WHAT IS HYPERBOLIC GEOMETRY?

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Euclid's five postulates of plane geometry are stated in [1, Section 2] as follows.

(1) Each pair of points can be joined by one and only one straight line segment.
(2) Any straight line segment can be indefinitely extended in either direction.
(3) There is exactly one circle of any given radius with any given center.
(4) All right angles are congruent to one another.
(5) If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which the angles are less than two right angles.

Mathematicians began in the 19th century to investigate the consequences of denying the fifth postulate, which is equivalent to the postulate that for any point off a given line there is a unique line through the point parallel to the given line, rather than trying to deduce it from the other four. Hyperbolic geometry, in which the parallel postulate does not hold, was discovered independently by Bolyai and Lobachesky as a result of these investigations. In this note we describe various models of this geometry and some of its interesting properties, including its triangles and its tilings.

1. Points and lines

We begin by giving a particular description of n dimensional hyperbolic space: the *hyperboloid model.* The *points* are the members of the set

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}$$

where $x = (x_1, \ldots, x_{n+1})$. In terms of the Lorentz bilinear form

$$\langle x, y\rangle_{\mathsf{L}} = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1} \tag{1.1}$$

the points of $\mathbb{H}^n$ are the points $x \in \mathbb{R}^{n+1}$ for which $\langle x, x\rangle_{\mathsf{L}}^2 = -1$ and $x_{n+1} > 0$. Write

$$\langle x, y\rangle_{\mathsf{E}} = x_1 y_1 + \cdots + x_n y_n \tag{1.2}$$

for the Euclidean norm on $\mathbb{R}^n$.

Lines are defined by the points they contain: a *line* in $\mathbb{H}^n$ is any non-empty set of the form $\mathbb{H}^n \cap P$ where P is a two-dimensional plane in $\mathbb{R}^{n+1}$ that passes through the origin. (See Figure 1.) It is immediate that, given any two distinct points, there is a unique line containing both points. Note also that a line is always a smooth curve $\mathbb{R} \to \mathbb{R}^{n+1}$.

Lines L_1 and L_2 in $\mathbb{H}^2$ are *parallel* if they are disjoint. In stark contrast with Euclidean geometry, for any line L and any point x not in L, there are infinitely many lines passing through x that are parallel to L.

Example 1.3. Consider the collection $\mathcal{L}$ of lines passing through $(0,0,1)$ and defined by a plane containing the point $(t,1,0)$ for some $-1 \le t \le 1$. If α is big enough then the line passing through $(0, \alpha, \sqrt{1+\alpha^2})$ and defined by the plane with normal $\langle 0, -\sqrt{1+\alpha^2}, \alpha\rangle$ is parallel to every line in $\mathcal{L}$.

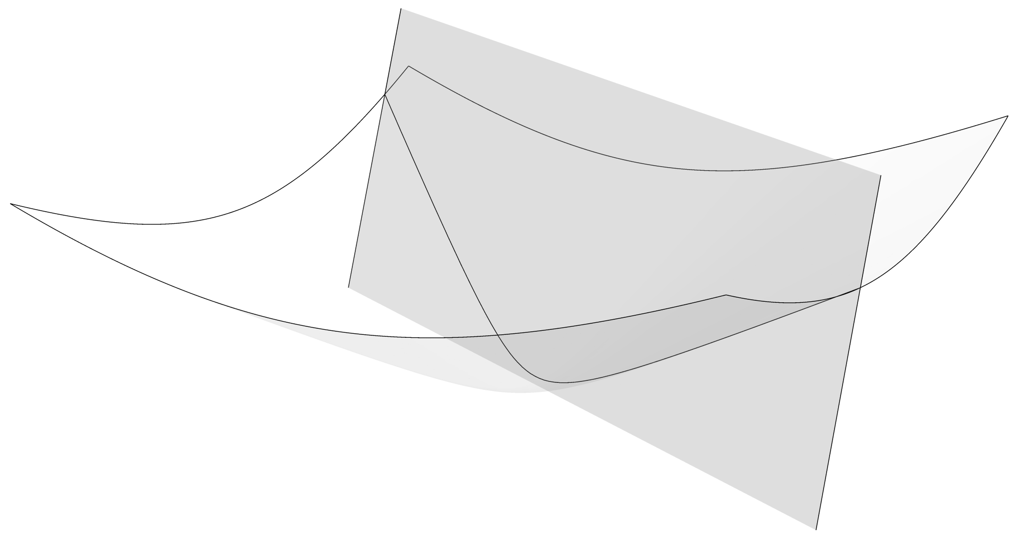

FIGURE 1. A line in the hyperboloid model $\mathbb{H}^2$ defined by a plane in $\mathbb{R}^3$.

