminus[G,G]$) $U_\lambda f$ is "almost" orthogonal to f (that is, $\big|\langle U_\lambda f, f\rangle\big|$ is small enough). Then we will slightly change f in order to make $U_\lambda f$ to be strictly orthogonal to f for all $\lambda \in \mathrm{FP}_{2c}(B^\pm)\setminus[G,G]$. This will imply $U_\alpha f \perp U_\beta f$ for all $\alpha,\beta \in \mathrm{FP}_c(B^\pm)$ with $\alpha^{-1}\beta \notin [G,G]$.

Let us introduce some notation. Let $F = F(z_1,z_2,\ldots)$ be the free group generated by the symbols $z_1,z_2,\ldots$. For a reduced word $w \in F$ let *the weight of* w be the maximal n for which z_n appears in w. Given $c \in \mathbb{N}$, we will denote by F_c the subset of F consisting of the reduced words in which each of $z_1,z_2,\ldots$ appears not more than c times.

For $w \in F$, we denote by $\deg_{z_n} w$ the total degree of $z_n \in w$ (for example, $\deg_{z_2}(z_1^{-2}z_2^3 z_3^4 z_2^{-8} z_1^6) = -5$). Let $w \in F$ have weight n. We will say that w *is degenerate* if $\deg_{z_n} w = 0$. We will denote by w_0 the element of F which is obtained from w by erasing all entries of z_n (as well as those of z_n^{-1}). Notice that w is degenerate if and only if $w \equiv w_0 \bmod [F,F]$.

Given a sequence $B = \{g_1,g_2,\ldots\}$ of elements of G and $w \in F$, we will denote by

$w(B)$ the element of G which is obtained from w by replacing each z_i by the corresponding g_i. Clearly, $\{w(B) \mid w \in F_c\} = FP_c(B^{\pm})$.

Lemma 9. *Let w be an element of F of weight $n+1$, let $g_1, \ldots, g_n \in G$, and let the mapping $\varphi\colon G \longrightarrow G$ be defined by the formula $\varphi(g) = w(g_1, \ldots, g_n, g)$. If w is nondegenerate, then $\varphi(G)$ is not contained in any subgroup of infinite index in G.*

Proof. This follows from the fact that the set $\varphi(G)$ is *syndetic* in G: there are finitely many elements $h_1, \ldots, h_k \in G$ such that $\bigcup_{l=1}^{k} h_l\varphi(G) = G$. Furthermore, we will show that for any nonzero $m \in \mathbb{Z}$ there are $h_1, \ldots, h_k \in G$ such that for any $r, s \in \mathbb{N}$ with $s \leq r$, any $v \in F$ of weight r with $\deg_{z_s} v = m$, and any $g_1, \ldots, g_{s-1}, g_{s+1}, \ldots, g_r \in G$, for the mapping $\psi\colon G \longrightarrow G$ given by $\psi(g) = v(g_1, \ldots, g_{s-1}, g, g_{s+1}, \ldots, g_r)$ one has $G = \bigcup_{l=1}^{k} h_l\psi(G)$. We will use induction on the nilpotency level of G; let $H = G/[G,G]$ and let $h'_1, \ldots, h'_k \in H$ satisfy our statement for the group H and given m. H is a finitely generated abelian group, thus H/H^m is finite; let $h''_1, \ldots, h''_l \in G$ be representatives of the cosets of H^m in H. The mapping ψ satisfies $\psi(g) \equiv g^m v(g_1, \ldots, g_{s-1}, \mathbf{1}_G, g_{s+1}, \ldots, g_r) \bmod(H)$. Thus, for any $p \in G$ there are $g \in G$ and $i \leq l$ such that $q = p(h''_i\psi(g))^{-1} \in H$. Consider the mapping $\eta\colon H \longrightarrow H$, $\eta(h) = h''_i\psi(gh)\psi(g)^{-1}(h''_i)^{-1}$. By induction hypothesis, there are $h \in H$ and $j \leq k$ such that $q = h'_j\eta(h)$. We then have $p = q(h''_i\psi(g)) = h'_jh''_i\psi(gh)$, and, hence, every $p \in G$ is representable as $p = h'_jh''_i\psi(g)$, $1 \leq j \leq k$, $1 \leq i \leq l$. ∎

The first part of the proof of Theorem 3 is given by the following lemma, whose formulation and proof utilize the notation introduced above.

