Failure of the Roth theorem for solvable groups of exponential growth

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Abstract

We show that for any finitely generated solvable group of exponential growth one can find a measure preserving action for which the multiple recurrence theorem fails, and a measure preserving action for which the ergodic Roth theorem fails. This contrasts the positive results established in [L] and [BL] for nilpotent group actions.

1. Introduction.

Let T and S be invertible measure preserving transformations of a probability measure space $(X,\mathcal{B},\mu)$. The following two facts were recently established in [L] and [BL]:
(a) (multiple recurrence) if T and S generate a nilpotent group then for any $A\in\mathcal{B}$ with $\mu(A)>0$ there exists $n\in\mathbb{N}$ such that $\mu(A\cap T^nA\cap S^nA)>0$;
(b) (convergence) if T and S generate a nilpotent group then for any $f,g\in L^\infty(X,\mathcal{B},\mu)$, $\lim\limits_{r-l\to\infty}\frac{1}{r-l}\sum\limits_{n=l}^{r-1}f(T^nx)g(S^nx)$ exists in L^2-norm.
(For $S=T^2$, (b) reduces to Furstenberg's *ergodic Roth theorem* ([F1],[F2]), while the statement (a), via Furstenberg's correspondence principle ([F2]), implies the combinatorial Roth theorem, namely the fact that any set $E\subseteq\mathbb{N}$ having positive upper density $\overline{d}(E)=\limsup\limits_{r\to\infty}\frac{|E\cap\{1,\ldots,r\}|}{r}$ contains arithmetic progressions of length 3.) These *nilpotent* results naturally lead to the following question: do the statements (a) and (b) remain true if the group generated by T and S is not virtually nilpotent? (A group is *virtually nilpotent* if it has a nilpotent subgroup of finite index.) In [BL] we brought some examples showing that (a) and (b) may fail if T and S generate a solvable group. (An example of similar nature pertaining to non-recurrence in topological setup appears already in Furstenberg's book, [F2] p. 40. See also [Be], page 283, for an example involving a non-solvable group.)

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The counterexamples mentioned above exhibit, actually, a stronger negative behavior. Namely, in these examples even the averages $\frac{1}{r}\sum_{n=0}^{r-1}\int f(T^n x)g(S^n x)\,d\mu$ were shown to be divergent. Also, it was shown that it may happen that for a set $A\in\mathcal{B}$ with $\mu(A)>0$, $\mu(T^nA\cap S^nA)=0$ for all $n>0$. Motivated by these examples, we conjectured that *any* non-virtually nilpotent solvable group has a representation by measure preserving transformations which furnish counterexamples to (a) and (b). Note that a finitely generated solvable group has exponential growth if and only if it is not virtually nilpotent (see, for example, [R]). The goal of this paper is to affirm the conjectures made in [BL] by showing that for solvable groups of exponential growth the Roth-type theorems fail in a strong way:

Theorem. *Let G be a finitely generated solvable group of exponential growth.*
(A) (non-recurrence) *There exist a measure preserving action T of G on a finite measure space $(X,\mathcal{B},\mu)$, elements $a_1,a_2\in G$ and a set $A\in\mathcal{B}$ with $\mu(A)>0$ such that*

$$T(a_1^n)A\cap T(a_2^n)A=\emptyset$$

for all $n\neq 0$.
(B) (non-convergence) *For any sequence of intervals $\{[l_m,r_m]\}_{m=1}^{\infty}$ with $r_m-l_m\to\infty$ there exist a measure preserving action T of G on a finite measure space $(X,\mathcal{B},\mu)$, elements $a_1,a_2\in G$ and a set $A\in\mathcal{B}$ such that*

$$\lim_{m\to\infty}\frac{1}{r_m-l_m}\sum_{n=l_m}^{r_m-1}\mu\big(T(a_1^n)A\cap T(a_2^n)A\big)$$

does not exist.