2. The hyperbolic line

The hyperbolic line $\mathbb{H}^1 = \{x \in \mathbb{R}^2 : x_1^2 - x_2^2 = -1, x_2 > 0\}$ is analogous to the circle: both consist of vectors of a fixed length with respect to some bilinear form on $\mathbb{R}^2$. Suppose that $p : \mathbb{R} \to \mathbb{H}^1$ is a smooth curve with $p(0) = (0, 1)$. We have $p_1(t)^2 - p_2(t)^2 = -1$ and therefore

$$\langle p(t), p'(t)\rangle_{\mathsf{L}} = p_1(t)p_1'(t) - p_2(t)p_2'(t) = 0$$

for all t upon differentiating. Thus the vectors $(p_1'(t), p_2'(t))$ and $(p_2(t), p_1(t))$ are parallel. This lets us write

$$(p_1'(t), p_2'(t)) = k(t)(p_2(t), p_1(t)) \tag{2.1}$$

for some function $k(t)$. Suppose now that our curve has constant speed. Then

$$p_2'(t)^2 - p_1'(t)^2 = k(t)^2(p_2(t)^2 - p_1(t)^2) = k(t)^2$$

is constant and by a linear transformation in $\mathbb{R}$ we can assume that $k(t) = 1$ for all $t \in \mathbb{R}$. Thus (2.1) implies that p satisfies the differential equation

$$\begin{pmatrix} p_1'(t) \\ p_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$$

with the initial condition $p_1(0) = 0, p_2(0) = 1$. The solution of this initial value problem is

$$\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \left[\exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}$$

where

$$\begin{aligned} \sinh t &= t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots = \frac{e^t - e^{-t}}{2} \\ \cosh t &= 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = \frac{e^t + e^{-t}}{2} \end{aligned} \tag{2.2}$$

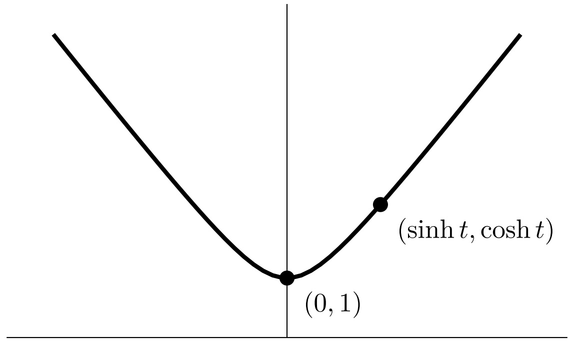

FIGURE 2. The hyperbolic line $\mathbb{H}^1$.

for all $t \in \mathbb{R}$. Thus $\mathbb{H}^1$ can be parameterized using the hyperbolic trigonometric functions sinh and cosh as shown in Figure 2. Putting $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we obtain from

$$\begin{pmatrix} \cosh(t+s) & \sinh(t+s) \\ \sinh(t+s) & \cosh(t+s) \end{pmatrix} = \exp((t+s)J) = \exp(tJ)\exp(sJ) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}$$

the expected addition formulae.

The hyperbolic trigonometric functions cosh and sinh are analogous to the trigonometric functions cos and sin. The matrix

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

is a *hyperbolic rotation*. Just as circular rotations preserve areas of sectors, the hyperbolic rotations preserve areas of hyperbolic sectors, where a *hyperbolic sector* is any region in $\mathbb{R}^2$ bounded by $\mathbb{H}^1$ and two distinct lines from the origin to $\mathbb{H}^1$. The area bounded by $\mathbb{H}^1$, the x_2 axis and the line between $(0,0)$ and $(\sinh t, \cosh t)$ can easily be calculated as follows

$$\int_0^{\sinh t} \sqrt{1+x_1^2}\,\mathrm{d}x_1 - \frac{\sinh t \cosh t}{2} = \int_0^t (\cosh u)^2\,\mathrm{d}u - \frac{\sinh(2t)}{4} = \frac{1}{2}\int_0^t 1 + \cosh(2t)\,\mathrm{d}u - \frac{\sinh t \cosh t}{2} = \frac{t}{2}$$

where we have used the substitution $u = \operatorname{arcsinh} x_1$. Combined with the addition formulas, this implies that hyperbolic rotations preserve the areas of hyperbolic sectors.