Lemma 10. *For any $b \in \mathbb{N}$ and any mapping $\delta\colon F_b \longrightarrow \mathbb{R}_+$, $w \mapsto \delta_w > 0$, there is an infinite sequence $B = \{g_1, g_2, \ldots\}$ satisfying condition (A) of Theorem 3 and such that for every $w \in F_b$ and $\gamma = w(B)$, one has*

$$U_\gamma \mathcal{L}_{s_i} \perp \mathcal{L}_{s_j} \text{ for all } i, j = 1, \ldots, m,\ i \neq j,$$

and, for every $i = 1, \ldots, m$,
(∗) if $H_i \neq G$, then $U_\gamma \mathcal{L}_{s_i} \perp \mathcal{L}_{s_i}$ for all $\gamma \notin [G,G]$;
(∗∗) if $H_i = G$, then $\left|\langle U_\gamma f_i, f_i\rangle\right| < \delta_w$ for all $\gamma \notin K_i$.

Proof. We will construct B inductively: assuming that $g_1, g_2, \ldots g_n \in B$ have been already chosen, we will show that, to satisfy the conclusion of the lemma, g_{n+1} can be taken from an intersection of finitely many subsets of density one in G. (When choosing g_1 we start with the empty B.)

First of all, by our assumption, $G/[G,G] \simeq \mathbb{Z}^d$. If the images of $g_1, \ldots, g_n, g_{n+1}$ in the vector space $(G/[G,G]) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^d$ (that is, when $G/[G,G]$ is considered as the sublattice $\mathbb{Z}^d$ in $\mathbb{Q}^d$) are in general position (that is, any d of these elements span $\mathbb{Q}^d$), then any d elements of $g_1, \ldots, g_n, g_{n+1}$ generate a subgroup of finite index in $G/[G,G]$ and so, in G itself (see [BL1] Lemma 1.10). Thus, assuming that $g_1, \ldots, g_n$ are in general position, in order to satisfy condition (A) of Theorem 3 we have to choose g_{n+1} so that it will be in the general position with respect to $g_1, \dots, g_n$, that is, from a subset of density one in G.

Let $w \in F_b$ have weight $n+1$. Then $\varphi(g) = w_0(g_1, \dots, g_n)^{-1} w(g_1, \dots, g_n, g)$ defines a polynomial mapping $\varphi: G \longrightarrow G$. Define $V_g = U_{\varphi(g)}$. Fix i, $1 \le i \le m$. Consider two cases:

$(*)$ $H_i \ne G$.

(a) If $\varphi(G) \not\subseteq H_i$, then, since H_i is closed in G, by Theorem 8, for any $t \in S$ one has $V_g(\mathcal{L}_{s_i}) \perp \mathcal{L}_t$ for almost all $g \in G$. This implies that for all $j = 1, \dots, m$, $U_{w(g_1,\dots,g_n,g)}(\mathcal{L}_{s_i}) \perp \mathcal{L}_{s_j}$ for almost all $g \in G$.

(b) If $\varphi(G) \subseteq H_i$, we have $w(g_1, \dots, g_n, g) \equiv w_0(g_1, \dots, g_n) \bmod H_i$ for all $g \in G$ and so, $U_{w(g_1,\dots,g_n,g)}(\mathcal{L}_{s_i}) = U_{w_0(g_1,\dots,g_n)}(\mathcal{L}_{s_i})$. Now, since H_i is closed in G, the index of H_i in G is infinite. It follows from Lemma 9 that $U_{\varphi(G)} \not\subseteq H_i$ for nondegenerate w. Thus, w is degenerate and we also have $w \equiv w_0 \bmod [F,F]$. It follows by induction on n that for $w \notin [F,F]$ we have $U_{w(g_1,\dots,g_n,g)}(\mathcal{L}_{s_i}) \perp \mathcal{L}_{s_j}$, $j = 1, \dots, m$, for almost all $g \in G$. In the case $w \in [F,F]$, we have $\gamma = w(g_1, \dots, g_{n+1}) \in [G,G]$ for any choice of $g_{n+1} \in G$.

$(**)$ $H_i = G$. Then E_i and, so, K_i are normal in G. Since U is weakly mixing, $E_i \ne G$.