We will actually be proving the following fact, from which Theorem 1.0 clearly follows:

Theorem. *Let G be a finitely generated solvable group of exponential growth. For any partition $R\cup P=\mathbb{Z}\setminus\{0\}$ there exist an action T of G on a probability measure space $(X,\mathcal{B},\mu)$ and a set $A\in\mathcal{B}$ of positive measure such that $T(b_0)A\cap T(b_n)A=\emptyset$ whenever $n\in R$ and $\mu\big(T(b_0)A\cap T(b_n)A\big)\geq\frac{1}{6}$ for any $n\in P$.*

The main ingredient in the proof of Theorem 1.0 is a purely algebraic fact (Theorem 1.0 below) which allows us to reduce the construction of a counterexample to the case where the group G is of one of the two "standard" types. Such a reduction is possible because of the following observation:

Lemma. *Let $\tilde{G}$ be either a subgroup or a factor-group of a group G. For any measure preserving action T of $\tilde{G}$ on a probability measure space $(X,\mathcal{B},\mu)$, a set $A\in\mathcal{B}$ and elements $a_1,a_2\in\tilde{G}$ there exist a measure preserving action S of G on a probability measure space $(Y,\mathcal{D},\nu)$, a set $B\in\mathcal{D}$ and elements $b_1,b_2\in G$ such that $\nu\big(S(b_1^n)B\cap S(b_2^n)B\big)=\mu\big(T(a_1^n)A\cap T(a_2^n)A\big)$ for all $n\in\mathbb{Z}$.*

Indeed, if $\tilde{G}$ is a factor-group of G and $\eta\colon G \longrightarrow \tilde{G}$ is the factor map, we take $(Y,\mathcal{D},\nu) = (X,\mathcal{B},\mu)$, $S = T\circ\eta$, $B = A$, $b_1 \in \eta^{-1}(a_1)$ and $b_2 \in \eta^{-1}(a_2)$. If $\tilde{G}$ is a subgroup of G, we take S to be the action of G induced by T: we define $Y = \bigl\{\varphi\colon G \longrightarrow X : \varphi(ba) = T(b)\varphi(a)$ for all $b \in \tilde{G}$, $a \in G\bigr\}$ and $S(c)\colon Y \longrightarrow Y$ by $(S(c)\varphi)(a) = \varphi(ac)$, $c, a \in G$. Then $Y \simeq X^{\tilde{G}\backslash G}$, and S preserves the product measure ν on Y. The projection $\pi\colon Y \longrightarrow X$, $\pi(\varphi) = \varphi(\mathbf{1}_G)$, turns (X,T) into a factor of $(Y, S|_{\tilde{G}})$, and we take $B = \pi^{-1}(A)$, $b_1 = a_1$ and $b_2 = a_2$.

We are therefore free to replace our group G by its subgroup or a factor-group. Given a solvable group of exponential growth, we will extract from it a sub-factor-group (that is, a subgroup of a factor group) of a very special form which still has exponential growth and for which we will be able to establish Theorem 1.0. Let H be an abelian group and let $\mathbf{a}$ be an automorphism of H. We will denote by $\mathbf{a}[H]$ the extension of H by $\mathbf{a}$: $\mathbf{a}[H]$ is the group generated by H and an additional element a such that $a^{-1}ba = \mathbf{a}b$, $b \in H$. (The group $\mathbf{a}[H]$ can be seen as the set $\bigl\{a^k b :\ k \in \mathbb{Z},\ b \in H\bigr\}$ with the product defined by $(a^k b)(a^l c) = a^{k+l}(\mathbf{a}^l b)c$.) $G = \mathbf{a}[H]$ is a 2-step solvable group: H is a normal subgroup of G and G/H is the cyclic group generated by aH. Here are the descripitions of the two types of groups, obtainable in this way, that will be utilized in the proof of Theorem 1.0:

Type 1: lamplighter group. Let p be a prime integer and let H be the direct sum of countably many copies of $\mathbb{Z}_p = \mathbb{Z}/(p\mathbb{Z})$, indexed by $\mathbb{Z}$: $H = \bigoplus_{\mathbb{Z}} \mathbb{Z}_p$. Let $\ldots, b_{-1}, b_0, b_1, \ldots$ be the natural basis of H, let $b = b_0$, and let $\mathbf{a}$ be the coordinate shift on H: $\mathbf{a}b_n = b_{n+1}$, $n \in \mathbb{Z}$. Define $G = \mathbf{a}[H]$; it is a solvable group of exponential growth.