3. Angles and distances

In this section we describe how to measure the distance between two points in $\mathbb{H}^n$. We begin by measuring the lengths of smooth curves in $\mathbb{H}^n$. This is done by integrating the speed, which is measured using the inner product $\langle \cdot, \cdot \rangle_{\mathsf{L}}$, along the curve. For this to make sense we need the following lemma. Write

$$\langle x, y \rangle_{\mathsf{E}} = x_1 y_1 + \cdots + x_n y_n$$

for the Euclidean inner product on $\mathbb{R}^n$.

Lemma 3.1. *Let $\phi : [a,b] \to \mathbb{H}^n$ be a smooth curve. Then $\langle \phi'(t), \phi'(t) \rangle_{\mathsf{L}} \geq 0$ for all $a < t < b$.*

Proof. Write $\phi(t) = (\phi_1(t), \ldots, \phi_{n+1}(t))$ and $\psi(t) = (\phi_1(t), \ldots, \phi_n(t))$. Since $\phi(t) \in \mathbb{H}^n$ for all t we have

$$\phi_1(t)^2 + \cdots + \phi_n(t)^2 - \phi_{n+1}(t)^2 = -1 \tag{3.2}$$

and therefore

$$2\phi_1(t)\phi_1'(t) + \cdots + 2\phi_n(t)\phi_n'(t) - 2\phi_{n+1}(t)\phi_{n+1}'(t) = 0 \tag{3.3}$$

for all $a < t < b$ upon differentiating. In other words $\langle \phi(t), \phi'(t) \rangle_{\mathsf{L}} = 0$ for all $a < t < b$.

Fix $a < t < b$. If $\phi'_{n+1}(t) = 0$ then

$$\langle \phi'(t), \phi'(t) \rangle_{\mathsf{L}} = \langle \psi'(t), \psi'(t) \rangle_{\mathsf{E}} \geq 0$$

and the result is immediate, so assume otherwise. We have

$$\langle \psi(t), \psi(t) \rangle_{\mathsf{E}} \cdot \langle \psi'(t), \psi'(t) \rangle_{\mathsf{E}} \geq \langle \psi(t), \psi'(t) \rangle_{\mathsf{E}}^2 = \left(\phi_{n+1}(t) \phi'_{n+1}(t) \right)^2$$

by the Cauchy-Schwarz inequality and (3.3) so

$$\langle \psi(t), \psi(t) \rangle_{\mathsf{E}} \cdot \langle \psi'(t), \psi'(t) \rangle_{\mathsf{E}} \geq (1 + \langle \psi(t), \psi(t) \rangle_{\mathsf{E}})\, \phi'_{n+1}(t)^2$$

upon using (3.2). Rearranging gives

$$\langle \psi(t), \psi(t) \rangle_{\mathsf{E}} \cdot \langle \phi'(t), \phi'(t) \rangle_{\mathsf{L}} \geq \phi'_{n+1}(t)^2 > 0$$

which implies $\langle \psi(t), \psi(t) \rangle_{\mathsf{E}} \neq 0$ so $\langle \phi'(t), \phi'(t) \rangle_{\mathsf{L}} \geq 0$. □

We now define

$$\ell(\phi) = \int_a^b \langle \phi'(t), \phi'(t) \rangle_{\mathsf{L}}^{1/2} \, \mathrm{d}t$$

to be the *length* of a smooth curve $\phi : [a, b] \to \mathbb{H}^n$. Put another way

$$(\mathrm{d}s)^2 = (\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_n)^2 - (\mathrm{d}x_{n+1})^2 \tag{3.4}$$

is the infinitesimal arc length in $\mathbb{H}^n$. The distance between two points is the length of the shortest path between them: define

$$\mathsf{d}(x, y) = \inf\{\ell(\phi) : \phi : [a, b] \to \mathbb{H}^n \text{ smooth}, \phi(a) = x, \phi(b) = y\}$$

for all x, y in $\mathbb{H}^n$. This is a metric on $\mathbb{H}^n$.

It is natural to ask what the *geodesics* of this metric are. That is, given distinct points in $\mathbb{H}^n$, is there a smooth curve realizing the distance between them? In fact, the geodesics in $\mathbb{H}^n$ are precisely the lines defined in Section 1. One can show that this would not be the case if we measured the lengths of curves using the Euclidean inner product: the geodesics in $\mathbb{H}^n$ for (1.2) are not the lines defined in Section 1.