(a) Let $\varphi(G) \subseteq H_i \setminus E_i$. Then $V_g(\mathcal{L}_{s_i}) = \mathcal{L}_{s_i}$ for all $g \in G$, and V is weakly mixing on $\mathcal{L}_{s_i}$. By induction on n, $U_{w(g_1,\dots,g_n,g)}(\mathcal{L}_{s_i}) = U_{w_0(g_1,\dots,g_n)}(\mathcal{L}_{s_i}) \perp \mathcal{L}_{s_j}$ for all $j \ne i$ and all $g \in G$.

If also $U_{w_0(g_1,\dots,g_n)}(\mathcal{L}_{s_i}) \ne \mathcal{L}_{s_i}$, then $U_{w(g_1,\dots,g_n,g)} f_i \perp f_i$. Otherwise, by Theorem 8, for every $h \in \mathcal{L}_{s_i}$ one has $\left|\langle V_g f_i, h \rangle\right| < \delta_w$ for almost all $g \in G$. Applying this to $h = U_{w_0(g_1,\dots,g_n)^{-1}} f_i$, we obtain $\left|\langle U_{w(g_1,\dots,g_n,g)} f_i, f_i \rangle\right| < \delta_w$ for almost all $g \in G$. Clearly, $w(g_1, \dots, g_n, g) \notin E_i$ for such g.

(b) Now, let $\varphi(G) \subseteq E_i$. Then, again, $\mathcal{L}_{s_i}$ is V_G-invariant and so, $U_{w(g_1,\dots,g_n,g)}(\mathcal{L}_{s_i}) \perp \mathcal{L}_{s_j}$ for all $j \ne i$ and all $g \in G$. V is scalar on $\mathcal{L}_{s_i}$, and hence, $U_{w(g_1,\dots,g_n,g)} f_i = a U_{w_0(g_1,\dots,g_n)} \varphi(g) f_i$, $a = a(g) \in \mathbb{C}$, $|a| = 1$.

The inclusion $\varphi(G) \subseteq E_i$ implies $w(g_1, \dots, g_n, g) \equiv w_0(g_1, \dots, g_n) \bmod E_i$ for all $g \in G$. Since E_i has infinite index in G, by Lemma 9, $\varphi(G) \subseteq E_i$ is possible only for degenerate w, which implies $w \equiv w_0 \bmod [F,F]$ and $w(g_1, \dots, g_n, g) \equiv w_0(g_1, \dots, g_n) \bmod K_i$ for all $g \in G$. By induction on n, $\left|\langle U_{w_0(g_1,\dots,g_n)} f_i, f_i \rangle\right| < \delta_w$, except for the case $w_0(g_1, \dots, g_n) \in K_i$. In this last case, $w(g_1, \dots, g_n, g) \in K_i$ for all $g \in G$.

In either case, elements g of G which may serve as g_{n+1} for the fixed w can be chosen from an intersection of finitely many sets of density one in G, which has density one itself. Since F_b contains finitely many elements of weight $n+1$, g_{n+1} can be taken from a set of density one. ■

We pass now to the second part of the proof, which is quite analogous to the proof of the abelian case given in [BKM1]. Fix $\varepsilon > 0$ and put $\varepsilon' = \min\left\{\frac{\varepsilon}{\|f\|}, 1\right\}$. Choose positive numbers δ_w, $w \in F_{6c}$, satisfying $\sum_{w \in F_{6c}} \delta_w < \frac{\varepsilon'}{8}$. Using Lemma 10 find an infinite sequence

$B = \{g_1, g_2, \ldots\}$ satisfying condition (A) of Theorem 3 and such that for every $w \in F_{6c}$ and $\gamma = w(B)$ one has $U_\gamma \mathcal{L}_{s_i} \perp \mathcal{L}_{s_j}$ for all $i, j = 1, \ldots, m$, $i \neq j$, and for each $i = 1, \ldots, m$ either $(*)$ or $(**)$ of Lemma 10 holds.