Type 2: group of affine transformations. Let V be a finite dimensional $\mathbb{Q}$-vector space, let $b \in V$ and let $\mathbf{a}$ be an invertible linear transformation of V for which b is cyclic: $\mathrm{Span}_{\mathbb{Q}}\{\mathbf{a}^n b : n \in \mathbb{Z}\} = V$. Let G be the group of affine transformations of V generated by $\mathbf{a}$ and the translation by b. In this case $G = \mathbf{a}[H]$, where H is the group generated by the vectors $\mathbf{a}^n b$, $n \in \mathbb{Z}$. G is nilpotent iff the transformation $\mathbf{a}$ is unipotent: $(\mathbf{a} - \mathrm{Id}_V)^m = 0$ for some $m \in \mathbb{N}$, and G is virtually nilpotent iff $\mathbf{a}^d$ is unipotent for some $d \in \mathbb{N}$. We will say that the automorphism $\mathbf{a}$ is *almost unipotent on* H if $\mathbf{a}^d$ is unipotent on H for some $d \in \mathbb{N}$. We will say that G is of type 2 if $\mathbf{a}$ is not almost unipotent on H and so, G is a solvable group of exponential growth.

Theorem. *Let G be a finitely generated solvable group of exponential growth. Then G has a sub-factor-group of either type 1 or of type 2.*

Theorem 1.0 is proved in the next section; Section 3 is devoted to the proof of Theorem 1.0.

2. Proof of Theorem 1.0.

We first introduce some notation. Let G be a group. The subgroup of G generated by (the union of) subsets $S_1, \ldots, S_k \subseteq G$ will be denoted by $\langle S_1, \ldots, S_k\rangle$. For $a, b \in G$ let $[b,a] = b^{-1}a^{-1}ba$ and $[b,a]_m = \bigl[[b,a]_{m-1}, a\bigr]$ for $m = 2, 3, \ldots$ We will say that $a \in G$ is *Engel with respect to* $b \in G$ if there exists $m \in \mathbb{N}$ such that $[b,a]_m = \mathbf{1}_G$, and that a *is Engel with respect to* $S \subseteq G$ if a is Engel with respect to each $b \in S$. When a is Engel with respect to G it is said to be *an Engel element of* G.

Theorem. (See, for example, [Sch]VI.8.g) *The Engel elements of a solvable group G form the maximal locally nilpotent normal subgroup of G,* the Hirsch-Plotkin radical of G.

(A group is *locally nilpotent* if all of its finitely generated subgroups are nilpotent.)

We will say that $a \in G$ is *almost Engel with respect to* $b \in G$ if there exists $d \in \mathbb{N}$ such that a^d is Engel with respect to b, and that a *is almost Engel with respect to* $S \subseteq G$ if a is almost Engel with respect to each $b \in S$.

Let F be a normal subgroup of G. We will not distinguish between an element $a \in G$ and the coset $aF \in G/F$. We will say that a *is (almost) Engel with respect to* $b \in G$ *modulo* F if a is (almost) Engel with respect to b in G/F, that is, if there exists $m \in \mathbb{N}$ such that $[b, a]_m \in F$ (respectively, $[b, a^d]_m \in F$ for some $d \in \mathbb{N}$). Let G have solvability class r and let $G_1, \ldots, G_r$ be the commutator subgroups of G: $G_1 = G$ and $G_{i+1} = [G_i, G_i]$ for $i = 1, 2, \ldots, r$, with $G_{r+1} = \{\mathbf{1}_G\}$. We will prove the following:

Proposition. *Let G be a finitely generated solvable group of class r such that for any $a \in G$ and any $i \leq r$, a is almost Engel with respect to G_i modulo G_{i+1}. Then G is virtually nilpotent.*

In Lemmas 2.0–2.0 below we keep the assumptions of Proposition 2.0, namely that for any $i = 1, \ldots, r$, any $a \in G$ is almost Engel with respect to any $b \in G_i \setminus G_{i+1}$ modulo G_{i+1}. For each $a \in G$ and $b \in G_i \setminus G_{i+1}$ we may therefore fix $d(a, b), m(a, b) \in \mathbb{N}$ such that $\left[b, a^{d(a,b)}\right]_{m(a,b)} \in G_{i+1}$.