The inner product $\langle \cdot, \cdot \rangle_{\mathsf{L}}$ also gives us a way to measure angles between intersecting curves. Suppose ϕ and ψ are curves in $\mathbb{H}^n$ that intersect at some time t. Since the Lorentz form is positive definite on tangent vectors (by Lemma 3.1) we can define by

$$\cos\theta = \frac{\langle \phi'(0), \psi'(0) \rangle_{\mathsf{L}}}{\|\phi'(0)\|_{\mathsf{L}} \|\psi'(0)\|_{\mathsf{L}}} \tag{3.5}$$

the angle θ between ϕ and ψ at $t = 0$. This measurement of angles is not the usual one: away from $(0, \ldots, 0, 1)$, the Euclidean angle between the vectors $\phi'(0)$ and $\psi'(0)$ in $\mathbb{R}^{n+1}$ will not equal to the angle determined by (3.5). In the next section, we will see some models of hyperbolic space that are *conformal*, which means that the angles we measure with our Euclidean protractors are the same as the angles determined by the hyperbolic geometry we are studying.

4. Models

There are many other models of n dimensional hyperbolic space. Most can be obtained from the hyperboloid model by some geometric projection in $\mathbb{R}^{n+1}$. See Figure 5 in [1] for a schematic of how the various projections are related. We begin by describing two conformal models.

4.1. **The Poincaré ball model.** The points of the Poincaré ball model are the points in the open unit ball

$$\mathbb{D}^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 < 1, x_{n+1} = 0\}$$

of radius 1. The map between $\mathbb{H}^n$ and $\mathbb{D}^n$ is the central projection from the point $(0, \ldots, 0, -1)$. This is the map that identifies a point $x \in \mathbb{H}^n$ with the one and only point on $\mathbb{D}^n$ that lies on the line connecting x with $(0, \ldots, 0, -1)$. One can easily check that this projection corresponds to the map

$$(x_1, \ldots, x_n, x_{n+1}) \mapsto \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{x_{n+1} + 1}, 0 \right)$$

from $\mathbb{H}^n$ to $\mathbb{D}^n$. It is also possible to check that lines in $\mathbb{H}^n$ correspond to diameters of $\mathbb{D}^n$ and arcs in $\mathbb{D}^n$ of circles that are orthogonal to the boundary of $\mathbb{D}^n$. The arc length in $\mathbb{D}^n$ is given by

$$(\mathrm{d}s)^2 = \frac{(2\,\mathrm{d}x_1)^2 + \cdots + (2\,\mathrm{d}x_n)^2}{\left(1 - (x_1^2 + \cdots + x_n^2)\right)^2}$$

which, as a scalar multiple of the Euclidean arc length $(\,\mathrm{d}x_1)^2 + \cdots + (\,\mathrm{d}x_n)^2$ at each point on $\mathbb{D}^n$, is a *conformal* metric. One consequence of being conformal is that, for each $x \in \mathbb{D}^n$, the angle between two tangent vectors v and w at x as measured by the inner product

$$\langle v, w \rangle_x = \frac{4v_1w_1 + \cdots + 4v_nw_n}{(1 - (x_1^2 + \cdots + x_n^2))^2}$$

agrees with the Euclidean angle between v and w.

Three hyperbolic geodesics in the Poincaré ball model of hyperbolic space are shown in Figure 3. The three intersection points define a triangle. By drawing the Euclidean straight lines between the intersection points we can see that the sum of the interior angles of the triangle is strictly less that 180°. We will see later that as a consequence $\mathbb{H}^2$ has a much richer family of tilings by regular polygons than $\mathbb{R}^2$ does.

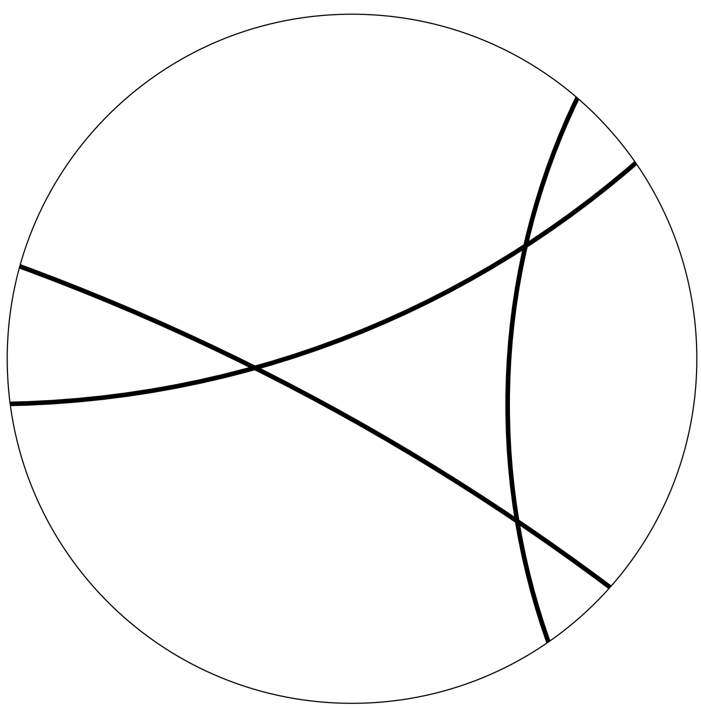

FIGURE 3. Three hyperbolic geodesics in the Poincaré ball model of the hyperbolic plane. The three intersection points define a triangle.