Given a set $\Gamma \subseteq G$, we define $\Gamma^k = \{g_1 \ldots g_k \mid g_1, \ldots, g_k \in \Gamma\}$, $k \in \mathbb{N}$, and $\Gamma^{-1} = \{g^{-1} \mid g \in \Gamma\}$. Let $\Gamma = \mathrm{FP}_c(B^\pm)$; then $\Gamma^{-1} = \Gamma$, and $\Gamma^k \subseteq \mathrm{FP}_{kc}(B^\pm) = \{w(B) \mid w \in F_{kc}\}$ for any $k \in \mathbb{N}$. Notice that, for every $i = 1, \ldots, m$,

$$\sum_{\gamma \in \Gamma^6 \setminus K_i} \left|\langle U_\gamma f_i, f_i \rangle\right| < \frac{\varepsilon'}{8}. \tag{2}$$

Fix i, $1 \leq i \leq m$. We will look for $\tilde{f}_i$ in the form

$$\tilde{f}_i = f_i + \sum_{\xi \in \Gamma^2} x_\xi U_\xi f_i \tag{3}$$

with $x_\xi \in \mathbb{C}$ small enough, in order to have

$$U_\lambda \tilde{f}_i \perp \tilde{f}_i \text{ for all } \lambda \in \Gamma^2 \setminus [G, G]. \tag{4}$$

This will imply

$$\langle U_\alpha \tilde{f}_i, U_\beta \tilde{f}_i \rangle = \langle \tilde{f}_i, U_{\alpha^{-1}\beta} \tilde{f}_i \rangle = 0$$

for all $\alpha, \beta \in \Gamma$ with $\alpha^{-1}\beta \notin [G, G]$. After we will have found $\tilde{f}_i$ for all $i = 1, \ldots, m$, we will put $\tilde{f} = r_1 \tilde{f}_1 + \ldots + r_m \tilde{f}_m$ (where $r_1, \ldots, r_m$ are as in (1)). Since for any $\alpha, \beta \in \Gamma$ and $\xi, \eta \in \Gamma^2$ one has $\xi^{-1}\alpha^{-1}\beta\eta \in \mathrm{FP}_{6c}(B^\pm)$, we will have

$$\langle U_\alpha U_\xi f_i, U_\beta U_\eta f_j \rangle = \langle f_i, U_{\xi^{-1}\alpha^{-1}\beta\eta} f_j \rangle = 0$$

for all $1 \leq i, j \leq m$, $i \neq j$. This implies $U_\alpha \tilde{f}_i \perp U_\beta \tilde{f}_j$ for all $\alpha, \beta \in \Gamma$ and all $1 \leq i, j \leq m$, $i \neq j$. Summarizing, we will have $\langle U_\alpha \tilde{f}, U_\beta \tilde{f} \rangle = 0$ for all $\alpha, \beta \in \Gamma$ with $\alpha^{-1}\beta \notin [G, G]$.

Our goal is to find $\tilde{f}_i \in \mathcal{L}_{s_i}$ in the form (3) to get (4). If i is such that $(*)$ of Lemma 10 takes place, then (4) is already satisfied for $\tilde{f}_i = f_i$. Thus, let us assume that $(**)$ takes place for the fixed i. The subspace $\mathcal{L}_{s_i}$ is then U_G-invariant, the subgroup K_i is normal in G, and its action on $\mathcal{L}_{s_i}$ is scalar: for $\gamma \in K_i$, $U_\gamma|_{\mathcal{L}_{s_i}} = a_\gamma \,\mathrm{Id}_{\mathcal{L}_{s_i}}$ with $a_\gamma \in \mathbb{C}$, $|a_\gamma| = 1$. The set Γ^2 is partitioned into equivalence classes modulo K_i; choose a set D of representatives of these classes, assuming that $\lambda \in D$ implies $\lambda^{-1} \in D$. As the representative for the class $\Gamma^2 \cap K_i$ we take e, and put $D' = D \setminus \{e\}$. Now, since $\lambda_1 \equiv \lambda_2 \bmod K_i$ implies $U_{\lambda_1} \tilde{f}_i = a_{\lambda_2^{-1}\lambda_2} U_{\lambda_2} \tilde{f}_i$, and since $K_i \subseteq [G, G]$, to obtain (4) it suffices to have

$$U_\lambda \tilde{f}_i \perp \tilde{f}_i \text{ for all } \lambda \in D'. \tag{5}$$

We will look for $\tilde{f}_i$ in the form

$$\tilde{f}_i = f_i + \sum_{\xi \in D} x_\xi U_\xi f_i, \tag{6}$$

with $x_\xi \in \mathbb{C}$, $\xi \in D$, satisfying

$$x_{\xi^{-1}} = \overline{x_\xi}, \quad \xi \in D, \tag{7}$$

and

$$\sum_{\xi \in D} |x_\xi| < \varepsilon'. \tag{8}$$

It will follow from (8) that $\|\tilde{f}_i - f_i\| < \varepsilon'$, which implies

$$\|\tilde{f} - f\| = \Big\|\sum_{i=1}^m r_i \tilde{f}_i - \sum_{i=1}^m r_i f_i\Big\| < \varepsilon' \sqrt{\sum_{i=1}^m r_i^2} = \varepsilon' \|f\| \le \varepsilon.$$