Lemma. *If $a \in G$ is Engel with respect to $b \in G$, then a^k is Engel with respect to b for every $k \in \mathbb{Z}$.*

Proof. a is Engel with respect to the solvable group $H = \langle a, b \rangle$, and by Theorem 2.0, Engel elements of H form a group. ■

Lemma. *Let $a \in G$, let S be a finite subset of G and let $H = \langle a, S \rangle$. There exists a finite set $S' \subseteq [H, H]$ such that the group $H' = \langle S, S' \rangle$ is normal in H.*

Proof. We put $R_1 = R_{-1} = S \cup S^{-1}$,

$$R_i = \left\{ [c, a^{d(a,c)}]_{m(a,c)} : \ c \in R_{i-1} \right\}, \ i = 2, \ldots, r+1;$$
$$P_i = \left\{ [c, a^{d(a,c)}]_n : \ c \in R_i, \ n = 1, \ldots, m(a, c) - 1 \right\}, \ i = 1, \ldots, r;,$$
$$S_i = \left\{ [c, a^k], \ \left[[c, a^{d(a,c)}]_n, a^k \right] : \ c \in R_i, \ n = 1, \ldots, m(a, c) - 1, \ k = 1, \ldots, d(a, c) - 1 \right\}, \ i = 1, \ldots, r;$$
$$R_{-i} = \left\{ [c, a^{-d(a^{-1},c)}]_{m(a^{-1},c)} : \ c \in R_{-(i-1)} \right\}, \ i = 2, \ldots, r+1;$$
$$P_{-i} = \left\{ [c, a^{-d(a^{-1},c)}]_n : \ c \in R_{-i}, \ n = 1, \ldots, m(a^{-1}, c) - 1 \right\}, \ i = 1, \ldots, r;,$$
$$S_{-i} = \left\{ [c, a^{-k}], \ \left[[c, a^{-d(a^{-1},c)}]_n, a^{-k} \right] : \ c \in R_{-i}, \ n = 1, \ldots, m(a^{-1}, c) - 1, \ k = 1, \ldots, d(a^{-1}, c) - 1 \right\}, \ i = 1, \ldots, r;$$

and $S' = \bigcup_{i=2}^{r}(R_i \cup R_{-i}) \cup \bigcup_{i=1}^{r}(S_i \cup P_i \cup S_{-i} \cup P_{-i})$. We have $S' \subseteq [H,H]$. Also note that, by the definition of $d(a,c)$ and $m(a,c)$, we have $R_2, R_{-2} \in G_2$, $R_3, R_{-3} \in G_3$, etc. In particular, $R_{r+1} = R_{-(r+1)} = \{\mathbf{1}_G\}$.

We have to show that $H' = \langle S, S' \rangle$ is normal in H. Every element b of H' has form

$$b = a^{k_0} c_1 a^{k_1} c_2 \dots a^{k_{r-1}} c_r a^{k_r}, \quad \text{where } c_1, \dots, c_r \in S \cup S^{-1} \text{ and } k_0, \dots, k_r \in \mathbb{Z} \text{ with } \textstyle\sum_{i=0}^{r} k_i = 0. \tag{2.1}$$

We will check that all elements of the form (2.1) are in H'; clearly, this will imply the normality of H. It suffices to show that for any $c \in S \cup S^{-1} = R_1$ and any $k \in \mathbb{Z}$ one has $ca^k = a^k h$ with $h \in H'$, that is, $a^{-k} c a^k \in H'$.

Assume that $k > 0$; for $k < 0$ the proof is similar (one simply replaces a by a^{-1} and R_i, P_i, S_i by the corresponding R_{-i}, P_{-i}, S_{-i}). We will prove by induction on k that for any $i \leq r$ and any $c \in R_i \cup P_i$ one has $a^{-k} c a^k \in H'$. Let $c \in R_i \cup P_i$; put $d = d(a,c)$ if $c \in R_i$ and $d = d(a,c')$ if $c \in P_i$ is obtained as $[c', a^{d(a,b)}]$ with $c' \in R_i$. If $k < d$, we have $a^{-k} c a^k = c[c, a^k]$ with $[c, a^k] \in S_i$. If $k \geq d$, we have $a^{-k} c a^k = a^{-(k-d)} c [c, a^d] a^{k-d} = a^{-(k-d)} c a^{k-d} a^{-(k-d)} [c, a^d] a^{k-d}$. By induction on k, $a^{-(k-d)} c a^{k-d} \in H'$. Also, $[c, a^d] \in P_i$ or $\in R_{i+1}$, and again, by induction on k, $a^{-(k-d)} [c, a^d] a^{k-d} \in H'$. ∎

Let us remind that a group H is *polycyclic* if it possesses a finite series $\{\mathbf{1}_H\} = H_{m+1} \subset H_m \subset \ldots \subset H_1 = H$ such that for each j, H_{j+1} is a normal subgroup of H_j and H_j/H_{j+1} is cyclic. Among solvable groups, the polycyclic groups are characterized by the property that any subgroup of a polycyclic group is finitely generated.