4.2. **The Poincaré half plane model.** The Poincaré half space model is another conformal model of $\mathbb{H}^n$. As a set it is

$$\mathbb{U}^n = \{x \in \mathbb{R}^{n+1} : x_1 = 1, x_{n+1} > 0\}$$

and the map from $\mathbb{U}^n$ to $\mathbb{D}^n$ can be described as a composition of two projections: the first is a central projection from $\mathbb{D}^n$ onto the upper half-ball

$$\{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 < 1, x_{n+1} > 0\}$$

using the point $(0, \ldots, 0, -1)$ and the second is a central projection from the upper half-ball to $\mathbb{U}^n$ using the point $(-1, 0, \ldots, 0)$. The hyperbolic metric in $\mathbb{D}^n$ is

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}x_2)^2 + \cdots + (\mathrm{d}x_{n+1})^2}{x_{n+1}^2}$$

so the half space model is conformal.

When $n = 2$ we can identify $\mathbb{D}^2$ with the interior of the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and identify $\mathbb{U}^2$ with the upper half-plane $\{z \in \mathbb{C} : \operatorname{im} z > 0\}$. With these descriptions the map $\mathbb{D}^2 \to \mathbb{U}^2$ is just the fractional linear transformation

$$z \mapsto i \cdot \frac{1+z}{1-z}$$

and its inverse

$$z \mapsto \frac{z-i}{z+i}$$

is the map $\mathbb{U}^2 \to \mathbb{D}^2$. In terms of $z = x + iy$ the metric is

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{y^2}. \tag{4.1}$$

Under these maps the boundary of the unit disc corresponds to $\mathbb{R} \cup \{\infty\}$, the point 1 on the boundary of $\mathbb{D}^2$ being mapped to ∞. Since fractional linear transformations preserve angles in $\mathbb{C}$, the half space model is also conformal. Preservation of angles in $\mathbb{C}$ also implies that the geodesics in $\mathbb{U}^2$ are the vertical lines in $\mathbb{U}^2$ and the arcs of those circles whose centers lie on $\mathbb{R}$.

5. The isometry group

Given any mathematical object, it is useful to consider the isomorphisms of that object. In our context the isomorphisms are the isometries: the invertible maps that preserve all distances. When trying to prove a result that only depends on metric properties, one can use isometries to make simplifying assumptions. For example, if we know there are enough isometries to move any point to any other point, then we can use an isometry to move any given line in the Poincaré disc model so that it passes through the origin. We begin this section by considering the isometries of $\mathbb{H}^n$.

Since $\mathbb{H}^n$ and its metric were defined in terms of the Lorentz inner product (1.1), any map that preserves it is a candidate isometry of $\mathbb{H}^2$. In particular, any member of the *indefinite orthogonal group*

$$\mathsf{O}(n,1) = \{A \in \mathsf{GL}(n+1,\mathbb{R}) : \langle Ax, Ay\rangle_{\mathsf{L}} = \langle x, y\rangle_{\mathsf{L}} \text{ for all } x, y \in \mathbb{R}^{n+1}\}$$

is a candidate. However, not every member of $\mathsf{O}(n,1)$ preserves the condition $x_{n+1} > 0$ in the definition of $\mathbb{H}^n$. For example, the diagonal matrix with entries $1, \ldots, 1, -1$ interchanges the two sheets of the hyperbola $x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1$. Write

$$\mathsf{O}^+(n,1) = \{A \in \mathsf{O}(n,1) : (Ax)_{n+1} > 0 \text{ for all } x \in \mathbb{H}^n\}$$

for the subgroup of $\mathsf{O}(n,1)$ that preserves $\mathbb{H}^n$. Every linear transformation in $\mathsf{O}^+(n,1)$ is an isometry of $\mathbb{H}^n$. In fact, these are the only isometries of $\mathbb{H}^n$.

Theorem 5.1. *The isometry group of* $\mathbb{H}^n$ *is* $\mathsf{O}^+(n,1)$.

Proof. See [1, Theorem 10.1]. ∎

Write $\mathrm{Iso}(\mathbb{H}^n)$ for the isometry group of $\mathbb{H}^n$. We can write down all the isometries in $\mathrm{Iso}(\mathbb{H}^1) = \mathsf{O}^+(1,1)$.