We will also require that

$$\|\tilde{f}_i\| = 1. \tag{9}$$

For $\gamma \in G$ we denote $b_\gamma = \langle f_i, U_\gamma f_i \rangle$. Then

$$b_\gamma = \overline{a_\gamma} \quad \text{for } \gamma \in K_i \tag{10}$$

and, by (2),

$$\sum_{\gamma \in \Gamma^6 \setminus K_i} |b_\gamma| < \frac{\varepsilon'}{8}. \tag{11}$$

For $\lambda \in \Gamma^2$ we have

$$\begin{aligned}
\langle \tilde{f}_i, U_\lambda \tilde{f}_i \rangle &= \Big\langle f_i + \sum_{\xi \in D} x_\xi U_\xi f_i, U_\lambda f_i + U_\lambda \big(\sum_{\eta \in D} x_\eta U_\eta f_i \big) \Big\rangle \\
&= b_\lambda + \sum_{\xi \in D} x_\xi b_{\xi^{-1}\lambda} + \sum_{\xi \in D} \overline{x_\eta} b_{\lambda\eta} + \sum_{\xi,\eta \in D} x_\xi \overline{x_\eta} b_{\xi^{-1}\lambda\eta} \\
&= b_\gamma + \sum_{\xi \in D} x_\xi (b_{\xi^{-1}\gamma} + b_{\gamma\xi^{-1}}) + \sum_{\xi,\eta \in D} x_\xi \overline{x_\eta} b_{\xi^{-1}\gamma\eta}.
\end{aligned}$$

Let $b'_\lambda = \begin{cases} -\frac{1}{2} b_\lambda, & \text{if } \lambda \ne e \\ 0, & \text{if } \lambda = e. \end{cases}$ Then the equalities $\langle \tilde{f}_i, U_\lambda \tilde{f}_i \rangle = 0$ for all $\lambda \in D'$ are equivalent to the following system of equations for unknowns $\{x_\xi\}_{\xi \in D}$:

$$\frac{1}{2} \sum_{\xi \in D} x_\xi (b_{\xi^{-1}\lambda} + b_{\lambda\xi^{-1}}) + \frac{1}{2} \sum_{\xi,\eta \in D} x_\xi \overline{x_\eta} b_{\xi^{-1}\lambda\eta} = b'_\lambda, \quad \lambda \in D. \tag{12}$$

(Condition (9) corresponds to the case $\lambda = e$ in (12).) The problem of solving the system (12) can be interpreted as a fixed-point problem. Namely, we consider the space

$$l'_{\mathrm{symm}}(D) = \big\{ x = (x_\xi)_{\xi \in D} \;\big|\; x_{\xi^{-1}} = \overline{x_\xi},\ \|x\| := \textstyle\sum_{\xi \in D} |x_\xi| < \infty \big\}$$

of conjugate-symmetric summable functions on D. In this space we define a (nonlinear) mapping Φ by setting, for any $x \in l'_{\text{symm}}(D)$, $\Phi(x) = y = (y_\lambda)_{\lambda\in D}$ with

$$y_\lambda = b'_\lambda - \frac{1}{2}\sum_{\xi\in D\setminus\{\lambda\}} (b_{\xi^{-1}\lambda} + b_{\lambda\xi^{-1}})x_\xi - \frac{1}{2}\sum_{\xi,\eta\in D} b_{\xi^{-1}\lambda\eta}x_\xi\overline{x_\eta}. \tag{13}$$

And if a point $x = (x_\xi) \in l'_{\text{symm}}(D)$ is a fixed-point of Φ, $\Phi(x) = x$, then (x_ξ) is a solution of the system (12), and so, the vector $\tilde{f} \in \mathcal{H}$ defined by (6) satisfies Theorem 3.