Lemma. *G is polycylic.*

Proof. We will prove that every finitely generated subgroup H of G (in particular, G itself) is polycyclic. Let $H \subseteq G_i$ and $H = \langle a_1, \dots, a_k, S \rangle$ with $S \subseteq G_{i+1}$, $|S| < \infty$. By Lemma 2.0 there exists a finite $S' \subseteq G_{i+1}$ such that $H' = \langle a_2, \dots, a_k, S, S' \rangle$ is normal in H. By the double induction on decreasing $i = r, r-1, \dots$ and increasing $k = 1, 2, \dots$, H' is polycyclic. Since H/H' is cyclic (it is generated by a_1), H is also polycyclic. ∎

Lemma. *If an element $a \in G$ is Engel with respect to G_i modulo G_{i+1} for each $i = 1, \dots, r$, then a is Engel with respect to G.*

Proof. Take any $b \in G$. Since a is Engel with respect to G_1 modulo G_2, there exists $m_1 \in \mathbb{N}$ such that $[b,a]_{m_1} \in G_2$. Since a is Engel with respect to G_2 modulo G_3, there exists $m_2 \in \mathbb{N}$ such that $[b,a]_{m_1+m_2} = [[b,a]_{m_1}, a]_{m_2} \in G_3$. And so on, till $[b,a]_{m_1+\ldots+m_r} = \mathbf{1}_G$. ∎

Lemma. *Let $F \subseteq H$ be normal subgroups of G such that H/F is abelian and finitely generated. If $a \in G$ is Engel modulo F with respect to a set of generators of H/F, then a is Engel with respect to H modulo F.*

Proof. The mapping $b \mapsto [b,a]$ induces a self-homomorphism $\tau\colon H/F \longrightarrow H/F$, and $\tau^m(b) = [b,a]_m \bmod F$, $m \in \mathbb{N}$. Let $b_1, \dots, b_s$ be generators of H/F, let m_j, $j = 1, \dots, s$, be such that $[b_j, a]_{m_j} \in F$, and let $m = \max\{m_1, \dots, m_s\}$. Then $\tau^m(b_j) = [b_j, a]_m = \mathbf{1}_{H/F}$ for all $j = 1, \dots, s$, and so, is trivial on H/F. ∎

Proof of Proposition 2.0. Since G is polycyclic by Lemma 2.0, every subgroup of G is finitely generated. Take $i \le r$; let G_i be generated by $b_1, \ldots, b_s$ and let $d_i = \prod_{j=1}^{s} d(a, b_j)$. By Lemma 2.0, a^{d_i} is Engel modulo G_{i+1} with respect to $b_1, \ldots, b_s$. By Lemma 2.0, a^{d_i} is Engel with respect to G_i modulo G_{i+1}.

Now let $d = \prod_{i=1}^{r} d_n$. By Lemma 2.0, a^d is Engel with respect to G_i modulo G_{i+1} for every i. By Lemma 2.0, a^d is Engel with respect to G.

Let E be the Hirsch-Plotkin radical of G (see Theorem 2.0 above). E is a locally nilpotent group, and is finitely generated by Lemma 2.0; hence, E is nilpotent. We have shown that for any $a \in G$ there exists $d \in \mathbb{N}$ such that $a^d \in E$, that is, all elements of G/E have finite orders. Since G/E is polycyclic, this implies that G/E is finite. Hence, G is virtually nilpotent. ∎

Proof of Theorem 1.0. Let G be a finitely generated solvable group of exponential growth. By Proposition 2.0 there exist $a \in G$, $i \in \mathbb{N}$ and $b \in G_i$ such that a is not almost Engel with respect to b modulo G_{i+1}. Put $\tilde{G} = \langle a, b \rangle / G_{i+1}$. Clearly, the group $H = \langle a^{-n} b a^n,\ n \in \mathbb{Z} \rangle$ is normal in $\tilde{G}$, and $\tilde{G}/H = \langle a \rangle$. Since $b \in G_i$ and G_i is normal in G, $H \subseteq G_i/G_{i+1}$ and so, is abelian. The element a acts on H by conjugation, $c \mapsto a^{-1}ca$ for $c \in H$. Let us use additive notation for H and denote the action of a on H by $\mathbf{a}$: $\mathbf{a}c = a^{-1}ca$, $c \in H$. This turns H into a $\mathbb{Z}[\mathbf{a}, \mathbf{a}^{-1}]$-module; as such, H is spanned by a single element b and so, has rank 1. Since $\mathbb{Z}[\mathbf{a}, \mathbf{a}^{-1}]$ is a Noetherian ring, H is a Noetherian module. In $\tilde{G}$, a is not almost Engel with respect to b; in additive notation this means that $(\mathbf{a}^d - \mathrm{Id}_H)^m b \neq 0$ for all $m, d \in \mathbb{N}$, and so, $\mathbf{a}$ is not almost unipotent on H.