Example 5.2. Fix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathsf{O}^+(1,1)$. Since A preserves the Lorentz form we have $\langle Ax, Ax\rangle_{\mathsf{L}} = \langle x, x\rangle_{\mathsf{L}}$ for all x in $\mathbb{R}^2$. Choosing $x = (1,1)$ gives $ab - cd = 0$. Thus the vectors (a,c) and $(b,-d)$ are Euclidean perpendicular, so (a,c) and (d,b) are parallel. We also know that $A\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is in $\mathbb{H}^1$ so we can write $b = \sinh t$ and $d = \cosh t$ for some $t \in \mathbb{R}$. Thus

$$A = \begin{pmatrix} k\cosh t & \sinh t \\ k\sinh t & \cosh t \end{pmatrix}$$

for some $k \in \mathbb{R}$. Lastly, taking $x = (1,0)$ and noting that $\langle Ax, Ax\rangle_{\mathsf{L}} = \langle x, x\rangle_{\mathsf{L}}$ gives $k = \pm 1$. In other words we have shown that

$$\left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

generates $\mathsf{O}^+(1,1)$ in $\mathsf{GL}(2,\mathbb{R})$.

We can identify n copies of $\mathsf{O}^+(1,1)$ in $\mathsf{O}^+(n,1)$, each copy fixing all but one of the coordinates $x_1, \ldots, x_n$. These copies correspond to hyperbolic rotations in the (x_i, x_{n+1}) plane. There is also a copy of the orthogonal group

$$\mathsf{O}(n) = \{A \in \mathsf{GL}(n) : \langle Ax, Ay\rangle_{\mathsf{E}} = \langle x, y\rangle_{\mathsf{E}} \text{ for all } x, y \in \mathbb{R}^n\}$$

in $\mathsf{O}^+(n,1)$ corresponding to rotations about the x_{n+1} axis in $\mathbb{R}^{n+1}$. We can use these transformations to prove that $\mathsf{O}^+(n,1)$ acts transitively on $\mathbb{H}^n$.

Proposition 5.3. *The group $\mathsf{O}^+(n,1)$ acts transitively on $\mathbb{H}^n$.*

Proof. Fix $x \in \mathbb{H}^n$. Write $y = (x_1, \ldots, x_n)$ and choose a rotation $B \in \mathsf{O}(n)$ such that $By = (0, \ldots, 0, \|y\|_{\mathsf{E}})$. Putting

$$R = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

we see that $Rx = (0, \ldots, 0, \|y\|_{\mathsf{E}}, x_{n+1})$. Write

$$(0, \ldots, 0, \|y\|_{\mathsf{E}}, x_{n+1}) = (0, \ldots, 0, \sinh t, \cosh t)$$

for some $t \in \mathbb{R}$. Let

$$A^{-1} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

and put

$$S = \begin{pmatrix} I_{n-1} & 0 \\ 0 & A \end{pmatrix}$$

so that $SRx = (0, \ldots, 0, 1)$ as desired. $\square$

When a group acts transitively, a natural question is to compute the stabilizer of a point. We next show that the stabilizer of $(0, \ldots, 0, 1)$ is the subgroup $\mathsf{O}(n)$ of rotations about the x_{n+1} axis. This lets us identify $\mathbb{H}^n$ with the coset space $\mathsf{O}^+(n,1)/\,\mathsf{O}(n)$, giving us a model of $\mathbb{H}^n$ as a homogeneous space.

In order to prove that $\mathsf{O}(n)$ is the stabilizer it is convenient to write J for the diagonal matrix with entries $(1, \ldots, 1, -1)$ and to note that $A \in \mathsf{O}(n,1)$ if and only if $A^{\mathsf{T}} J A = J$ where A^{T} is the transpose of A. Thus we can write $A^{-1} = JA^{\mathsf{T}}J$ whenever $A \in \mathsf{O}(n,1)$. Since $\mathsf{O}(n,1)$ is a group, for any $A \in \mathsf{O}(n,1)$ we have

$$J = (A^{-1})^{\mathsf{T}} J (A^{-1}) = (JA^{\mathsf{T}}J)^{\mathsf{T}} J (JA^{\mathsf{T}}J) = JAJA^{\mathsf{T}}J$$

so $J = AJA^{\mathsf{T}}$, which proves that $\mathsf{O}(n,1)$ is closed under transposition.