The existence of a fixed point $x \in l'_{\text{symm}}(E)$ will be proved (cf. [BKM1]) by a contraction mapping argument. Namely, one shows that
(i) the space $l'_{\text{symm}}(D)$ is invariant under Φ;
(ii) if $\frac{\varepsilon'}{2} \le \rho \le \varepsilon'$, the ball $B_\rho = \{x \in l'_{\text{symm}}(E) \mid \|x\| \le \rho\}$ is invariant under Φ;
(iii) if $\rho < (1 - \frac{\varepsilon'}{8})(1 + \frac{\varepsilon'}{8})^{-1}$, then Φ is contractive in B_ρ; more precisely,

$$\big\|\Phi(x) - \Phi(x')\big\| \le \theta\|x - x'\|, \quad x, x' \in B_\rho,$$

where $\theta = \rho(1 + \frac{\varepsilon'}{8}) + \frac{\varepsilon'}{8} < 1$.

In order to prove (i) let us estimate $\|\Phi(x)\|$. By (13) we have

$$\|\Phi(x)\| = \sum_{\lambda\in D} |y_\lambda| \le \|\Sigma_0\| + \|\Sigma_1\| + \|\Sigma_2\|, \tag{14}$$

where Σ_0, Σ_1 and Σ_2 are, respectively, the constant, the linear and the quadratic parts of Φ. We have, by (11) and the inclusion $D' \subseteq \Gamma^2 \setminus K_i$,

$$\begin{aligned}
\|\Sigma_0\| &\le \frac{1}{2}\sum_{\lambda\in D'} |b_\lambda| \le \frac{\varepsilon'}{16},\\
\|\Sigma_1\| &= \frac{1}{2}\sum_{\lambda\in D}\Big|\sum_{\substack{\xi\in D\\ \xi\neq\lambda}} (b_{\xi^{-1}\lambda} + b_{\lambda\xi^{-1}})x_\xi\Big| \le \frac{1}{2}\sum_{\xi\in D}|x_\xi| \sum_{\substack{\lambda\in D\\ \lambda\neq\xi}} \big(|b_{\xi^{-1}\lambda}| + |b_{\lambda\xi^{-1}}|\big)\\
&\le \sum_{\xi\in D}|x_\xi| \sum_{\lambda\in\Gamma^2\setminus K_i} |b_\lambda| \le \frac{\varepsilon'}{8}\|x\|
\end{aligned}$$

and, since $\xi^{-1}\lambda\eta \in K_i$ may happen only for $\lambda = \lambda_{\eta^{-1}\xi} \in D$ which represents the coset $\eta^{-1}\xi K_i$, by (10) and (11),

$$\begin{aligned}
\|\Sigma_2\| = \frac{1}{2}\sum_{\lambda\in D}\Big|\sum_{\xi,\eta\in D} b_{\xi^{-1}\lambda\eta}x_\xi\overline{x_\eta}\Big| &\le \frac{1}{2}\sum_{\xi,\eta\in D}|x_\xi||x_\eta|\Big(|a_{\xi^{-1}\lambda_{\eta^{-1}\xi}\eta}| + \sum_{\substack{\lambda\in D\\ \lambda\neq\lambda_{\eta^{-1}\xi}}} |b_{\xi^{-1}\lambda\eta}|\Big)\\
&\le \frac{1}{2}\sum_{\xi,\eta\in D}|x_\xi||x_\eta|\Big(1 + \sum_{\gamma\in\Gamma^6\setminus K_i} |b_{\xi^{-1}\lambda\eta}|\Big) \le \frac{1}{2}\Big(1 + \frac{\varepsilon'}{8}\Big)\|x\|^2.
\end{aligned}$$

Therefore,

$$\|\Phi(x)\| \le \frac{\varepsilon'}{16} + \frac{\varepsilon'}{8}\|x\| + \frac{1}{2}\Big(1+\frac{\varepsilon'}{8}\Big)\|x\|^2 < \infty \quad \text{for } x \in l'_{\mathrm{symm}}(D).$$