If H has torsion, we represent H as a tower $0 = H_0 \subset H_1 \subset \ldots \subset H_k = H$, where for each $i = 1, \ldots, k$, $N_i = H_i/H_{i-1}$ is a $\mathbb{Z}[\mathbf{a}, \mathbf{a}^{-1}]$-module of rank 1 and either is torsion free or is annihilated by a prime integer p: $pN_i = 0$. (Such a tower exists since H is Noetherian.) If $\mathbf{a}$ were almost unipotent on each of $N_1, \ldots, N_k$, then $\mathbf{a}$ would be almost unipotent on H. Let us replace H by one of $N_1, \ldots, N_k$ on which $\mathbf{a}$ is not almost unipotent, and denote by b a generator of H over $\mathbb{Z}[\mathbf{a}, \mathbf{a}^{-1}]$. We have two cases:

1) H is annihilated by a prime integer p: $pH = 0$. Then H is a $\mathbb{Z}_p$-vector space. Put $b_n = \mathbf{a}^n b$, $n \in \mathbb{Z}$. If $\ldots, b_{-1}, b_0, b_1, \ldots$ are linearly dependent over $\mathbb{Z}_p$ then, since $\mathbf{a}$ is an automorphism of H, H has finite dimension over $\mathbb{Z}_p$ and so, is finite. In this case $\mathbf{a}$ is almost unipotent, since some its power is identical. Hence, there is no relations between b_n, $n \in \mathbb{Z}$, and so, $H \simeq \mathbb{Z}_p[\mathbf{a}, \mathbf{a}^{-1}]$. The group $\langle H, a \rangle = \mathbf{a}[H]$ is therefore a group of type 1, a lamplighter group.

2) H is torsion-free. Again, let $b_n = \mathbf{a}^n b$, $n \in \mathbb{Z}$. If $\ldots, b_{-1}, b_0, b_1, \ldots$ are linearly independent over $\mathbb{Z}$, then $H \simeq \mathbb{Z}[\mathbf{a}, \mathbf{a}^{-1}]$; by factorizing H by $2H$ we turn it into $\mathbb{Z}_2[\mathbf{a}, \mathbf{a}^{-1}]$, and $\langle H, a \rangle = \mathbf{a}[H]$ into the corresponding lamplighter group. If b_n, $n \in \mathbb{Z}$, are linearly dependent over $\mathbb{Z}$, the $\mathbb{Q}$-vector space $V = H \otimes \mathbb{Q}$ is finite dimensional. Since H has no torsion, the natural mapping $H \longrightarrow V$ is an embedding. It follows that the action of $\mathbf{a}$ on V is not almost unipotent and so, the group $\langle H, a \rangle = \mathbf{a}[H]$ is of type 2. ∎

3. Proof of Theorem 1.0.

In light of Lemma 1.0 and Theorem 1.0, the proof of Theorem 1.0 is reduced to the case where G is a group of either type 1 or 2. In both cases $G = \mathbf{a}[H]$, where H is an abelian group and $\mathbf{a}$ is an automorphism of H possessing a cyclic element $b \in H$. Denoting the element of G corresponding to $\mathbf{a}$ by a, we have $\mathbf{a}c = a^{-1}ca$ for any $c \in H$.

We take $a_1 = a^d$, $a_2 = ba^db^{-1}$ and for $n \in \mathbb{Z}$ put $b_n = \mathbf{a}^{dn}b = a^{-dn}ba^{dn}$, with a nonzero integer d to be specified later. Then for any measure preserving action T of G on a measure space $(X, \mathcal{B}, \mu)$ and a set $A \in \mathcal{B}$ one has

$$\begin{aligned} T(a_1^n)A \cap T(a_2^n)A = T(a_1^n)\big(A \cap T(a_1^{-n}a_2^n)A\big) = T(a_1^n)\big(A \cap T(a^{-dn}ba^{dn}b^{-1})A\big) \\ = T(a_1^n)\big(A \cap T(b_nb^{-1})A\big) = T(a_1^n)\big(A \cap T(b_nb_0^{-1})A\big), \quad n \in \mathbb{Z}. \end{aligned}$$

When dealing solely with H we will use the additive notation, so that $b_nb_0^{-1}$ becomes $b_n - b_0$.