Proposition 5.4. *The stabilizer of $\xi = (0, \ldots, 0, 1)$ under the action of $\mathsf{O}^+(n,1)$ on $\mathbb{H}^n$ is subgroup $\mathsf{O}(n)$ of rotations about the x_{n+1} axis.*

Proof. Fix $A \in \mathsf{O}^+(n,1)$ with $A\xi = \xi$. We have $A^{-1}\xi = \xi$ so $A^{\mathsf{T}}(J\xi) = J\xi$. It follows that

$$A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

for some $n \times n$ matrix B. The fact that A preserves $\langle \cdot, \cdot \rangle_{\mathsf{L}}$ is now just the statement that B preserves $\langle \cdot, \cdot \rangle_{\mathsf{E}}$. □

The diagonal matrix in $\mathsf{O}^+(2,1)$ with diagonal entries $(-1,1,1)$ corresponds to a rotation in the plane $x_1 = 0$. In $\mathbb{H}^2$ it can be thought of as a reflection in the line determined by the plane $x_1 = 0$. Combined with transitivity of $\mathrm{Iso}(\mathbb{H}^2)$, this allows us to reflect $\mathbb{H}^2$ across any geodesic as follows: first shift the geodesic to interesct the plan $x_1 = 0$ and then use $\mathsf{O}(2)$ to place the geodesic entirely within the plane $x_1 = 0$; after applying the reflection in the plane $x_1 = 0$, undo the preceding operations.

We conclude this section by describing the isometries of the Poincaré half space model $\mathbb{U}^2$ of the hyperbolic plane. There is an action of

$$\mathsf{SL}(2,\mathbb{R}) = \{A \in \mathsf{GL}(2,\mathbb{R}) : \det A = 1\}$$

on $\mathbb{U}^2$ defined by *fractional linear transformations*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

for all z in $\mathbb{U}^2$. Note that this is well-defined because if $z = x + iy$ then the imaginary part of $\frac{az+b}{cz+d}$ is $\frac{y}{|cz+d|^2}$. One can check that this is an action by isometries, which means that each fractional linear transformation preserves the metric (4.1).

Define subgroup of $\mathsf{SL}(2,\mathbb{R})$ by

$$K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} \cong \mathsf{O}(2)$$

$$A = \left\{ \mathsf{A}^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} \cong (0,\infty)$$

$$N = \left\{ \mathsf{U}^r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbb{R} \right\} \cong \mathbb{R}$$

and note that $\mathsf{U}^r z = z + r$ and $\mathsf{A}^t z = e^{2t} z$ so $\mathsf{U}^r \mathsf{A}^t z = e^{2t} z + r$. Thus the subgroup NA acts transitively on $\mathbb{H}^2$ and that the subgroup of NA that fixes i is the trivial subgroup. It is straightforward to check that K is the stabilizer of i. By considering the natural action of $\mathsf{SL}(2,\mathbb{R})$ on ordered bases of $\mathbb{R}^2$, one can prove that $\mathsf{SL}(2,\mathbb{R}) = NAK$. Thus we can identify $\mathbb{U}^2$ with $\mathsf{SL}(2,\mathbb{R})/\mathsf{O}(2)$. This might suggest that $\mathsf{SL}(2,\mathbb{R})$ and $\mathsf{O}^+(2,1)$ are isomorphic, but this is not quite the case.

6. Area

In this section we discuss the measurement of area in the hyperbolic plane using the Poincaré half space model. From the metric on $\mathbb{U}^2$ we can compute that

$$\mathrm{dm} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

is the corresponding area form. We first show that $\mathrm{Iso}(\mathbb{U}^2)$ preserves area. This material is from [2].

Proposition 6.1. *The isometries of* $\mathbb{U}^2$ *preserve area.*

Proof. Fix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathsf{PSL}(2,\mathbb{R})$. It suffices to prove that

$$\int f \,\mathrm{dm} = \int f \circ A \,\mathrm{dm}$$

for every continuous function $f : \mathbb{U}^2 \to \mathbb{R}$ with compact support. But, writing $z = x + iy$, the right-hand side is

$$\int f\left(\frac{az+b}{cz+d}\right)\frac{1}{y^2}\,\mathrm{d}x\,\mathrm{d}y$$

and if we make the substitution $w = Az$ we get the left-hand side. □

The principle result of this section is that the area of a triangle is completely determined by its angles. Of course, this is in stark contrast with Euclidean geometry. We begin with the area of a triangle having one vertex at infinity.