Furthermore,

$$\begin{aligned} y_{\lambda^{-1}} &= b'_{\lambda^{-1}} - \frac{1}{2}\sum_{\xi\in D\setminus\{\lambda^{-1}\}} (b_{\xi^{-1}\lambda^{-1}} + b_{\lambda^{-1}\xi^{-1}})x_\xi - \frac{1}{2}\sum_{\xi,\eta\in D} b_{\xi^{-1}\lambda^{-1}\eta} x_\xi \overline{x_\eta} \\ &= \overline{b'_\lambda} - \frac{1}{2}\sum_{\xi\in D\setminus\{\lambda\}} (b_{\xi\lambda^{-1}} + b_{\lambda^{-1}\xi})x_{\xi^{-1}} - \frac{1}{2}\sum_{\xi,\eta\in D} \overline{b_{\eta^{-1}\lambda\xi}} x_\xi \overline{x_\eta} \\ &= \overline{b'_\lambda} - \frac{1}{2}\sum_{\xi\in D\setminus\{\lambda\}} (\overline{b_{\lambda\xi^{-1}}} + \overline{b_{\xi^{-1}\lambda}})\overline{x_\xi} - \frac{1}{2}\sum_{\eta,\xi\in D} \overline{b_{\eta^{-1}\lambda\xi}}\, \overline{x_\eta} x_\xi = \overline{y_\lambda}, \end{aligned}$$

that is, $y = \Phi(x)$ is conjugate-symmetric for $x \in l'_{\mathrm{symm}}(D)$, and so, $y \in l'_{\mathrm{symm}}(D)$. This proves (i).

If $x \in B_\rho$, that is, $\|x\| \le \rho$, then for any $\frac{\varepsilon'}{2} \le \rho \le 1$ (and under the assumption $\varepsilon' \le 1$) one has

$$\|\Phi(x)\| \le \frac{\varepsilon'}{16} + \frac{\varepsilon'}{8}\rho + \frac{1}{2}\Big(1+\frac{\varepsilon'}{8}\Big)\rho^2 \le \Big(\frac{2}{16} + \frac{\varepsilon'}{8} + \frac{1}{2} + \frac{\varepsilon'}{16}\Big)\rho < \rho.$$

This proves (ii).

To prove (iii) take $x = (x_\xi)$, $x' = (x'_\xi)$ in B_ρ and let $q = \|x - x'\|$. Similarly to (14),

$$\|\Phi(x) - \Phi(x')\| \le \|\sigma_1\| + \|\sigma_2\|,$$

where σ_1 and σ_2 correspond, respectively, to the linear and the quadratic terms. We have

$$\|\sigma_1\| = \frac{1}{2}\sum_{\lambda\in D}\Big|\sum_{\substack{\xi\in D\\ \xi\neq\lambda}} (b_{\xi^{-1}\lambda} + b_{\lambda\xi^{-1}})(x_\xi - x'_\xi)\Big| \le \sum_{\xi\in D}|x_\xi - x'_\xi| \sum_{\lambda\in\Gamma^2\setminus K_i} |b_\lambda| \le \frac{\varepsilon'}{8}q$$

and

$$\begin{aligned} \|\Sigma_2\| &= \frac{1}{2}\sum_{\lambda\in D}\Big|\sum_{\xi,\eta\in D} b_{\xi^{-1}\lambda\eta}(x_\xi\overline{x_\eta} - x'_\xi\overline{x'_\eta})\Big| \\ &\le \frac{1}{2}\sum_{\xi,\eta\in D} |x_\xi\overline{x_\eta} - x_\xi\overline{x'_\eta}| + |x_\xi\overline{x'_\eta} - x'_\xi\overline{x'_\eta}|\Big(1 + \sum_{\gamma\in\Gamma^6\setminus K_i} |b_{\xi^{-1}\lambda\eta}|\Big) \\ &\le \frac{1}{2}\Big(\sum_{\xi,\eta\in D} |x_\xi||x_\eta - x'_\eta| + \frac{1}{2}\sum_{\xi,\eta\in D} |x_\xi||x_\eta - x'_\eta|\Big)\Big(1+\frac{\varepsilon'}{8}\Big) \le \rho\Big(1+\frac{\varepsilon'}{8}\Big)q. \end{aligned}$$

Finally,

$$\big\|\Phi(x) - \Phi(x')\big\| \le \frac{\varepsilon'}{8}q + \rho\Big(1+\frac{\varepsilon'}{8}\Big)q = \theta q.$$

Now, since for $0 < \varepsilon' \le 1$ one has $(1 - \frac{\varepsilon'}{8})\big(\varepsilon'(1+\frac{\varepsilon'}{8})\big)^{-1} \ge \frac{7}{9}$, any $\rho \in \big(\frac{\varepsilon'}{2}, \frac{7\varepsilon'}{9}\big)$ fits both (ii) and (iii). And for such ρ, by the Contractive Mapping Theorem, Φ has a (unique) fixed point in B_ρ. This proves Theorem 3.

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