Let $R \cup P$ be a partition of $\mathbb{Z} \setminus \{0\}$. In view of Lemma 1.0 it is enough to construct a measure preserving action T of H and a set A of positive measure such that

$$A \cap T(b_n - b_0)A = \emptyset \text{ for } n \in R \text{ and } \mu\big(A \cap T(b_n - b_0)A\big) \geq \tfrac{1}{6} \text{ for } n \in P. \tag{3.1}$$

We define T to be an action of H by rotations on $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, identified with $[0, 1)$ and equipped with the standard Lebesgue measure, and $A = \big[0, \frac{1}{3}\big)$. Namely, let $T(c)x = x + \alpha(c)$, $c \in H$, $x \in \mathbb{S}$, where α is a homomorphism from H to $\mathbb{S}$, that is, a character of H. Denote $\alpha_n = \alpha(b_n)$, $n \in \mathbb{Z}$, then the condition (3.1) takes the form

$$|\alpha_n - \alpha_0| \geq \tfrac{1}{3} \text{ for } n \in R \text{ and } |\alpha_n - \alpha_0| \leq \tfrac{1}{6} \text{ for } n \in P, \tag{3.2}$$

where for $x \in \mathbb{S}$ we denote $|x| = \min\{x, 1 - x\}$.

First let G have type 1, that is, $G = \mathbf{a}[H]$ where $H = \bigoplus_{\mathbb{Z}} \mathbb{Z}_p$ with p a prime integer, and $\mathbf{a}$ acts on H as the coordinate shift. We put $d = 1$, then $\{\dots, b_{-1}, b_0, b_1, \dots\}$ is the standard basis in H over $\mathbb{Z}_p$. Therefore the only restriction on the choice of elements $\alpha_n \in \mathbb{S}$ is $p\alpha_n = 0$, $n \in \mathbb{Z}$. To satisfy (3.2), we put $\alpha_n = 0$ for $n = 0$ and $n \in P$, and $\alpha_n = \frac{1}{2}$ if $p = 2$, $\alpha_n = \frac{p-1}{2p}$ if $p \geq 3$ for $n \in R$.

Now assume that G is of type 2, that is, assume that $\mathbf{a}$ is a non-almost unipotent automorphism of a finite dimensional $\mathbb{Q}$-vector space V, $b \in V$ is cyclic for $\mathbf{a}$ and $H = \langle \mathbf{a}^n b \rangle_{n \in \mathbb{Z}}$. Let $p(t) = m_rt^r + m_{r-1}t^{r-1} + \ldots + m_0$ be the minimal polynomial of $\mathbf{a}^d$, which we normalize so that $m_0, \dots, m_r$ are integers, $\gcd(m_0, \dots, m_r) = 1$ and $m_r > 0$.

We say that a sequence $\{\alpha_n\}_{n \in \mathbb{Z}}$ in $\mathbb{S}$ is *admissible* if $\alpha_n = \alpha(b_n)$, $n \in \mathbb{Z}$, for some character α of H. $\{\alpha_n\}_{n \in \mathbb{Z}}$ is admissible if α_n satisfy every relation with integer coefficients that b_n satisfy. Let $k_{-N}, \dots, k_N$ be integers and let $q(t) = \sum_{i=0}^{2N} k_i t^{N+i}$. Then one has $k_{-N}b_{-N} + \ldots + k_Nb_N = k_{-N}\mathbf{a}^{-N}b + \ldots + k_N\mathbf{a}^Nb = 0$ iff $q(\mathbf{a}^d)b = 0$. Since b is cyclic for $\mathbf{a}$ this implies $q(\mathbf{a}^d) = 0$, and thus $q(t) = p(t)q_1(t)$, where q_1 has integer coefficients since the content of $p(t)$ is 1. It follows that $\{\alpha_n\}_{n \in \mathbb{Z}}$ is admissible iff α_n satisfy the induction relation

$$m_r\alpha_{n+r} + m_{r-1}\alpha_{n+r-1} + \ldots + m_0\alpha_n = 0 \tag{3.3}$$

for all $n \in \mathbb{Z}$.