Theorem 6.2. *Let a, b be points in $\mathbb{U}^2$. The are of the triangle determined by a, b and ∞ is $\pi - (\alpha + \beta)$ where α and β are the angles at a and b respectively.*

Proof. The geodesic between a and b is a circle centered at $t \in \mathbb{R}$ with radius $r > 0$. Write $a = t + re^{i\theta_1}$ and $b = t + re^{i\theta_2}$. By relabeling a and b if necessary, we can assume that $\pi > \theta_1 > \theta_2 > 0$. The area of the triangle is then

$$\int_{t+r\cos\theta_1}^{t+r\cos\theta_2}\int_{\sqrt{r^2-(x-t)^2}}^{\infty}\frac{1}{y^2}\,\mathrm{d}y\,\mathrm{d}x = \theta_1 - \theta_2$$

and one can verify geometrically that $\alpha + \theta_1 = \pi$ and $\beta = \theta_2$. □

Theorem 6.3 (Lambert's formula)**.** *Let a, b, c be distinct points in $\mathbb{U}^2$ not all lying on the same line. Then the area of the triangle they define is $\pi - (\alpha + \beta + \gamma)$ where α, β and γ are the interior angles of the triangle.*

Proof. By rotating the hyperbolic plane we can assume that no two of the points a, b, c have the same real part. Extend the geodesic from b to c so that it intersects the real axis and let w be the intersection point. The triangle determined by a, c and w has interior angles $\pi - \gamma$, 0, and δ for some $\delta > 0$ so its area is $\pi - ((\pi - \gamma) + \delta) = \gamma - \delta$. The triangle determined by a, b and w has interior angles β, 0 and $\alpha + \delta$ so its area is $\pi - (\alpha + \beta + \delta)$. Thus the area of the triangle determined by a, b and c is the difference of these two areas, which is

$$\pi - (\alpha + \beta + \delta) - (\gamma - \delta) = \pi - (\alpha + \beta + \gamma)$$

as desired. □

One can use this result to show that every triangle determined by three points on the boundary of $\mathbb{U}^2$ has area π. Since the Poincaré disc model is also conformal, the same area formula applies there.

7. Tilings

It is well-known that only three of the regular polygons (the triangle, the square and the hexagon) can be used to tile the Euclidean plane. This is because an integer number of polygons need to fit around every vertex, and the only regular polygons with this property are the triangle, the square and the hexagon.

We have already seen that the interior angles of a hyperbolic triangle always sum to less that π. Moreover, the larger the triangle, the smaller the sum of the angles. Thus by using large enough triangles we can fit any integer number of triangles around a point. As a consequence, there is an infinitely richer variety of tilings of the hyperbolic plane than of the Euclidean plane.

We describe here how to tile $\mathbb{D}^2$ by triangles so that $n \geq 7$ triangles meet at every vertex. Let $1, \omega, \omega^2$ be the roots of unity. For each $0 < t < 1$ consider the triangle in the Poincaré disc model determined by the points $t, t\omega$ and $t\omega^2$. As $t \to 0$ the area of the triangle tends to 0 and as $t \to 1$ the area tends to 1. By Lambert's formula for the area of a triangle, as $t \to 0$ the interior angle tends to $2\pi/6$ and as $t \to 1$ the interior angle tends to 0. Thus for any $k \geq 7$ we can choose t such that the interior angle of the corresponding triangle T_k is exactly $2\pi/k$. By reflecting T_k across each of its sides we obtain three more triangles. Reflecting in the new sides adds more triangles and, proceeding by induction, we can tile all of $\mathbb{D}^2$ this way.

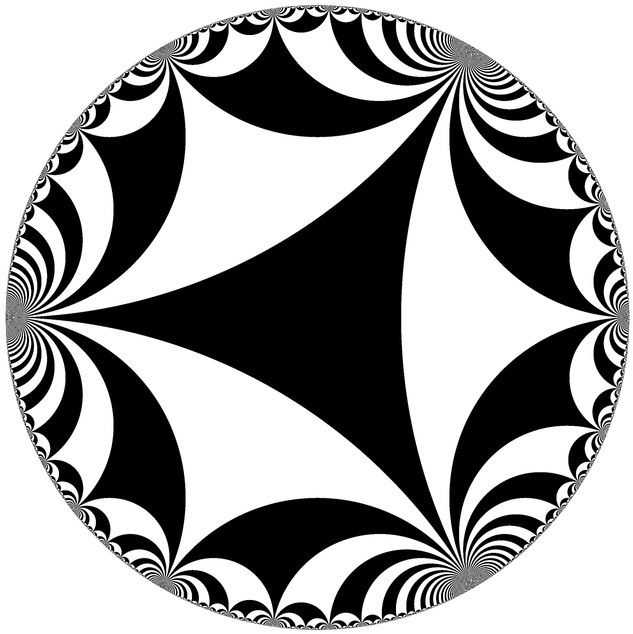

FIGURE 4. A tiling of $\mathbb{D}^2$ by triangles with all vertices at the boundary. [3]

References

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