We consider two cases.

Case 1: all eigenvalues of $\mathbf{a}$ have modulus 1. After an appropriate choice of $d \in \mathbb{N}$ we may assume that $m_r = m(d)_{r(d)} \geq 3$. Indeed, the assumption that $\mathbf{a}$ is not almost unipotent means that not all eigenvalues of $\mathbf{a}$ are roots of unity. The Kronecker lemma (which states: *an algebraic integer of modulus* 1 *whose every conjugate has modulus* 1 *is a root of unity*) implies that there is an eigenvalue λ of $\mathbf{a}$ that is not an algebraic integer. Let M be the set of algebraic integers contained in the field $\mathbb{Q}(\lambda)$. M is a finitely generated $\mathbb{Z}$-module, and for any value of d we have $m(d)_{r(d)}\lambda^d \in M$. Thus, if $m(d)_{r(d)} \leq 2$ for all $d \in \mathbb{N}$, then all powers λ^d, $d \in \mathbb{N}$, of λ are contained in the finitely generated $\mathbb{Z}$-module $\frac{1}{2}M$, which contradicts the choice of λ.

Since all roots of $p(t)$ have modulus 1 we have $|m_0| = m_r \geq 3$. For $n = 0, \ldots, r-1$ put $\alpha_n = 0$ if $n = 0$ or $n \in P$ and $\alpha_n = \frac{1}{2}$ if $n \in R$. Then we can choose by induction α_n for $n \geq r$ and $n < 0$ according to (3.3) each in the corresponding interval of length $\frac{1}{3}$ in order that (3.2) be satisfied.

Case 2: $\mathbf{a}$ has an eigenvalue of modulus $\neq 1$. By taking d, either positive or negative, large enough we get that $p(t)$ has a root λ with $|\lambda| \geq 7$. The following lemma, with $\delta = \frac{1}{12}$, $\beta_n = 0$ for $n = 0$ and $n \in P$ and $\beta_n = \frac{1}{2}$ for $n \in R$, yields the desired sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$.

Lemma. *Let $\delta > 0$ and assume that $p(t)$ has a root $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1 + \frac{1}{2\delta}$. Then for every sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ in $\mathbb{S}$ there exists an admissible sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ such that $|\alpha_n - \beta_n| \leq \delta$ for all $n \in \mathbb{Z}$.*

Proof. In view of the compactness of the set of admissible sequences with respect to the pointwise convergence it is enough to show that, given $N \in \mathbb{N}$, there exists an admissible sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ such that $|\alpha_n - \beta_n| \leq \delta$ for $-N \leq n \leq N$.

Build a finite sequence $\{z_n\}_{n=-N}^{N}$ in $\mathbb{C}$ satisfying

$$|z_n - z_{n-1}| \leq \tfrac{1}{2}|\lambda|^n \text{ for } -N < n \leq N \text{ and } \mathrm{Re}(\lambda^n z_n) \bmod 1 = \beta_n \text{ for } -N \leq n \leq N$$

in the following way. Choose first z_{-N} with $\mathrm{Re}(\lambda^{-N} z_{-N}) \bmod 1 = \beta_{-N}$. Assuming that $z_{-N}, \ldots, z_{n-1}$ are defined take $y_n \in \mathbb{R}$ with $\left|y_n - \mathrm{Re}(\lambda^n z_{n-1})\right| \leq \frac{1}{2}$ and $y_n \bmod 1 = \beta_n$. Define $z_n = z_{n-1} + \lambda^{-n}\big(y_n - \mathrm{Re}(\lambda^n z_{n-1})\big)$, then $|z_n - z_{n-1}| \leq \frac{1}{2}|\lambda|^n$ and $\mathrm{Re}(\lambda^n z_n) \bmod 1 = \beta_n$.

Now take $z = z_N$ and put $\alpha_n = \mathrm{Re}(\lambda^n z) \bmod 1 \in \mathbb{S}$, $n \in \mathbb{Z}$. The sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ is then admissible, and we have

$$|\lambda^n z - \lambda^n z_n| \leq |\lambda^n(z - z_n)| \leq \frac{1}{2}\big(|\lambda|^{-1} + \ldots + |\lambda|^{-(N-n)}\big) \leq \frac{1}{2(|\lambda| - 1)} \leq \delta,$$

and so $|\alpha_n - \beta_n| \leq \delta$ for $-N \leq n \leq N$. ∎

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